On the Recent Breakthrough On Dynamical Higgs Mechanism For dRGT Gravity: How Finally it is a Framework permitting to resolve the Strong Coupling Problem

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Abstract. In a recent paper, we claimed to propose a solution to the strong coupling break-down problem of massive gravity by making the theory massless above some energy-scale.

Although interesting, it seems in light of the valuable work of Bonifacio, Hinterbichler and Rosen (*Constraints on a Gravitational Higgs Mechanism*), that our proposed mehanism fails in its aim.

At first glance, this may seem to spell the end for our model; fortunately things are rather more subtle because our model has indeed all the necessary tools to solve the strong coupling problem of massive gravity.

Indeed, we are going to demonstrate in this paper, that for some appropriate expressions of the free tensor $\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$ of our recently proposed model; it becomes possible to cleverly raise up the strong coupling scale of dRGT massive gravity without spoiling its ghost-free and local character.

More precisely, through a concrete example, we will prove in the present paper that our recently proposed framework of massive gravity is well and truly a framework permitting to resolve both the strong coupling break-down problem as well as the van Dam-Veltman-Zakharov (vDVZ) discontinuity of dRGT massive gravity; we achieve this breakthrough by highlighting the first rather simple and improved four-dimensional dRGT massive gravity capable of overcoming both strong coupling problem and vDVZ discontinuity.

1. Introduction

As well known [1, 2, 3], the strong coupling break-down problem of massive gravity is a long-standing theoretical problem that appears very hard to be resolved.

In a recent paper [4], we showed how for example graviton can acquire mass (dynamically) through the electroweak phase transition [5, 6, 7]; better we claimed that this mechanism resolve the strong coupling problem of massive gravity by making the model massless above its strong coupling scale $\Lambda_3 = \left[m^2 M_P \right]^{\frac{1}{3}}$.

Unfortunately, as notified to me by Hinterbichler, one of the authors of the paper *Constraints on a Gravitational Higgs Mechanism* [1]; it seems that simply permitting to massive gravity to become dynamically massless above its strong coupling scale Λ_3 not suffices to overcome the strong coupling break-down problem, especially if the invoked higgs mechanism is based on fields/particles with spin less than two [1] as it is the case in my model [4].

Of course, based on this observation, one might be tempted to abandon the model [4] since seeming incapable of resolving the strong coupling problem; however, that would be jumping to conclusions since things are fortunately rather more subtle.

Indeed as we are going to see, it is possible to permit to the model [4] to respect its initial aim of resolving strong coupling problem by choosing appropriate expression for the free tensor $\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$ of the model; better we will see that the obtained "improved" dRGT massive gravity handles also vDVZ discontinuity.

Before broaching the demonstration; it is important to recall that all the main problems of dRGT massive gravity (strong coupling problem and vDVZ discontinuity) come from the longitudinal scalar mode $\pi(x) = \pi$ of the graviton [8, 9, 10, 11].

More exactly it is the absence of independent kinetic term for π that gives rise to the problems, and all the difficulty resides in the fact that there is no known simple manner of adding independent kinetic term for π in the action of four-dimensional dRGT theory without spoiling the ghost-free and/or local character of the theory.

To overcome this difficulty, the four-dimensional dRGT gravity is sometimes embedded into a greater five-dimensional bulk like for example the embedding model of Gabadadze, Older and Pirtskhalava [12] which cleverly surmounts both strong coupling problem and vDVZ discontinuity; a model of which the core trick is to succeed in modifying salutary (within some energy range and from five-dimensional contributions) the dynamics of the scalar mode π of the dRGT graviton without "noticeable" altering the ghost free and local character of the theory.

Unfortunately the model [12] of Gabadadze, Older and Pirtskhalava although very interesting is not a pure four-dimensional dRGT gravity; whence the *big challenge* of getting for pure four-dimensional dRGT gravity, a trick salutary modifying the dynamics of the scalar mode π of the dRGT graviton without altering the ghost-free and local character of the theory remained still open.

In this paper, we are going to show that our recent model [4] permits to overcome this *big challenge* as long as we assume appropriate expression for the free tensor $\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$ of the model [4].

More preisely, we are going to show that for some expressions of $\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$; our model [4] will yield an "improved" dRGT massive gravity (derivable from variational principle) for which "only" the dynamics of the scalar mode π undergoes salutary change in the decoupling limit without spoiling the general ghost-free and local character of the theory; a change which permits to resolve both strong coupling problem and vDVZ discontinuity.

