Linear Transformations on Vector Multi-Spaces

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Abstract

A Smarandache multi-space is a union of spaces satisfying certain conditions. If the spaces being united are vector spaces, then we have a vector multi-space [3]. In [4] we defined the notion of a multi-matrix and operations with multi-matrices, and in this paper we shall define linear mappings on a vector multi-space and study some of their properties.

1 Introduction

The notion of a multi-space was introduced by Florentin Smarandache as follows:

Definition 1.1 ([2]). *For any integer i, $1 \leq i \leq n$, let A_i be a set with a system of laws L_i, denoted by (A_i, L_i). The union*

$$\widetilde{A} = \bigcup_{i=1}^{n} (A_i; L_i)$$

is called a multi-space.

When the spaces A_i are vector spaces over the fields F_i, we have the notion of a vector multi-space.

Definition 1.2 ([2]). *Let*

$$\widetilde{V} = \bigcup_{i=1}^{k} V_i$$

be a complete multi-space with the set of binary operations

$$O(\widetilde{V}) = \{(+_i, \cdot_i),\ 1 \leq i \leq m\},$$

and let

$$\widetilde{F} = \bigcup_{i=1}^{k} F_i$$

be a multi-field space with the set of binary operations

$$O(\widetilde{F}) = \{(+_i, \times_i),\ 1 \leq i \leq k\}.$$

If, for any integers i, j, $1 \leq i, j \leq k$, and

$$\forall a, b, c \in \widetilde{V}, \qquad k_1, k_2 \in \widetilde{F},$$

the following properties hold:

(i) $(V_i, +_i, \cdot_i)$, $1 \leq i \leq k$, *is a vector space over the field F_i, with vector addition $+_i$ and scalar multiplication $\cdot_i$;*

(ii)

$$(a +_i b) +_j c = a +_i (b +_j c);$$

(iii)

$$(k_1 +_i k_2) \cdot_j a = (k_1 \cdot_j a) +_i (k_2 \cdot_j a),$$

whatever the operations $+_i, \cdot_j \in O(\widetilde{V})$ *may be,*

then it is called a multi-vector space over the multi-field space $\widetilde{F}$*, with the set of binary operations* $O(\widetilde{V})$*, and is denoted by* $(\widetilde{V}; \widetilde{F})$*.*

For the subsets $\widetilde{V_1} \subset \widetilde{V}$ and $\widetilde{F_1} \subset \widetilde{F}$, if $\widetilde{F_1}$ is a multi-field and $(\widetilde{V_1}; \widetilde{F_1})$ is a multi-vector space, then we shall call $(\widetilde{V_1}; \widetilde{F_1})$ a multi-vector subspace.

In paper [4], the notion of a multi-matrix, the type of an element, and operations with multi-matrices were introduced.

Definition 1.3 ([4]). *Let*

$$\widetilde{R} = \bigcup_{i=1}^{n} R_i, \qquad O(\widetilde{R}) = \{(+_i, \times_i),\ 1 \le i \le n\}$$

be a complete multi-ring with a fixed numbering of the rings, let $l, m \in \mathbb{N}^*$*, and let*

$$T \in M_{l,m}(\{1, 2, \ldots, n\})$$

be a matrix with l *rows,* m *columns, and entries from the set* $\{1, 2, \ldots, n\}$*. The mapping*

$$A : \{1, 2, \ldots, l\} \times \{1, 2, \ldots, m\} \to \widetilde{R}$$

is called a multi-matrix of type $(l \times m, T)$ *with entries in* $\widetilde{R}$*.*

We shall denote $A(i, j)$ by A_{ij}, and

$$A_{ij} \in R_{T_{ij}}.$$

Definition 1.4. *Let* $a \in \widetilde{R}$*. The type of the element* a *is the smallest index of the ring to which* a *belongs. The mapping*

$$t : \widetilde{R} \to \{1, 2, \ldots, n\}, \qquad t(a) = k,$$

where k *is the type of* a*, is called the type mapping.*

In this paper we shall consider complete vector multi-spaces defined over complete multi-fields.

