

EVEN-ODD SUBSERIES OF THE MULTIPLE ZETA VALUE

$$\zeta(2, 1)$$

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ABSTRACT. The Multiple Zeta Value $\zeta(2, 1)$ is a double sum over two positive integer indices. If these indices are restricted by parity to cover either only even or only odd positive integers, the Multiple Zeta Value splits into four non-overlapping subseries. The manuscript evaluates these four series in terms of the Riemann Zeta function, π and $\log 2$.

1. NOMENCLATURE

The Multiple Zeta Values (MZV's) $\zeta(s, 1)$ are double sums of reciprocal powers of positive integers [9, 25],

Definition 1. (MZV of length 2)

$$(1) \quad \zeta(s, 1) \equiv \sum_{n \geq 2} \frac{1}{n^s} \sum_{m=1}^{n-1} \frac{1}{m}.$$

Remark 1. The $\zeta(.,.)$ in this manuscript are not Hurwitz Zeta-functions.

By sampling only the terms with even or odd n or m we define for example the $o - o$ subseries

$$(2) \quad \zeta_{oo}(s, 1) \equiv \sum_{n=3,5,7,\dots} \frac{1}{n^s} \sum_{m=1,3,5,\dots}^{n-2} \frac{1}{m} = \sum_{n=3,5,7,\dots} \frac{1}{n^s} O_{(n-1)/2}.$$

Definition 2.

$$(3) \quad O_l \equiv \sum_{k=1}^l \frac{1}{2k-1}$$

are the odd harmonic numbers.

Adding all four subseries of the disjoint parities recovers the full series $\zeta(s)$ [6, A002117]:

$$(4) \quad \zeta_{oo}(2, 1) + \zeta_{eo}(2, 1) + \zeta_{oe}(2, 1) + \zeta_{ee}(2, 1) = \zeta(2, 1) = \zeta(3).$$

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2. A DOUBLY ODIIOUS SUBSERIES OF $\zeta(2, 1)$

The exponent $s = 2$ defines

$$(5) \quad \zeta_{oo}(2, 1) \equiv \sum_{n=3,5,7,9,\text{odd}} \sum_{m=1,3,5,7,\dots}^{n-2} \frac{1}{n^2 m} = \sum_{n \geq 1} \sum_{m=1}^n \frac{1}{(2n+1)^2 (2m-1)}.$$

This matches (almost) the known series [23, (32)][14, (2.6)]

$$(6) \quad \sum_{n \geq 1} \frac{O_n}{(2n-1)^j} = \frac{1}{4} [\zeta(j+1) + \eta(j+1)] + \frac{1}{2} [\zeta(j) + \eta(j)] \log 2 \\ - \frac{1}{2^{j+1}} \sum_{k=1}^{j/2-1} (2^{2k} - 1) \zeta(2k) \zeta(1-2k+1).$$

Definition 3.

$$(7) \quad \eta(z) \equiv \sum_{n \geq 1} \frac{(-)^{n+1}}{n^z} = (1 - 2^{1-z}) \zeta(z)$$

is the Dirichlet eta function.

At $j = 2$ the formula reads in particular

$$(8) \quad \sum_{n \geq 1} \frac{O_n}{(2n-1)^2} = \frac{7}{16} \zeta(3) + \frac{1}{8} \pi^2 \log 2.$$

With a compensation for the $n = 1$ component along the diagonal of the triangular grid of n and k this becomes [6, A173624][8, 9.521.1]

$$(9) \quad S_{oo}(2, 1) = \sum_{n \geq 2} \frac{1}{(2n-1)^2} \sum_{k=1}^{n-1} \frac{1}{2k-1} = \sum_{n \geq 1} \frac{1}{(2n-1)^2} \sum_{k=1}^n \frac{1}{2k-1} - \sum_{n \geq 1} \frac{1}{(2n-1)^3} \\ = \frac{7}{16} \zeta(3) + \frac{1}{8} \pi^2 \log 2 - \frac{7}{8} \zeta(3)$$

$$(10) \quad \boxed{S_{oo}(2, 1) = -\frac{7}{16} \zeta(3) + \frac{1}{8} \pi^2 \log 2 \approx 0.329236162849817068243549448 \dots}$$