The paper is organised as follow, in the section 2, I make a brief recall of our recent model [4] and briefly discuss why in its original form, it probably fails to solve the strong coupling problem; In the section 3, I show how to get improved dRGT massive gravity quite capable of overcoming both strong coupling problem and vDVZ discontinuity. I devote the Section 4 for short discussion on the breakthrough.

The signature convention $(-+++)$ will be assumed for the metric; likewise Einstein's summation convention will be assumed.

2. Brief Recall

2.1. Brief Recall of the Model

In the recent paper [4], I achieved from variational principle, the modified gravity theory reading:

$$G_{\mu\nu} = \kappa T_{\mu\nu} - m^2 \ \mathcal{U}_{\mu\nu}(\Phi) + k(\Phi)A_\mu A_\nu + k(\Phi)\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A) \tag{1}$$

$$\nabla_\mu \left[\frac{\partial \mathcal{U}(g,\varphi^\rho)}{\partial(\partial_\mu \varphi^\rho)} \right] = \nabla^\mu \mathcal{U}_{\mu\nu} = 0 \tag{2}$$

$$\nabla^\mu \left[k(\Phi)A_\mu A_\nu + k(\Phi)\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A) - m^2 \ \mathcal{U}_{\mu\nu}(\Phi) \right] = 0 \tag{3}$$

where of course $G_{\mu\nu}$ denotes the Einstein tensor; κ the Einstein gravitational constant; $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}}$ the energy-momentum tensor of standard matter; m a free constant; A_μ an auxiliary variable; $\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$ a free symmetric tensor that can depend on the standard matter fields, the metric, the Stückelberg fields φ^ρ etc... and/or their derivatives, as well as of the auxiliary vector field A_μ; with $\mathcal{U}_{\mu\nu}$ the energy-momentum tensor associated to the potential of dRGT massive gravity, it reads:

$$\mathcal{U}_{\mu\nu} = \frac{\delta(\mathcal{U}(g,\varphi^\rho)\sqrt{-g})}{\sqrt{-g}\delta g^{\mu\nu}} = K_{\mu\nu} - [K]g_{\mu\nu} - [1 + 3\alpha_3]\left[K_{\mu\nu}^2 - [K]K_{\mu\nu} - \tfrac{1}{2}\mathcal{U}_2 g_{\mu\nu} \right]$$

$$+3[4\alpha_4 + \alpha_3]\left[K_{\mu\nu}^3 - [K]K_{\mu\nu}^2 + \tfrac{1}{2}K_{\mu\nu}\mathcal{U}_2 - \tfrac{1}{6}\mathcal{U}_3 g_{\mu\nu} \right] \tag{4}$$

while $\mathcal{U}_{\mu\nu}(\Phi)$ is the tensor obtained by replacing into $\mathcal{U}_{\mu\nu}$, the fudicial/Minkowski metric tensor $f_{\rho\beta}$ by the tensor $\Phi_{\rho\beta}$ reading:

$$\Phi_{\rho\beta} = \begin{pmatrix} e^{-[\varphi_0 k(\Phi)\varphi_0]}f_{00} & e^{-[\varphi_0 k(\Phi)\varphi_1]}f_{01} & e^{-[\varphi_0 k(\Phi)\varphi_2]}f_{02} & e^{-[\varphi_0 k(\Phi)\varphi_3]}f_{03} \\ e^{-[\varphi_1 k(\Phi)\varphi_0]}f_{10} & e^{-[\varphi_1 k(\Phi)\varphi_1]}f_{11} & e^{-[\varphi_1 k(\Phi)\varphi_2]}f_{12} & e^{-[\varphi_1 k(\Phi)\varphi_3]}f_{13} \\ e^{-[\varphi_2 k(\Phi)\varphi_0]}f_{20} & e^{-[\varphi_2 k(\Phi)\varphi_1]}f_{21} & e^{-[\varphi_2 k(\Phi)\varphi_2]}f_{22} & e^{-[\varphi_2 k(\Phi)\varphi_3]}f_{23} \\ e^{-[\varphi_3 k(\Phi)\varphi_0]}f_{30} & e^{-[\varphi_3 k(\Phi)\varphi_1]}f_{31} & e^{-[\varphi_3 k(\Phi)\varphi_2]}f_{32} & e^{-[\varphi_3 k(\Phi)\varphi_3]}f_{33} \end{pmatrix} \tag{5}$$

where e is the Euler's number; φ^ρ the Stückelberg fields; f_{00}, f_{01} etc... the components of

the fudicial metric $f_{\rho\beta}$ and $k(\Phi)$ a function of the standard model Higgs field Φ, reading:

$$k(\Phi) = \left[2B(e^{\Phi_1}, 1+e^{\Phi_2}) - B(e^{\Phi_2}, e^{\Phi_2})\right]^2 \qquad (6)$$

