Time Bidirectionality and the Coherent Superposition of Time-Reversal Partners: An Axiomatic Construction

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Abstract: An axiomatic framework for the temporal structure of quantum pure states is formulated. In addition to time-reversal symmetry, two ontological postulates are introduced: binary reality and coherent composition. A physically complete pure state is required to contain two distinct time-reversal-related branches with a definite relative phase. We prove that, for a time-reversal-covariant Hamiltonian, the reversed branch obtained by applying the antiunitary time-reversal operator to the time-reflected forward solution satisfies the same Schrödinger equation. Linearity and coherent composition then imply a coherent superposition of the two partners. The theorem does not require different energy eigenvalues: the partners may lie in the same degenerate eigenspace. The result is therefore a conditional theorem of the enlarged axiomatic framework rather than a derivation of quantum superposition from standard time-reversal symmetry alone.

Keywords: time reversal; antiunitary operator; coherent superposition; time bidirectionality; quantum foundations

1 Introduction

There exists a long-standing and deep tension between the time-reversal symmetry of microscopic dynamical equations and the arrow of time in macroscopic experience. In standard quantum mechanics, time reversal is represented by an antiunitary operator; this structure guarantees that transition probabilities are preserved and that the time-reversed dynamics bears a definite correspondence to the original dynamics [1]. However, symmetry of the laws under time reversal does not automatically entail that an actually occurring quantum state must simultaneously contain both a forward and a backward history. The latter proposition amounts to an additional stipulation concerning the ontological structure of the quantum state.

Existing time-symmetric quantum theories typically restore symmetry between prediction and retrodiction through a joint description of initial and final conditions. The time-symmetric measurement analysis proposed by Aharonov, Bergmann, and Lebowitz shows that time-symmetric probability expressions can be constructed by restricting the ensemble with both pre-selection and post-selection [2]. Nevertheless, two-state or two-boundary formulations are not equivalent to treating two branches as coherent components of a single pure state. Modern studies of symmetry in quantum evolution also indicate that extending time reversal from unitary dynamics to the full set of quantum operations requires either imposing additional restrictions on the allowed processes or modifying the i nterpretation of the operations [3]. Therefore, any theory that postulates the “simultaneous reality of forward and backward histories” should clearly distinguish standard symmetry from an added ontological axiom.

This paper adopts an axiomatic rather than an empirical-inductive approach. The research objective is not to derive the superposition principle from standard quantum mechanics alone, but to determine whether the proposition “a physical pure state is a coherent superposition of time-reversal partners” follows as a rigorous conclusion once three axioms are explicitly accepted. To avoid conceptual confusion, this paper does not invoke the time–energy uncertainty relation, nor does it require the time-reversal partners to possess different energies.

2 Mathematical Framework and Terminology

2.1 Hilbert space, rays, and antiunitary operators

Consider a closed quantum system described by a complex Hilbert space $\mathcal{H}$. A normalized state vector $|\psi\rangle \in \mathcal{H}$ satisfies $\langle\psi|\psi\rangle = 1$. Since vectors that differ only by a global phase represent the same pure state, we denote by $[\psi]$ the ray in Hilbert space determined by $|\psi\rangle$, and define

$$[\psi_1] = [\psi_2] \Leftrightarrow |\psi_2\rangle = e^{i\theta}|\psi_1\rangle, \theta \in \mathbb{R} \tag{1}$$

The time-reversal operator is denoted by $\hat{T}$. It is antiunitary, i.e., for arbitrary complex numbers a, b and arbitrary state vectors $|\varphi\rangle$、$|\chi\rangle$,

$$\hat{T}(\mathrm{a}|\varphi\rangle + b|\chi\rangle) = \mathrm{a} * \hat{T}|\varphi\rangle + b * \hat{T}|\chi\rangle, \langle\hat{T}\varphi|\hat{T}\chi\rangle = \langle\chi|\varphi\rangle \tag{2}$$

In an irreducible time-reversal sector one typically has $\hat{T}^2 = \varepsilon\mathrm{I}$, with $\varepsilon = +1$ or -1. In this paper, “two branches are distinct” is defined to mean that they do not belong to the same ray, without the a priori requirement that they be orthogonal. If one further requires that a single measurement be capable of distinguishing them perfectly, then the orthogonality condition $\langle\psi_F|\psi_B\rangle = 0$ should be added.

