Bills and coins in prime world in view of H.F. Scherk's theorem

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Abstract

We present expressions for primes of type

$$p_{4n-6} = (1 \pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5) \pm p_6 \pm p_8 \pm ... \pm p_{4n-10} + p_{4n-8}, n \geq 4$$

with greater predictability range in comparison with classical H.F. Scherk's theorem.

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1 Introduction

In 1833 two empirical theorems (today we say "conjectures") were stated by H.F. Scherk: "Every prime of odd rank can be composed by addition and subtraction of all the smaller primes, each taken once. Every prime of even rank can be composed similarly, except that the next earlier prime is doubled" [2]. At that times number 1 was treated as prime, so the corresponding formulae can be written in today's notation as:

$$\begin{aligned} p_{2n} &= 1 \pm p_1 \pm p_2 \pm ... \pm p_{2n-2} \pm p_{2n-1}, \\ p_{2n+1} &= 1 \pm p_1 \pm p_2 \pm ... \pm p_{2n-1} \pm 2p_{2n}. \end{aligned} \tag{1}$$

These formulae are valid for all natural numbers n by suitable choice of plus and minus signs. H.F. Scherk's conjecture was proved only 95 years later by S.S. Pillai [3]. Since this, several new proofs were published [1, 4, 5].

As for all natural numbers $n > 5$ there are at least two primes between n and $2n$, we get $p_{n+2} < 2p_n$ for all $n \geq 4$. In this article we try to obtain (1) analogues, based on this criteria.

2 Theorems and results

Theorem 2.1. *If $p_1, p_2, ...$ is a prime number sequence, then for all natural numbers $n \geq 4$ and by suitable choice of plus and minus signs each even natural number $2k < p_{4n-6}$ can be written in the form*

$$2k = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5) \pm p_6 \pm p_8 \pm ... \pm p_{4n-10} + p_{4n-8}. \tag{2}$$

Proof. Lower limit can be calculated manually. Thus, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$, $p_7 = 17$, $p_8 = 19$, $p_9 = 23$, $p_{10} = 29$. For $n = 4$ we have $p_{4n-10} = p_6 = 13$, $p_{4n-8} = p_8 = 19$ and $p_{4n-6} = p_{10} = 29$. It can easily be checked that the sequence $\pm 2 \pm 3 \pm 5 \pm 7 \pm 11 \pm 13 + 19$ gives all even natural numbers up to $28 = p_{4n-6} - 1$. Shorter sequences do not work, thus $\pm 2 \pm 3 \pm 5 \pm 7 \pm 11 + 13$ did not give 4 and $\pm 2 \pm 3 \pm 5 + 7$ did not give 10.

Now suppose that Theorem 1 holds for any natural $n \geq 4$ and $2k < p_{4n-2}$ is an even natural number. As $p_{n+2} < 2p_n$ holds for all natural $n \geq 4$, we have $p_{4n-2} < 2p_{4n-4}$ and, as $2k < p_{4n-2}$, number $0 < 2k < 2p_{4n-4}$. Then

$$-p_{4n-4} < 2k - p_{4n-4} < p_{4n-4}$$

and, by suitable choice of + and – signs, we have

$$0 < \pm(2k - p_{4n-4}) < p_{4n-4}.$$

Again, $p_{4n-4} < 2p_{4n-6}$ and we have

$$0 \leq \pm(\pm(2k - p_{4n-4}) - p_{4n-6}) < p_{4n-6}.$$

As $\pm(\pm(2k - p_{4n-4}) - p_{4n-6})$ now is an even natural number (or 0) and it is less than p_{4n-6}, in accordance with our inductive assumption it can be written as

$$\pm(\pm(2k - p_{4n-4}) - p_{4n-6}) = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5) \pm p_6 \pm p_8 \pm ... \pm p_{4n-10} + p_{4n-8}.$$

Then, by suitable choice of + and – signs,

$$2k = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5) \pm p_6 \pm p_8 \pm ... \pm p_{4n-10} \pm p_{4n-8} \pm p_{4n-6} + p_{4n-4}.$$

This proves Theorem 1 for $n + 1$ and, by induction, for all natural numbers $n > 4$. □

Theorem 2.2. *If $p_1, p_2, ...$ is a prime number sequence, then for all natural numbers $n \geq 4$ and by suitable choice of plus and minus signs each odd natural number $2k + 1 \leq p_{4n-5}$ can be written in the form*

$$2k + 1 = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6) \pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + p_{4n-7}. \tag{3}$$

Proof. Proof is analogous as for Theorem 1. □

Corollary 2.3. *If $p_1, p_2, ...$ is a prime number sequence, then for all natural numbers $n \geq 4$ and by suitable choice of plus and minus signs equation*

$$p_{4n-5} = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6) \pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + p_{4n-7} \tag{4}$$

holds. The same for all odd primes $p < p_{4n-5}$.