Another representation of (5) via resummation [17, (3.34)] is

$$(11) \quad S_{oo}(2, 1) = \frac{1}{8} \sum_{n \geq 1} \sum_{m=0}^{n-1} \frac{1}{(n+1/2)^2 (m+1/2)} = \frac{1}{8} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(m+k+1/2)^2} \frac{1}{m+1/2}.$$

By converting the factors into Pochhammer symbols with

$$(12) \quad \frac{1}{z+n} = \frac{1}{z} \frac{(z)_n}{(1+z)_n},$$

the double sum is a sum over generalized hypergeometric functions [19]

$$(13) \quad \zeta_{oo}(2, 1) = \frac{1}{8} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \left[\frac{1}{k+1/2} \frac{(k+1/2)_m}{(k+3/2)_m} \right]^2 \frac{1}{1/2} \frac{(1/2)_m}{(3/2)_m} \\ = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{(k+1/2)^2} {}_4F_3 \left(\begin{matrix} k+1/2, k+1/2, 1/2, 1 \\ k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right).$$

One can lift contiguously the upper parameters $k+1/2$ or $1/2$ such that the order of the hypergeometric function is smaller by cancellation of lower and upper parameters [7, (A2.2)]:

$$(14) \quad {}_4F_3 \left(\begin{matrix} k+1/2, k+1/2, 1/2, 1 \\ k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right) \\ = (k+1/2) {}_4F_3 \left(\begin{matrix} k+3/2, k+1/2, 1/2, 1 \\ k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right) - \frac{1}{2} {}_4F_3 \left(\begin{matrix} k+1/2, k+1/2, 3/2, 1 \\ k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right) \\ = (k+1/2) {}_3F_2 \left(\begin{matrix} k+1/2, 1/2, 1 \\ k+3/2, 3/2 \end{matrix} \mid 1 \right) - \frac{1}{2} {}_3F_2 \left(\begin{matrix} k+1/2, k+1/2, 1 \\ k+3/2, k+3/2 \end{matrix} \mid 1 \right).$$

The first term on the right hand side is a difference of digamma-values: [16, 7.4.4.33]

$$(15) \quad {}_3F_2 \left(\begin{matrix} k+1/2, 1/2, 1 \\ k+3/2, 3/2 \end{matrix} \mid 1 \right) = \frac{\frac{1}{2}(k+\frac{1}{2})}{k} [\psi(k+1/2) - \psi(1/2)].$$

The second term is a trigamma function [16, 7.4.4.34][1]

$$(16) \quad {}_3F_2 \left(\begin{matrix} k+1/2, k+1/2, 1 \\ k+3/2, k+3/2 \end{matrix} \mid 1 \right) = (k+1/2)^2 \psi'(k+1/2).$$

Both terms back into (14) yields

$$(17) \quad {}_4F_3 \left(\begin{matrix} k+1/2, k+1/2, 1/2, 1 \\ k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right) \\ = (k+1/2)^2 \frac{1}{2k^2} [\psi(k+1/2) - \psi(1/2)] - \frac{1}{2k} (k+1/2)^2 \psi'(k+1/2),$$

and this back into (13) yields

$$(18) \quad \zeta_{oo}(2, 1) = \sum_{k \geq 1} \frac{1}{8k^2} [\psi(k+1/2) - \psi(1/2)] - \frac{1}{8k} \psi'(k+1/2).$$

On the right hand side we insert $\psi(1/2) = -\gamma - 2 \log 2$ [5, (2.13)][12] and [3, (7a)]

$$(19) \quad \sum_{k=1}^{\infty} \frac{1}{k^2} [\gamma + \psi(k+1/2)] = \frac{7}{2} \zeta(3) - 2 \log 2 \zeta(2).$$

The additional aspect obtained by comparing this right hand side with the right hand side of (9) is a representation of the sum of the trigamma-values [6, A400508]:

$$(20) \quad \boxed{7\zeta(3) - \pi^2 \log 2 = \sum_{k \geq 1} \frac{1}{k} \psi'(k+1/2) \approx 1.57330985 \dots,}$$