2.2 Time-reversal covariant dynamics

Let $\mathrm{H(t)}$ be a self-adjoint Hamiltonian; the forward state satisfies the Schrödinger equation. Taking t=0 as the reference point of time reversal, time-reversal symmetry is expressed as

$$\hat{T}\mathrm{H(t)}\hat{T}^{-1} = \mathrm{H(-t)} \tag{3}$$

For time-independent systems, Eq. (3) reduces to $\hat{T}\mathrm{H}\hat{T}^{-1}=\mathrm{H}$. If the Hamiltonian contains parameters that are odd under time reversal, such as external magnetic fields, then Eq. (3) must include the reversal of those parameters as well; this paper only treats closed systems in which all external parameters have already been incorporated into the transformation.

2.3 Branch states and the complete physical state

This paper strictly distinguishes between a “branch state” and the “complete physical state”. Given a forward branch $|\psi_F(t)\rangle$, its time-reversal partner is defined as

$$|\psi_B(t)\rangle := \hat{T}|\psi_F(-t)\rangle \tag{4}$$

Here B denotes the backward branch, and does not mean treating $e^{+iHt/\hbar}|\psi\rangle$ directly as another solution of the same forward Schrödinger equation. The complete physical state is denoted by $|\Psi(t)\rangle$; it should not be confused with any single branch.

3 Axiomatic System of Time Bidirectionality

Axiom 1 (Time-reversal symmetry)
The physical laws are symmetric under time reversal. For the closed quantum systems discussed in this paper, this requirement is realized by an antiunitary operator $\hat{T}$ together with the Hamiltonian covariance relation of Eq. (3).

Axiom 2 (Binary reality)
A physically complete quantum state must simultaneously contain a forward branch evolving from the past to the future and a backward branch evolving from the future to the past. There exists a forward branch $|\psi_F\rangle$ whose time-reversal partner $|\psi_B\rangle = \hat{T}|\psi_F\rangle$ belongs to a different ray:

$$[\psi_B] \neq [\psi_F] \tag{5}$$

Equation (5) requires the branches to be physically distinct, but it does not require the complete total state to also change into a different ray under time reversal. Time reversal may interchange the two internal constituents while leaving a suitable total state invariant.

Axiom 3 (Principle of coherent combination)
The simultaneously real forward and backward branches are not a classical probabilistic mixture, mutually independent state pairs, or merely two boundary conditions juxtaposed. Rather, they must jointly constitute the same pure quantum state and maintain a definite relative phase. Consequently, there exist time-independent complex coefficients a, b with $ab \neq 0$ such that the complete physical state takes the form

$$|\Psi(t)\rangle = \mathcal{N}[\mathrm{a}|\psi_F(t)\rangle + \mathrm{b}|\psi_B(t)\rangle] \tag{6}$$

where $\mathcal{N}$ is a normalization factor. Axiom 3 is the necessary bridge from “two components are simultaneously real” to “coherent superposition”. If this axiom were removed, binary reality would also permit incoherent mixed states or ordered state pairs, and coherent superposition could not be derived.

Ontological status of the axioms

Axiom 2 and Axiom 3 are definitional choices; they delineate the extension of the concept “physically complete quantum state” within the present framework. They do not modify the dynamics of standard quantum mechanics, nor do they claim that quantum states failing to satisfy these definitions do not exist in nature; they merely stipulate that, within this theory, any state that is not regarded as a coherent superposition of time-reversal partners does not enter the discussion of “complete physical states”. This choice renders the main theorem a conditional conclusion, rather than a new empirical assertion about nature.

4 Preliminary Lemmas

Lemma 1 (The time-reversal partner satisfies the same Schrödinger equation)

If $|\psi_F(t)\rangle$ satisfies $i\hbar\partial_t|\psi_F(t)\rangle = H(t)|\psi_F(t)\rangle$ and $\hat{T}H(t)\hat{T}^{-1} = H(-t)$, then $|\psi_B(t)\rangle$ defined by Eq. (4) also satisfies $i\hbar\partial_t|\psi_B(t)\rangle = H(t)|\psi_B(t)\rangle$.