2 Coordinate Systems

Definition 2.1. *Let* $(\widetilde{V}; \widetilde{F})$ *be a complete vector multi-space over the multi-field* $\widetilde{F}$*, where*

$$\widetilde{V} = \bigcup_{i=1}^{k} V_i, \qquad O(\widetilde{V}) = \{(+_i, \cdot_i),\ 1 \le i \le k\},$$

and

$$\widetilde{F} = \bigcup_{i=1}^{k} F_i, \qquad O(\widetilde{F}) = \{(+_i, \times_i),\ 1 \le i \le k\}$$

is a multi-field space. An ordered basis in $\widetilde{V}$ *is called a coordinate system or frame.*

Let $B = (e_1, e_2, \ldots, e_l)$ be a basis in $\widetilde{V}$, with $\dim(\widetilde{V}) = l$, and let $x \in \widetilde{V}$,

$$x = x_1 \cdot_1 e_1 +_1 x_2 \cdot_2 e_2 +_2 \cdots +_{l-1} x_l \cdot_l e_l.$$

The components $x_1, x_2, \ldots, x_l$ are called the coordinates of x in the basis $(e_1, e_2, \ldots, e_l)$, and we shall write

$$x = (x_1, x_2, \ldots, x_l).$$

Change of Coordinates

Let $B' = (e'_1, \ldots, e'_l)$ be another coordinate system in $\widetilde{V}$. We shall denote by $a_{1j}, a_{2j}, \ldots, a_{lj}$ the coordinates of the vector e'_j in the basis B:

$$\begin{aligned}
e'_1 &= a_{11} \cdot_{11} e_1 +_1 \cdots +_{l-1} a_{l1} \cdot_{l1} e_l, \\
e'_2 &= a_{12} \cdot_{12} e_1 +_1 \cdots +_{l-1} a_{l2} \cdot_{l2} e_l, \\
&\ \,\vdots \\
e'_l &= a_{1l} \cdot_{1l} e_1 +_1 \cdots +_{l-1} a_{ll} \cdot_{ll} e_l.
\end{aligned}$$

Here $\cdot_{ij}$ is the operation

$$\cdot_{ij} : F_{t(a_{ij})} \times V_{t(a_{ij})} \to V_{t(a_{ij})},$$

and $+_{ik}$ is the operation in the space V_p, where

$$p = t(a_{i1} \cdot_{i1} e_1 +_1 \cdots +_{k-1} a_{i,k-1} \cdot_{i,k-1} e_{k-1}).$$

We shall write, in condensed form,

$$e'_j = \sum_{i=1}^{l} a_{ij} e_i, \qquad j = 1, \ldots, l.$$

We have

$$\begin{aligned}
x &= \sum_{j=1}^{l} x'_j e'_j = \sum_{j=1}^{l} x'_j \left(\sum_{i=1}^{l} a_{ij} e_i \right) \\
&= \sum_{i=1}^{l} \left(\sum_{j=1}^{l} a_{ij} x'_j \right) e_i = \sum_{i=1}^{l} x_i e_i.
\end{aligned}$$

Whence, using the uniqueness of the expression of a multi-vector in a basis [3], it follows that

$$x_i = \sum_{j=1}^{l} a_{ij} x'_j, \qquad i = 1, 2, \ldots, l.$$

In matrix form, we have

$$\begin{bmatrix} x_1 \\ \vdots \\ x_l \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{l1} & \cdots & a_{ll} \end{bmatrix} \begin{bmatrix} x'_1 \\ \vdots \\ x'_l \end{bmatrix}.$$

The multi-matrix A is called the transition multi-matrix from the basis B to the basis B', and we shall write, in matrix form,

$$[x]_B = A[x]_{B'},$$

where

$$[x]_B = (x_1, \ldots, x_l)^T, \qquad [x]_{B'} = (x'_1, \ldots, x'_l)^T.$$

Let $B'' = (e''_1, \ldots, e''_l)$ be a third ordered basis in $\widetilde{V}$, and let $(x''_1, x''_2, \ldots, x''_l)$ be the coordinates of x in B''. Denoting by A_1 the transition multi-matrix from B' to B'', we have

$$[x]_{B'} = A_1[x]_{B''}.$$

From this relation and from $[x]_B = A[x]_{B'}$, we obtain

$$[x]_B = AA_1[x]_{B''}.$$

3 Linear Transformations on Multi-Vector Spaces

Definition 3.1. *Let $\widetilde{V}$ and $\widetilde{W}$ be complete vector multi-spaces over the same complete multi-field $\widetilde{F}$.*

A mapping $f : \widetilde{V} \to \widetilde{W}$ with the properties

(i)

$$f(x +_i y) = f(x) +_k f(y), \qquad i = t(x),\ k = t(f(x));$$

(ii)

$$f(\lambda \cdot_i x) = \lambda \cdot_t f(x), \qquad i = t(x),$$

is called a linear transformation or homomorphism.