In which e denotes the Euler's number or Napier's constant; $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ the beta function; $\Phi_1 = \frac{1}{4v}\sqrt{v^2 - [2v - \sqrt{2\Phi^\dagger\Phi}]\sqrt{\Phi^{\dagger*}\Phi + \Phi^\dagger\Phi^*}}$ and $\Phi_2 = \frac{\sqrt{2\Phi^\dagger\Phi}}{4v} - \frac{1}{4}$, with v a nonzero constant; Φ^* the complex conjugate of Φ and $\Phi^{\dagger*}$ the complex conjugate of $\Phi^\dagger$.

Of course, as already discussed in the paper [4]; the interest of this above function $k(\Phi)$ is that it worths zero only when $SU(2)$ symmetry breaks i.e. only after the electroweak phase transition where Φ develops the non-zero vacuum expectation value $\frac{v}{\sqrt{2}}$, what means that $\Phi_{\rho\beta}$ reduces to $f_{\rho\beta}$ or equivalenty $\mathcal{U}_{\mu\nu}(\Phi)$ reduces to $\mathcal{U}_{\mu\nu}$ only after the elctroweak symmetry breaking; what ensures that after the elctroweak symmetry breaking, the equations (1-3) will reduce to:

$$G_{\mu\nu} = \kappa T_{\mu\nu} - m^2 \mathcal{U}_{\mu\nu} \qquad (7)$$

$$\nabla_\mu\left[\frac{\partial \mathcal{U}(g,\varphi^\rho)}{\partial(\partial_\mu\varphi^\rho)}\right] = \nabla^\mu \mathcal{U}_{\mu\nu} = 0 \qquad (8)$$

which is nothing than standard dRGT massive gravty [2, 3, 13]; whence the affirmation that after eletroweak symmetry breaking, our model (1-3) yields the famous dRGT massive gravity.

We recall that the above standard dRGT massive gravity (7-8) can be derived from the action [2, 3, 13]:

$$S = \frac{M_P^2}{2} \int \left[R + 2\kappa L_M + m^2 \mathcal{U}(g, \varphi^\rho)\right]\sqrt{-g}d^4x \qquad (9)$$

where R denotes Ricci scalar; L_M the lagrangian of standard matter; $\kappa = \frac{1}{M_P^2}$ and $\mathcal{U}(g, \varphi^\rho) = \mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4$ the potential of dRGT massive gravity.

Let's note that the principal interest of (1-3) is that if after the eletroweak symmetry breaking it yields dRGT massive gravity; one can verify as done in the paper [4] that before the electroweak phase transition, it is well and truly a massless gravity which can be written as:

$$G_{\mu\nu} = \kappa T_{\mu\nu} + m^2 \; \mathcal{T}_{\mu\nu} \qquad (10)$$

$$\nabla_\mu\left[\frac{\partial \mathcal{U}(g,\varphi^\rho)}{\partial(\partial_\mu\varphi^\rho)}\right] = \nabla^\mu \mathcal{U}_{\mu\nu} = 0 \qquad (11)$$

$$\nabla^\mu \; \mathcal{T}_{\mu\nu} = 0 \tag{12}$$

where the tensor $\mathcal{T}_{\mu\nu}$ will, in for example the case $\alpha_3 = \alpha_4 = 0$ and $f_{\rho\beta} = \eta_{\rho\beta}$, read:

$$\mathcal{T}_{\mu\nu} = \frac{k(\Phi)}{m^2} \; \mathbb{A}_{\mu\nu} + f_\varphi \; \Phi_{\mu\nu} - \left[6 + 2f_\varphi \; \Phi - f_\varphi \sqrt{\Phi^\sigma_\rho \; \Phi^\rho_\sigma}\right] g_{\mu\nu} \; + \left[3 - 2\sqrt{f_\varphi \; \Phi}\right] \sqrt{f_\varphi g_{\mu\nu} \; \Phi_{\mu\nu}} \tag{13}$$

where $k(\Phi)$ is the function given by (6); $\mathbb{A}_{\mu\nu} = A_\mu A_\nu + \mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$; $f_\varphi = \partial_\rho \varphi^\rho \partial_\rho \varphi^\rho$, with φ^ρ the Stückelberg scalars fields; $\Phi = g^{\mu\nu} \; \Phi_{\mu\nu}$ with $\Phi_{\mu\nu}$ the tensor given by (5).