Proof.
Let s = -t. From Eq. (4) and the chain rule, noting that ds/dt = -1, we have

$$\partial_t|\psi_B(t)\rangle = -\hat{T}[\partial_s|\psi_F(s)\rangle]_{s=-t} \tag{7}$$

The forward Schrödinger equation gives $\partial_s|\psi_F(s)\rangle = -(i/\hbar)H(s)|\psi_F(s)\rangle$. Because $\hat{T}$ is antilinear, $\hat{T}(-i|v\rangle) = +i\hat{T}|v\rangle$; hence

$$\partial_t|\psi_B(t)\rangle = -(i/\hbar)\hat{T}H(-t)|\psi_F(-t)\rangle \tag{8}$$

Using $\hat{T}H(-t) = H(t)\hat{T}$, we obtain

$$i\hbar\,\partial_t|\psi_B(t)\rangle = H(t)|\psi_B(t)\rangle \tag{9}$$

This completes the proof.

Lemma 2 (Linear combinations preserve dynamical admissibility)
If both $|\psi_F(t)\rangle$ and $|\psi_B(t)\rangle$ satisfy the same linear Schrödinger equation, then for any time-independent complex coefficients a, b, their linear combination $a|\psi_F(t)\rangle + b|\psi_B(t)\rangle$ also satisfies that equation.

Proof.
By time differentiation and the linearity of $H(t)$,

$$i\hbar\partial_t(a|\psi_F\rangle + b|\psi_B\rangle) = aH|\psi_F\rangle + bH|\psi_B\rangle = H(a|\psi_F\rangle + b|\psi_B\rangle) \tag{10}$$

This completes the proof.

Lemma 3 (The coherence of a pure state is characterized by the off-diagonal terms of its density matrix)
Assume that both branches are normalized. The normalization factor of Eq. (6) satisfies

$$\mathcal{N}^{-2}=|a|^2+|b|^2+2\mathrm{Re}[a^*b\langle\psi_F|\psi_B\rangle] \tag{11}$$

Since both branches evolve under the same unitary dynamics, their inner product, and hence the right-hand side of Eq. (11), does not change with time. The density operator of the complete pure state is

$$\rho_\Psi=\mathcal{N}^2\{|a|^2|\psi_F\rangle\langle\psi_F|+|b|^2|\psi_B\rangle\langle\psi_B|+ab^*|\psi_F\rangle\langle\psi_B|+a^*b|\psi_B\rangle\langle\psi_F|\} \tag{12}$$

The last two terms in Eq. (12) are the coherent cross terms. The corresponding incoherent mixture would retain only the first two terms; therefore, what Axiom 3 demands is not a mere probabilistic juxtaposition of two histories, but the pure-state coherence that possesses a definable relative phase.

5 Main Theorem and Its Rigorous Proof

Main Theorem.
In a closed quantum system where Axiom 1 (time-reversal symmetry), Axiom 2 (binary reality), and Axiom 3 (principle of coherent combination) all hold, every admissible complete physical pure state must contain two time-reversal partners that belong to distinct rays and must constitute a coherent superposition with nonzero coefficients. This superposition state satisfies the same Schrödinger equation as the two branches.

Proof.
Take any admissible forward branch $|\psi_F(t)\rangle$. By Axiom 1 and Lemma 1, $|\psi_B(t)\rangle=\hat{T}|\psi_F(-t)\rangle$ is a legitimate solution under the same dynamics. By Axiom 2, $[\psi_B]\neq[\psi_F]$; hence the two branches do not belong to the same ray. By Axiom 3, the complete physical state is not a classical mixture or a mere juxtaposed pair of the two, but must be their coherent combination within a single pure state. Therefore, there exist $ab\neq0$ such that the state takes the form of Eq. (6). Equation (11) yields the normalization factor $\mathcal{N}$; by Lemma 2, $|\Psi(t)\rangle$ satisfies the same Schrödinger equation; by Eq. (12), its density matrix contains nonzero coherent cross terms. Hence the complete physical pure state is a coherent superposition of two distinct time-reversal partners. The proof is complete.

Logical conclusion.
Time bidirectionality and the principle of coherent combination together demand that a physical pure state be a coherent superposition of time-reversal partners.