From the trigamma integral representation [21][20]

$$(21) \quad \psi'(z) = \int_0^1 \frac{t^{z-1}}{t-1} \log t dt$$

we find by interchange of summation and integration that (20) is also the value of a logarithmic integral [24, (3.65)][15, 3.3.13]:

$$(22) \quad \sum_{k \geq 1} \frac{1}{k} \psi'(k+1/2) = \sum_{k \geq 1} \int_0^1 \frac{t^{k-1/2}}{k(t-1)} \log t dt = \int_0^1 \frac{\log(1-t)}{(1-t)\sqrt{t}} \log t dt \\ = \int_0^1 \frac{\log(1-t) \log t}{t\sqrt{1-t}} dt.$$

3. AN EVEN-ODIOUS SUBSERIES OF $\zeta(2, 1)$

Sampling even n and odd m defines

$$(23) \quad \zeta_{eo}(2, 1) \equiv \sum_{n=2,4,6,\dots} \frac{1}{n^2} \sum_{m=1,3,5,7,\dots}^{n-1} \frac{1}{m} = \frac{1}{4} \sum_{n \geq 1} \frac{1}{n^2} \sum_{m=1}^n \frac{1}{2m-1} = \frac{1}{4} \sum_{n \geq 1} \frac{1}{n^2} O_n$$

which are covered by [23, (31)]

$$(24) \quad \sum_{n \geq 1} \frac{O_n}{n^{2m}} = \frac{2^{2m+1}-1}{4} \zeta(2m+1) - \frac{1}{2} \sum_{j=1}^{m-1} (2^{2m-2j+1}-1) \zeta(2j) \zeta(2m-2j+1),$$

in particular at $m = 1$

$$(25) \quad \sum_{n \geq 1} \frac{O_n}{n^2} = \frac{7}{4} \zeta(3);$$

$$(26) \quad \boxed{\zeta_{eo}(2, 1) = \frac{7}{16} \zeta(3).}$$

4. AN EVEN-EVEN SUBSERIES OF $\zeta(2, 1)$

Sampling even n and even m defines

$$(27) \quad \zeta_{ee}(2, 1) \equiv \sum_{n=4,6,\dots} \frac{1}{n^2} \sum_{m=2,4,6,\dots}^{n-2} \frac{1}{m} = \frac{1}{8} \sum_{n \geq 2} \frac{1}{n^2} \sum_{m=1}^{n-1} \frac{1}{m} = \frac{1}{8} \sum_{n \geq 1} \frac{1}{n^2} \sum_{m=1}^n \frac{1}{m} - \frac{1}{8} \sum_{n \geq 1} \frac{1}{n^3} \\ = \frac{1}{8} \sum_{n \geq 1} \frac{1}{n^2} H_m - \frac{1}{8} \zeta(3).$$

Definition 4.

$$(28) \quad H_m \equiv \sum_{k=1}^m \frac{1}{k}$$

are the Harmonic Numbers.

That is known via [12]

$$(29) \quad \sum_{n \geq 1} \frac{H_n}{n^2} = 2\zeta(3).$$

$$(30) \quad \boxed{\rightsquigarrow \zeta_{ee}(2, 1) = \frac{1}{8}\zeta(3)}$$

5. THE ODDIOUS-EVEN SUBSERIES OF $\zeta(2, 1)$

$$(31) \quad \zeta_{oe}(2, 1) = \sum_{n=3,5,7,\dots} \frac{1}{n^2} \sum_{m=2,4,6,\dots}^{n-1} \frac{1}{m} = \frac{1}{2} \sum_{n \geq 1} \frac{1}{(2n+1)^2} \sum_{m=1}^n \frac{1}{m} = \frac{1}{8} \sum_{n \geq 1} \frac{1}{(n+1/2)^2} H_n$$

This is handled by [22]

$$(32) \quad \sum_{n=1}^{\infty} \frac{H_n}{(n+\alpha)^2} = \gamma\psi'(\alpha) + \psi(\alpha)\psi'(\alpha) - \frac{1}{2}\psi''(\alpha)$$

at $\alpha = 1/2$:

$$(33) \quad \zeta_{oe}(2, 1) = \frac{1}{8}[\gamma\psi'(1/2) + \psi(1/2)\psi'(1/2) - \frac{1}{2}\psi''(1/2)]$$

with polygamma special values [6, A020759, A102753][10, (3.7)]

$$(34) \quad \psi(1/2) = -\gamma - 2\log 2; \quad \psi'(1/2) = \frac{\pi^2}{2}; \quad \psi''(1/2) = -14\zeta(3).$$

$$(35) \quad \boxed{\zeta_{oe}(2, 1) = \frac{1}{8} [7\zeta(3) - \pi^2 \log 2] .}$$