Remark. $f(0_{\widetilde{V}}) = 0_{\widetilde{W}}$, since

$$f(0_{\widetilde{V}}) = f(0_k \cdot_k x) = 0_k \cdot_t f(x) = 0_l \in \widetilde{W}.$$

Here, by 0_V we mean an element $0_k \in \widetilde{V}$, and by 0_W an element $0_l \in \widetilde{W}$, in the case when it is not specified that all the elements $0_i \in V_i$ are identified. The following property holds:

$$0_l \cdot_1 x = 0_k, \qquad 0_l \in V_i, \quad x \in \widetilde{V}.$$

Indeed,

$$0_l \cdot_1 x = (0_l +_1 0_l) \cdot_1 x = 0_l \cdot_1 x +_1 0_l \cdot_1 x.$$

If $0_l \cdot_1 x = y_k$, then

$$y_k = 0_l \cdot_1 x +_1 y_k$$

and

$$0_l \cdot_1 x = y_k - y_k,$$

hence

$$0_l \cdot_1 x = 0_k.$$

Let $B = (e_1, e_2, \ldots, e_l)$ be a basis in $\widetilde{V}$, and let

$$x = \sum_{i=1}^{l} x_i e_i \in \widetilde{V}.$$

It follows that

$$f(x) = \sum_{i=1}^{l} x_i f(e_i).$$

Theorem 3.1. *There exists a unique linear transformation of the multi-vector space $\widetilde{V}$ of dimension l into the multi-vector space $\widetilde{W}$ which transforms a given basis into a given system of vectors in $\widetilde{W}$ of the same cardinality.*

Proof. Let $B = (e_1, e_2, \ldots, e_l)$ be a basis in $\widetilde{V}$, and let

$$x = \sum_{i=1}^{l} x_i e_i \in \widetilde{V}.$$

Then

$$f(x) = \sum_{i=1}^{l} x_i f(e_i).$$

It follows that f is determined by the vectors

$$f(e_1), f(e_2), \ldots, f(e_l).$$

$\square$

The Equations of a Linear Mapping

Let $B = (e_1, e_2, \dots, e_l)$ be a basis in $\widetilde{V}$, and let $\overline{B} = (\overline{e}_1, \dots, \overline{e}_m)$ be a basis in $\widetilde{W}$. The vector $f(e_j)$ is expressed in $\overline{B}$ as a linear combination of the vectors of the basis $\overline{B}$:

$$f(e_j) = \sum_{i=1}^{m} f_{ij} \overline{e}_i, \qquad j = 1, \dots, l.$$

The multi-matrix

$$[f]_{\overline{B}}^{B} = \begin{bmatrix} f_{11} & \cdots & f_{1l} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{ml} \end{bmatrix}$$

is called the multi-matrix associated with the mapping f in the bases B and $\overline{B}$.

Let

$$x = \sum_{j=1}^{l} x_j e_j$$

be an arbitrary multi-vector in $\widetilde{V}$. We have

$$\begin{aligned} f(x) = f\left(\sum_{j=1}^{l} x_j e_j\right) &= \sum_{j=1}^{l} \sum_{i=1}^{m} f_{ij} x_j \overline{e}_i \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{l} f_{ij} x_j\right) \overline{e}_i. \end{aligned}$$

Thus, with respect to the basis $\overline{B} = (\overline{e}_1, \dots, \overline{e}_m)$ in $\widetilde{W}$, the multi-vector $f(x)$ has the coordinates $y_1, \dots, y_m$ given by the formulas

$$y_i = \sum_{j=1}^{l} f_{ij} x_j, \qquad i = 1, \dots, m,$$

or, in matrix form,

$$[f(x)]_{\overline{B}} = [f]_{\overline{B}}^{B} [x]_B.$$

Theorem 3.2. *The image $f(X)$ of a vector multi-subspace X of the vector multi-space $\widetilde{V}$ is a vector multi-subspace of the vector multi-space $\widetilde{W}$.*

Proof. It is sufficient to show that, for any y_1 and y_2 in $f(X)$, and for any λ_1 and λ_2 in $\widetilde{F}$,

$$\lambda_1 \cdot_i y_1 +_k \lambda_2 \cdot_j y_2 \in f(X),$$

where

$$i = t(\lambda_1), \qquad j = t(\lambda_2), \qquad k = t(\lambda_1 \cdot_i y_1).$$

Since $y_1, y_2 \in f(X)$, there exist $x_1, x_2 \in X$ such that

$$f(x_1) = y_1, \qquad f(x_2) = y_2.$$

Since

$$\lambda_1 \cdot_i x_1 +_k \lambda_2 \cdot_j x_2 \in X,$$

it follows that

$$\begin{aligned} \lambda_1 \cdot_i y_1 +_k \lambda_2 \cdot_j y_2 &= \lambda_1 \cdot_i f(x_1) +_k \lambda_2 \cdot_j f(x_2) \\ &= f(\lambda_1 \cdot_i x_1 +_k \lambda_2 \cdot_j x_2) \in f(X). \end{aligned}$$