It should be emphasized that “together demand” in the main theorem carr ies a strict meaning: Axiom 2 supplies the two distinct branches, Axiom 3 stipulates that they be coherently combined as a pure state, and Axiom 1 guarantees that the time-reversal partner shares the same legitimate dynamical structure as the original branch. If any one of these components were absent, the conclusion could not be derived in the form presented above.

6 Corollaries and Special Cases

6.1 Time-reversal partners need not possess different energies

Consider a time-independent Hamiltonian that is invariant under time reversal, $\hat{T}\mathrm{H}\hat{T}^{-1}=\mathrm{H}$. If the forward branch is an energy eigenstate $H|E,\alpha\rangle = E|E,\alpha\rangle$, then, because E is real,

$$\mathrm{H}(\hat{T}|\mathrm{E},\alpha\rangle)=\hat{T}\mathrm{H}|\mathrm{E},\alpha\rangle=\mathrm{E}(\hat{T}|\mathrm{E},\alpha\rangle) \tag{13}$$

Hence the time-reversal partner also possesses energy E. If the two are distinct, they belong to the same degenerate energy subspace; any combination $a|E,\alpha\rangle + b\hat{T}|E,\alpha\rangle$ remains an eigenstate of H with eigenvalue E, and the energy variance is zero. The main theorem therefore requires only a coherent superposition of time-reversal partners, not a superposition of states with different energies.

6.2 $\hat{\mathbf{T}}^2 = +1$

When $\hat{T}^2 = +1$, there may exist time-reversal invariant rays in the Hilbert space that satisfy $\hat{T}|\psi\rangle \propto |\psi\rangle$ [4]. Axiom 2 excludes such single rays from serving as the seed of a complete physical branch, but it still permits the choice of states for which $\hat{T}|\psi_F\rangle$ is not proportional to $|\psi_F\rangle$. This exclusion is a selection rule imposed by the newly added ontological axiom, not a consequence of standard time-reversal symmetry.

6.3 $\hat{\mathbf{T}}^2 = -1$ and Kramers-type partners

When $\hat{T}^2 = -1$, any non-zero state is orthogonal to its time-reversal partner. Let $\mathrm{z} = \langle\psi|\hat{T}\psi\rangle$. From the antiunitary inner product relation,

$$\langle\hat{T}\psi|\hat{T}^2\psi\rangle = \langle\hat{T}\psi|\psi\rangle;\ \text{and}\ \hat{T}^2\psi = -\psi,\ \text{hence}\ \langle\hat{T}\psi|\psi\rangle = 0. \tag{14}$$

Therefore, such partners automatically satisfy the strong condition of perfect distinguishability. However, there exists no non-zero pure state ray satisfying $\hat{T}|\Psi\rangle \propto |\Psi\rangle$; the axiom system should therefore not additionally require the complete total state to be an “eigenstate” of $\hat{T}$. The correct requirement is that the pair of branches be closed under time reversal, not that the total state must remain invariant.

7 Examples

7.1 Free particle on a ring: partners with the same energy

Consider a free particle on a ring with moment of inertia I, $H=\hat{L}_z{}^2/(2I)$. The angular momentum eigenstates $|m\rangle$ satisfy

$$\hat{L}_z|m\rangle=m\hbar|m\rangle,\ \ H|m\rangle=(m^2\hbar^2/2I)|m\rangle \tag{15}$$

Time reversal yields $\hat{T}|m\rangle=|-m\rangle$. When $m\neq0$, the two states are orthogonal and distinguishable, yet possess the same energy. According to the three axioms, an admissible complete pure state may be written as

$$|\Psi_m\rangle=a|m\rangle+b|-m\rangle,ab\neq0 \tag{16}$$

Equation (16) is a coherent superposition of time-reversal partners, while still satisfying $H|\Psi_m\rangle=E_m|\Psi_m\rangle$. This example illustrates that the amended main conclusion is fully compatible with a definite energy.