In the same spirit as above, an alternative representation by hypergeometric series is

$$(36) \quad \begin{aligned} \zeta_{oe}(2, 1) &= \frac{1}{2} \sum_{n \geq 1} \frac{1}{(2n+1)^2} \sum_{m=1}^n \frac{1}{m} = \frac{1}{2} \sum_{m \geq 1} \sum_{n \geq m} \frac{1}{(2n+1)^2} \frac{1}{m} = \frac{1}{8} \sum_{m \geq 1} \sum_{n \geq m} \frac{1}{(n+1/2)^2} \frac{1}{m} \\ &= \frac{1}{8} \sum_{m \geq 1} \sum_{k \geq 0} \frac{1}{(m+k+1/2)^2} \frac{1}{m} = \frac{1}{8} \sum_{m \geq 0} \sum_{k \geq 0} \frac{1}{(m+k+3/2)^2} \frac{1}{m+1} \\ &= \frac{1}{8} \sum_{m \geq 0} \sum_{k \geq 0} \left[\frac{1}{k+3/2} \frac{(k+3/2)_m}{(k+5/2)_m} \right]^2 \frac{(1)_m}{(2)_m} = \frac{1}{8} \sum_{k \geq 0} \frac{1}{(k+3/2)^2} {}_4F_3 \left(\begin{matrix} k+3/2, k+3/2, 1, 1 \\ k+5/2, k+5/2, 2 \end{matrix} \mid 1 \right). \end{aligned}$$

The contiguous reduction is

$$(37) \quad \begin{aligned} &(k+1/2)_4 F_3 \left(\begin{matrix} k+3/2, k+3/2, 1, 1 \\ k+5/2, k+5/2, 2 \end{matrix} \mid 1 \right) \\ &= (k+3/2)_3 F_2 \left(\begin{matrix} k+3/2, 1, 1 \\ k+5/2, 2 \end{matrix} \mid 1 \right) - {}_3F_2 \left(\begin{matrix} k+3/2, k+3/2, 1 \\ k+5/2, k+5/2 \end{matrix} \mid 1 \right). \end{aligned}$$

The first term on the right hand side is [16, 7.4.4.33]

$$(38) \quad {}_3F_2 \left(\begin{matrix} k+3/2, 1, 1 \\ k+5/2, 2 \end{matrix} \mid 1 \right) = \frac{k+3/2}{k+1/2} [\psi(k+3/2) - \psi(1)].$$

The second term on the right hand side is [16, 7.4.4.34]

$$(39) \quad {}_3F_2 \left(\begin{matrix} k+3/2, k+3/2, 1 \\ k+5/2, k+5/2 \end{matrix} \mid 1 \right) = (k+3/2)^2 \psi'(k+3/2).$$

Both sides added read

$$(40) \quad {}_4F_3 \left(\begin{matrix} k+3/2, k+3/2, 1, 1 \\ k+5/2, k+5/2, 2 \end{matrix} \mid 1 \right) \\ = \frac{(k+3/2)^2}{(k+1/2)^2} [\psi(k+3/2) - \psi(1)] - \frac{1}{k+1/2} (k+3/2)^2 \left[\psi'(k+1/2) - \frac{1}{(k+1/2)^2} \right] \\ = \frac{(k+3/2)^2}{(k+1/2)^2} [\psi(k+3/2) - \psi(1)] - \frac{1}{k+1/2} (k+3/2)^2 \psi'(k+1/2) + \frac{1}{(k+1/2)^3} (k+3/2)^2.$$

Back into (36)

$$(41) \quad \zeta_{oe}(2, 1) = \frac{1}{8} \sum_{k \geq 0} \left[\frac{1}{(k+1/2)^2} [\psi(k+3/2) - \psi(1)] - \frac{1}{k+1/2} \psi'(k+1/2) + \frac{1}{(k+1/2)^3} \right].$$