□

Theorem 3.3. *The inverse image* $f^{-1}(Y)$ *of any vector multi-subspace* Y *of the vector multi-space* $\widetilde{W}$ *is a subspace of* $\widetilde{V}$*.*

Proof. Let $x_1, x_2 \in f^{-1}(Y)$ and $\lambda_1, \lambda_2 \in \widetilde{F}$. It follows that

$$f(\lambda_1 \cdot_i x_1 +_k \lambda_2 \cdot_j x_2) = \lambda_1 \cdot_i f(x_1) +_k \lambda_2 \cdot_j f(x_2) \in f^{-1}(Y).$$

□

Definition 3.2. *Let* $f : \widetilde{V} \to \widetilde{W}$ *be a linear transformation. Then*

$$\ker f = \bigcup_{i=1}^{n} f^{-1}(0_i)$$

is called the kernel of the mapping f*.*

Theorem 3.4. $\ker f$ *is a multi-vector subspace of* $\widetilde{V}$*.*

Proof. We shall show that, in a complete multi-ring, the following relations hold:

$$\forall k, l \in \{1, 2, \ldots, n\}, \qquad 0_k +_k 0_l = 0_i, \qquad \lambda_k \cdot_k 0_l = 0_j,$$

where $i, j \in \{1, 2, \ldots, n\}$.

Indeed, $\widetilde{V}$ is a complete multi-vector space and $0_k +_k 0_l \in \widetilde{V}$. Let

$$0_k +_k 0_l = x.$$

Since

$$0_k = 0_k +_k 0_k, \qquad 0_l = 0_l +_l 0_l,$$

we have

$$x = 0_k +_k 0_l = (0_k +_k 0_l) +_i (0_k +_k 0_l) = x +_i x.$$

From $x = x +_i x$ it follows that

$$x = 0_i, \qquad i = t(x).$$

Furthermore,

$$\lambda_k \cdot_k 0_l = \lambda_k \cdot_k (0_l +_l 0_l) = \lambda_k \cdot_k 0_l +_l \lambda_k \cdot_k 0_l,$$

hence

$$\lambda_k \cdot_k 0_l = 0_j.$$

Let $x_1, x_2 \in \ker f$ and $\lambda_1, \lambda_2 \in \widetilde{F}$. Then

$$\begin{aligned} f(\lambda_1 \cdot_1 x_1 +_k \lambda_2 \cdot_2 x_2) &= \lambda_1 \cdot_1 f(x_1) +_k \lambda_2 \cdot_2 f(x_2) \\ &= \lambda_1 \cdot_1 0_1 +_k \lambda_2 \cdot_2 0_2 \\ &= 0_i +_j 0_j = 0_k. \end{aligned}$$

Therefore,

$$\lambda_1 \cdot_1 x_1 +_k \lambda_2 \cdot_2 x_2 \in \ker f.$$

□

Theorem 3.5. *Let* $\widetilde{V}$ *and* $\widetilde{W}$ *be multi-vector spaces over the same multi-field* $\widetilde{F}$*. For every linear mapping* $f : \widetilde{V} \to \widetilde{W}$*, the following relation holds:*

$$\dim(\widetilde{V}) = \dim(\ker f) + \dim(f(\widetilde{V})).$$

Proof. Let $n = \dim(\widetilde{V})$ and $r = \dim(\ker f)$. We choose a basis of $\ker f$,

$$(e_1, e_2, \ldots, e_r),$$

which we extend to a basis

$$(e_1, e_2, \ldots, e_r, e_{r+1}, \ldots, e_n)$$

of the space $\widetilde{V}$. If

$$\begin{aligned} x &= \sum_{i=1}^{n} x_i f(e_i) \\ &= 0_{i_1} + 0_{i_2} + \cdots + 0_{i_k} + \sum_{i=r+1}^{n} x_i f(e_i) \\ &= 0_k + \sum_{i=r+1}^{n} x_i f(e_i), \end{aligned}$$

then it follows that

$$f(e_{r+1}), f(e_{r+2}), \ldots, f(e_n)$$

is a system of generators for $\operatorname{Im} f$, since these vectors are also linearly independent. Indeed, if

$$f(e_{r+1}), f(e_{r+2}), \ldots, f(e_n)$$

were linearly dependent, then it would follow that

$$\alpha_{r+1} f(e_{r+1}) + \cdots + \alpha_n f(e_n) = 0_k \in \widetilde{W}$$

and

$$\alpha_{r+1} e_{r+1} + \cdots + \alpha_n e_n \in \ker f.$$

Hence

$$\alpha_{r+1} e_{r+1} + \cdots + \alpha_n e_n = \alpha_1 e_1 + \cdots + \alpha_r e_r,$$

an impossible equality, because the vectors

$$e_1, e_2, \ldots, e_n$$

are independent. □

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