7.2 Spin-1/2: $\hat{T}^2=-1$

For a spin-1/2 degree of freedom in the absence of an external magnetic field, one may take $\hat{T}=-i\sigma_y K$, where K denotes the complex conjugation operator. Then $\hat{T}^2=-1$ and

$$\hat{T}|\uparrow\rangle=|\downarrow\rangle,\hat{T}|\downarrow\rangle=|\uparrow\rangle \tag{17}$$

Binary reality here naturally provides orthogonal partners, and the principle of coherent combination yields $a|\uparrow\rangle+b|\downarrow\rangle$. It should be noted that this does not yet fix the numerical values of a and b, nor does it specify which spatial direction is to be defined as the spin basis; such information must come from the specific Hamiltonian, boundary conditions, or the preparation procedure.

8 Logical Status, Explanatory Scope, and Open Problems

8.1 Conditional theorem rather than an independent derivation from standard quantum mechanics

The main theorem is deductively closed once the three axioms are granted, but the principle of coherent combination already incorporates into the premise that "the two branches must be combined as a single pure state". Hence the contribution of the present proof is to spell out the mathematical consequences of these premises, verify their compatibility with linear Schrödinger dynamics, and rule out illegitimate steps such as mistaking the backward exponential term for a forward solution; it is not a discovery or derivation of the standard quantum superposition principle from time-reversal symmetry alone.

8.2 Distinction from the two-state vector formalism

The time-symmetric pre- and post-selection theory typically describes the measurement interval with a state evolving from the past and a state evolving backward from the future [2]. Such a two-boundary object need not be equivalent to a coherent sum in a single Hilbert-space ket. Axiom 3 of this paper specifically chooses the latter, and thus constitutes an ontological assumption stronger than a generic two-boundary description. Whether the two yield equivalent or differing experimental probabilities cannot be established solely by the formal proof given here; independent measurement rules are required.

8.3 Quantities not yet determined by the axioms

The three axioms only require that a and b be nonzero and possess a definite relative phase, but they do not supply their numerical values, their dependence on system parameters, or a probabilistic interpretation. They also do not yield the Born rule, a collapse rule, a decoherence mechanism, or the uniqueness of branch selection. If different forward seeds produce the same complete total state, additional gauge conditions or dynamical principles are needed to remove the non-uniqueness of the decomposition.

8.4 Empirical testability

If the framework yields the same predictions as standard quantum mechanics for all observables, it mainly provides an ontological reformulation; if it is claimed that binary reality generates new interference terms or forbids certain ordinarily allowed pure states, then measurable branch labels, allowed observables, and experimental signals that differ from standard predictions must be specified. In particular, non-degenerate time-reversal invariant states in systems with $\hat{T}^2 = +1$ are legitimate in the standard theory, whereas Axiom 2 does not permit them to be regarded as complete ontological states; how to make this choice compatible with spectroscopic measurements and state tomography is a question that must be answered before the theory can become a testable physical model.

8.5 Scope of applicability

The proof given in this paper applies to closed systems whose evolution is unitarily generated by a self-adjoint Hamiltonian. Time reversal for open systems, non-unitary measurements, dissipative dynamics, and general quantum channels needs to be redefined at the superoperator level; existing work has shown that the time-reversal structure of unitary dynamics cannot be unconditionally extended to all quantum operations [3]. Hence the main theorem of this paper should not be directly applied to open systems or measurement processes.

9 Conclusion

Based on time-reversal symmetry, binary reality, and the principle of coherent combination, this paper establishes a clear axiomatic framework for quantum states. By defining the backward branch $|\psi_B(t)\rangle = \hat{T}|\psi_F(-t)\rangle$, we rigorously prove that it satisfies the same Schrödinger equation as the forward branch; employing the linearity of the dynamics and the principle of coherent combination, we then obtain that the complete physic al pure state must take the form $\mathcal{N}[a|\psi_F\rangle + b|\psi_B\rangle]$, with its coherent character confirmed by the off-diagonal terms of the density matrix.

The final conclusion is that time bidirectionality and the principle of coherent combination together demand that a physical pure state be a coherent superposition of time-reversal partners. This conclusion does not involve any necessity of different energies; time-reversal partners may possess the same energy. At the same time, this conclusion is a conditional consequence of the newly added axioms, and the superposition coefficients, relative phase, Born probabilities, or experimentally testable signals are not yet determined. Thus, what this paper presents is a mathematically closed axiomatic construction, rather than a complete predictive quantum theory.

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