Again with (21), the substitution $t = u^2$ and partial fraction decomposition

$$(42) \quad \sum_{k \geq 0} \frac{1}{k+1/2} \psi'(k+1/2) = - \sum_{k \geq 0} \int_0^1 \frac{t^{k-1/2}}{(k+1/2)(1-t)} \log t dt = - \int_0^1 \frac{\log \frac{1+\sqrt{t}}{1-\sqrt{t}}}{(1-t)t} \log t dt \\ = -4 \int_0^1 \frac{\log \frac{1+u}{1-u}}{(1-u^2)u} \log u du \\ = -4 \int_0^1 \log \frac{1+u}{1-u} \log u \left[\frac{1}{u} + \frac{1}{2} \frac{1}{1-u} - \frac{1}{2} \frac{1}{1+u} \right] du.$$

The integrated first term is [20]

$$(43) \quad \int_0^1 \frac{\log u}{u} \log \frac{1+u}{1-u} du = -\frac{7}{4} \zeta(3).$$

Splitting $\log \frac{1+u}{1-u} = \log(1+u) - \log(1-u)$, the others are also published [2]. Collecting all terms [6, A398394],

$$(44) \quad \boxed{\sum_{k \geq 0} \frac{1}{k+1/2} \psi'(k+1/2) = \frac{7}{2} \zeta(3) + \pi^2 \log 2.}$$

Another by-catch follows comparing the right hand sides of (35) and (41):

$$(45) \quad \boxed{\sum_{k \geq 0} \frac{1}{(k+1/2)^2} [\psi(k+3/2) + \gamma] = \frac{7}{2} \zeta(3).}$$

6. THE ODDIOUS $\zeta(3, 1)$

The MZV at $s = 3$ is $\zeta(3, 1) = \zeta(2)^2/10$ [6, A098198]. The hypergeometric evaluation of the doubly odious subsequence of $\zeta(3, 1)$ in the spirit of Sec. 2 starts as follows:

(46)

$$\begin{aligned}
 S_{oo}(3, 1) &= \sum_{n \geq 1} \sum_{m=1}^n \frac{1}{(2n+1)^3(2m-1)} = \frac{1}{16} \sum_{n \geq 1} \sum_{m=0}^{n-1} \frac{1}{(n+1/2)^3(m+1/2)} \\
 &= \frac{1}{16} \sum_{k \geq 1} \sum_{m \geq 0} \frac{1}{(m+k+1/2)^3(m+1/2)} \\
 &= \frac{1}{16} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \left[\frac{1}{k+1/2} \frac{(k+1/2)_m}{(k+3/2)_m} \right]^3 \frac{1}{1/2} \frac{(1/2)_m}{(3/2)_m} \\
 &= \frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{(k+1/2)^3} {}_5F_4 \left(\begin{matrix} k+1/2, k+1/2, k+1/2, 1/2, 1 \\ k+3/2, k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right).
 \end{aligned}$$

This time the contiguous relations for upper parameters $k+1/2$ and $1/2$ are [7, (A2.2)]

$$\begin{aligned}
 (47) \quad & k {}_5F_4 \left(\begin{matrix} k+1/2, k+1/2, k+1/2, 1/2, 1 \\ k+3/2, k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right) \\
 &= (k+1/2) {}_4F_3 \left(\begin{matrix} k+1/2, k+1/2, 1/2, 1 \\ k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right) - \frac{1}{2} {}_4F_3 \left(\begin{matrix} k+1/2, k+1/2, k+1/2, 1 \\ k+3/2, k+3/2, k+3/2 \end{matrix} \mid 1 \right).
 \end{aligned}$$

The first term on the right hand side is copied from (17), the second applies [16, 7.5.3.5]

$$\begin{aligned}
 (48) \quad & {}_5F_4 \left(\begin{matrix} k+1/2, k+1/2, k+1/2, 1/2, 1 \\ k+3/2, k+3/2, k+3/2, 3/2 \end{matrix} \mid 1 \right) \\
 &= (k+1/2)^3 \frac{1}{2k^3} [\psi(k+1/2) - \psi(1/2)] - \frac{1}{2k^2} (k+1/2)^3 \psi'(k+1/2) + \frac{1}{2k} \frac{(k+1/2)^3}{2} \psi''(k+1/2).
 \end{aligned}$$