Both theorems have simple explanation – sequences $\pm p_6 \pm p_8 \pm ... \pm p_{4n-10} \pm p_{4n-8}$ and $\pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + p_{4n-7}$ cannot generate all necessary even/odd numbers in specified limits; produced gaps are filled by initial fragments $(\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5)$ or $(\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6)$. Here smaller primes are functioning like coins/small change in money system, while p_6/p_7 and greater are bills.

Now we are ready to obtain analogues of equations (1).

As $p_{4n-6} - 1 < p_{4n-6}$ and is even, in accordance with Theorem 1 (equation 2) it can be written as

$$p_{4n-6} - 1 = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5) \pm p_6 \pm p_8 \pm ... \pm p_{4n-10} + p_{4n-8}.$$

Then $p_{4n-6} = (1 \pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5) \pm p_6 \pm p_8 \pm ... \pm p_{4n-10} + p_{4n-8}$.

As $p_{4n-4} < 2p_{4n-6}$, number $p_{4n-4} - p_{4n-6}$ is less than p_{4n-6} and is even, it can be written as

$$p_{4n-4} - p_{4n-6} = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5) \pm p_6 \pm p_8 \pm ... \pm p_{4n-10} + p_{4n-8}.$$

Then $p_{4n-4} = (1 \pm c_1 p_1 \pm c_2 p_2 \pm c_3 p_3 \pm c_4 p_4 \pm c_5 p_5) \pm c_6 p_6 \pm c_8 p_8 \pm ... \pm c_{4n-10} p_{4n-10} + 2p_{4n-8}$; here $c_i \in [0, 2], i = 1, 2, ..., 4n - 10$.

Number $p_{4n-5} - p_{4n-7} - 1$ is less than p_{4n-5} and is odd, so in accordance with Theorem 2 (equation 3) it can be written as

$$p_{4n-5} - p_{4n-7} - 1 = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6) \pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + p_{4n-7}.$$

Then $p_{4n-5} = (1 \pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6) \pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + 2p_{4n-7}$.

As $p_{4n-3} < 2p_{4n-5}$, number $p_{4n-3} - p_{4n-5} - 1$ is less than p_{4n-5} and is odd, it can be written as

$$p_{4n-3} - p_{4n-5} - 1 = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6) \pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + p_{4n-7}.$$

Then $p_{4n-3} = (1 \pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6) \pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + p_{4n-7} + p_{4n-5}$. As $p_{4n-5} = (\pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6) \pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + p_{4n-7}$ (from Corollary, equation (4)), we get $p_{4n-3} = (1 \pm c_1 p_1 \pm c_2 p_2 \pm c_3 p_3 \pm c_4 p_4 \pm c_5 p_5 \pm c_6 p_6) \pm c_7 p_7 \pm c_9 p_9 \pm ... \pm c_{4n-9} p_{4n-9} + 2p_{4n-7}$; here $c_i \in [0, 2], i = 1, 2, ..., 4n - 9$.

These results are combined in:

Theorem 2.4. *If $p_1, p_2, ...$ is a prime number sequence, then for all natural numbers $n \geq 4$ and by suitable choice of plus and minus signs the following formulae holds.*

$p_{4n-6} = (1 \pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5) \pm p_6 \pm p_8 \pm ... \pm p_{4n-10} + p_{4n-8}$.

$p_{4n-5} = (1 \pm p_1 \pm p_2 \pm p_3 \pm p_4 \pm p_5 \pm p_6) \pm p_7 \pm p_9 \pm ... \pm p_{4n-9} + 2p_{4n-7}$.

$p_{4n-4} = (1 \pm c_1p_1 \pm c_2p_2 \pm c_3p_3 \pm c_4p_4 \pm c_5p_5) \pm c_6p_6 \pm c_8p_8 \pm ... \pm c_{4n-10}p_{4n-10} + 2p_{4n-8}$;
here $c_i \in [0, 2], i = 1, 2, ..., 4n - 10$.

$p_{4n-3} = (1 \pm c_1p_1 \pm c_2p_2 \pm c_3p_3 \pm c_4p_4 \pm c_5p_5 \pm c_6p_6) \pm c_7p_7 \pm c_9p_9 \pm ... \pm c_{4n-9}p_{4n-9} + 2p_{4n-7}$;
here $c_i \in [0, 2], i = 1, 2, ..., 4n - 9$.

Of course, initial simplicity and elegance of (1) are lost, but predictability range now is greater.

References

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