Inserted into the right hand side of (46) [8, 8.361.7][21]:

$$\begin{aligned}
 (49) \quad & S_{oo}(3, 1) \\
 &= \frac{1}{16} \sum_{k \geq 1} \left[\frac{1}{k^3} [\psi(k+1/2) - \psi(1/2)] - \frac{1}{k^2} \psi'(k+1/2) + \frac{1}{2k} \psi''(k+1/2) \right] \\
 &= \frac{1}{16} \sum_{k \geq 1} \left[\frac{1}{k^3} [\psi(k+1/2) + \gamma + 2 \log 2] - \frac{1}{k^2} \psi'(k+1/2) + \frac{1}{2k} \psi''(k+1/2) \right] \\
 &= \frac{1}{16} (\gamma + 2 \log 2) \zeta(3) + \frac{1}{16} \sum_{k \geq 1} \left[\frac{1}{k^3} \psi(k+1/2) - \frac{1}{k^2} \psi'(k+1/2) + \frac{1}{2k} \psi''(k+1/2) \right].
 \end{aligned}$$

Remark 2. *The contiguous reduction is equivalent to partial fractions decomposition of (46):*

$$(50) \quad \frac{1}{(m+k+1/2)^3(m+1/2)} = \frac{4}{(2m+1)(2m+2k+1)k^2} - \frac{4}{(2m+2k+1)^2k^2} - \frac{8}{(2m+2k+1)^3k},$$

then performing the m -sums [8, 8.363.8]

$$(51) \quad \sum_{m=0}^{\infty} \frac{4}{(2m+1)(2m+2k+1)k^2} = \frac{\gamma + 2 \log 2 + \psi(k+1/2)}{k^3};$$

$$\sum_{m=0}^{\infty} -\frac{4}{(2m+2k+1)^2k^2} = -\frac{\psi'(k+1/2)}{k^2};$$

$$\sum_{m=0}^{\infty} -\frac{8}{(2m+2k+1)^3k} = \frac{\psi''(k+1/2)}{2k}.$$

The middle k -sum in (49) is [5, (8.5),(8.11)][2, Tab. III]:

$$(52) \quad \sum_{k \geq 1} \frac{1}{k^2} \psi'(k+1/2) = \sum_{k \geq 1} \frac{1}{k^2} \int_0^1 \frac{t^{k-1/2}}{t-1} \log t dt$$

$$= \int_0^1 \frac{\sqrt{t} {}_3F_2(1, 1, 1; 2, 2; t)}{t-1} \log t dt = \int_0^1 \frac{\text{Li}_2(t)}{(t-1)\sqrt{t}} \log t dt$$

$$= 4 \int_0^1 \frac{\text{Li}_2(u^2)}{u^2-1} \log u du$$

$$= 4 \int_0^1 [\text{Li}_2(u) + \text{Li}_2(-u)] \left[-\frac{1}{u+1} + \frac{1}{u-1} \right] \log u du.$$

(53)

$$\sum_{k \geq 1} \frac{1}{k^2} \psi'(k+1/2) = -\frac{121}{360} \pi^4 + 32 \text{Li}_4(1/2) + 28 \zeta(3) \log 2 - \frac{4}{3} \log^2 2 [\pi^2 - \log^2 2] \approx 1.1339899366 \dots$$

In the third k -sum, the integral representation of the polygamma functions is inserted [21][15, (1.153)]

$$(54) \quad \psi^{(k)}(z) = \int_0^1 \frac{t^{z-1}}{t-1} \log^k(t) dt, \quad k \geq 1,$$

and all these logarithmic integrals are published [2, Tab. III]:

$$(55) \quad \sum_{k \geq 1} \frac{1}{k} \psi''(k+1/2) = \sum_{k \geq 1} \frac{1}{k} \int_0^1 \frac{t^{k-1/2}}{t-1} \log^2 t dt = - \int_0^1 \frac{\log(1-t)}{(t-1)\sqrt{t}} \log^2 t dt$$

$$= - \int_0^1 \frac{\log(1-u^2)}{(u^2-1)u} \log^2(u^2) 2u du = -8 \int_0^1 \frac{\log(1-u^2)}{u^2-1} \log^2(u) du$$

$$= -4 \int_0^1 [\log(1+u) + \log(1-u)] \left[\frac{1}{u-1} - \frac{1}{u+1} \right] \log^2(u) du = -4 \left[\frac{\pi^4}{16} - 7 \log(2) \zeta(3) \right].$$

$$(56) \quad \boxed{\sum_{k \geq 1} \frac{1}{k} \psi''(k + 1/2) = 28 \log(2) \zeta(3) - \frac{\pi^4}{4} \approx -1.022606866165233 \dots}$$

Reduction of $S_{oo}(3, 1)$ to known constants is achieved by swapping the order of the two summations

$$(57) \quad \begin{aligned} S_{oo}(3, 1) &= \sum_{n \geq 1} \sum_{m=1}^n \frac{1}{(2n+1)^3(2m-1)} = \frac{1}{16} \sum_{n \geq 1} \sum_{m=0}^{n-1} \frac{1}{(n+1/2)^3(m+1/2)} \\ &= \frac{1}{16} \sum_{k \geq 1} \sum_{m \geq 0} \frac{1}{(m+k+1/2)^3(m+1/2)} \\ &= \frac{1}{16} \sum_{m \geq 0} \frac{1}{m+1/2} \sum_{k \geq 0} \frac{1}{(m+k+3/2)^3} \\ &= \frac{1}{16} \sum_{m \geq 0} \frac{1}{m+1/2} \sum_{k \geq 0} \left[\frac{1}{m+3/2} \frac{(m+3/2)_k}{(m+5/2)_k} \right]^3 \\ &= \frac{1}{16} \sum_{m \geq 0} \frac{1}{(m+1/2)(m+3/2)^3} {}_4F_3 \left(\begin{matrix} m+3/2, m+3/2, m+3/2, 1 \\ m+5/2, m+5/2, m+5/2 \end{matrix} \middle| 1 \right). \end{aligned}$$

The reduction to a polygamma function is [16, 7.5.3.5]:

$$(58) \quad \begin{aligned} S_{oo}(1, 3) &= -\frac{1}{32} \sum_{m \geq 0} \frac{1}{(m+1/2)(m+3/2)^3} (m+3/2)^3 \psi''(m+3/2) \\ &= -\frac{1}{32} \sum_{m \geq 0} \frac{1}{m+1/2} \psi''(m+3/2) = -\frac{1}{32} \sum_{m \geq 0} \frac{1}{m+1/2} \int_0^1 \frac{t^{m+1/2}}{t-1} \log^2 t dt \\ &= -\frac{1}{32} \int_0^1 \frac{\log \frac{1+\sqrt{t}}{1-\sqrt{t}}}{t-1} \log^2 t dt = -\frac{1}{4} \int_0^1 \frac{\log \frac{1+u}{1-u}}{u^2-1} \log^2(u) u du \\ &= -\frac{1}{8} \int_0^1 \log \frac{1+u}{1-u} \log^2(u) \left[\frac{1}{u+1} + \frac{1}{u-1} \right] du. \end{aligned}$$

All these logarithmic integrals are known [2]. Gathering all terms,

$$(59) \quad \boxed{S_{oo}(3, 1) = \frac{23}{5760} \pi^4 - \text{Li}_4(1/2) + \frac{1}{24} \log^2(2) [\pi^2 - \log^2(2)] \approx 0.0594411038619010762765 \dots}$$

where $\text{Li}_4(1/2) \approx 0.517479 \dots$ [6, A099218]. By plugging the constants (59), (53) and (56) into (49) one can solve for

$$(60) \quad \boxed{\sum_{k \geq 1} \frac{1}{k^3} \psi(k+1/2) = -\frac{53}{360} \pi^4 + 16 \text{Li}_4(1/2) - \frac{2}{3} \log^2(2) [\pi^2 - \log^2(2)] + \zeta(3)(12 \log 2 - \gamma).}$$

7. SUMMARY

The manuscript provides representations of $\zeta_{oo}(2, 1)$, $\zeta_{eo}(2, 1)$, $\zeta_{oe}(2, 1)$, $\zeta_{ee}(2, 1)$ and $\zeta_{oo}(3, 1)$ by constants related to the Riemann zeta-function, π , $\log 2$, and a polylogarithm.

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