In the paper [4], I had affirmed that the fact that (1-3) switches between massless and massive gravity through dynamical phase transition could permit to overcome the strong coupling break-down problem of the gotten massive gravity (7-8) if before entering in its strong coupling regime this last (7-8) give way place to the massless model (10-12). Unfortunately, this affirmation of mine did not take into account the valuable work of Bonifacio, Hinterbichler and Rosen [1] which seems show that a gravitational higgs mechanism based on fields/particles with spin less than two (like the mine [4]) not permits to improve the high-energy behavior (strong coupling scale) of massive gravity.

More precisely, there work [1] seems suggest that even in the case where the graviton of (1-3) is massless before electroweak phase transition; for sure, at leats one of the three physical degrees of freedom carried by the Stückelberg fields φ^ρ would be strongly coupled above scale $\Lambda_3 = \left[m^2 M_P\right]^{\frac{1}{3}}$ making finally the obtained massless theory (10-12) strongly coupled through φ^ρ.

In short, the work of Bonifacio, Hinterbichler and Rosen seems establish that rather than to try to raise up the strong coupling scale of dRGT massive gravity through a gravitational higgs mechanism, it would be better to find other tricks, and it is that we are going to do in this paper.

But before, let's quickly recall why from equation of motion viewpoint, $\Lambda_3 = \left[m^2 M_P\right]^{\frac{1}{3}}$ is the strong coupling scale of dRGT massive gravity, what is not a so hard task.

Indeed by considering the action (9) of dRGT gravity in decoupling limit [14]:

$$S = \int \left[-\frac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} + \frac{3}{2}\pi\Box\pi - \frac{\tilde{\alpha}}{\Lambda_3^3}X^{(1)}_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \frac{1}{\Lambda_3^6}\left(\frac{\tilde{\alpha}^2}{2} + \frac{\tilde{\beta}}{3}\right)X^{(2)}_{\mu\nu}\partial^\mu\pi\partial^\nu\pi \right.$$
$$\left. -\frac{\tilde{\beta}}{\Lambda_3^6}\left(\frac{\tilde{\alpha}}{\Lambda_3^3}\partial^\mu\pi\partial^\nu\pi - \tilde{h}^{\mu\nu}\right)X^{(3)}_{\mu\nu} + \frac{1}{M_p}\left(T_{\mu\nu}\tilde{h}^{\mu\nu} - \frac{\tilde{\alpha}}{\Lambda_3^3}T_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \pi T\right) \right]\sqrt{-g}d^4x \tag{14}$$

where $\tilde{\alpha} = 1 + \alpha_3$ and $\tilde{\beta} = \alpha_3 + \alpha_4$ are free constants; also we assumed $\kappa = \frac{1}{M_p^2}$ with M_p the Planck Mass; $\Box = \partial_\mu\partial^\mu$ the d'Alembertian; $T_{\mu\nu}$ the conserved source of gravity and $T = g^{\mu\nu}T_{\mu\nu}$ its trace, while:

$$X^{(1)}_{\mu\nu} = \frac{1}{2}\epsilon_\mu^{\;\alpha\rho\sigma}\epsilon_{\nu\;\rho\sigma}^{\;\beta}\partial_\alpha\partial_\beta\pi \tag{15}$$

$$X_{\mu\nu}^{(2)} = -\tfrac{1}{2}\epsilon_\mu{}^{\alpha\rho\gamma}\epsilon_\nu{}^{\beta\sigma}{}_\gamma \partial_\alpha\partial_\beta\pi\partial_\rho\partial_\sigma\pi \tag{16}$$

$$X_{\mu\nu}^{(3)} = \tfrac{1}{6}\epsilon_\mu{}^{\alpha\rho\gamma}\epsilon_\nu{}^{\beta\sigma\delta}\partial_\alpha\partial_\beta\pi\partial_\rho\partial_\sigma\pi\partial_\gamma\partial_\delta\pi \tag{17}$$

we can verify that the equations of motion of the helicity-2 mode $\hat{h}_{\mu\nu}$ and the helicity-0 mode π would respectively read [14]:

$$\mathcal{E}_{\mu\nu}^{\alpha\beta}\tilde{h}_{\alpha\beta} - \tfrac{\tilde{\beta}}{\Lambda_3^6}X_{\mu\nu}^{(3)} = \tfrac{T_{\mu\nu}}{M_p} \tag{18}$$

$$3\Box\pi + \tfrac{1}{\Lambda_3^3}f(\Box\pi) = \tfrac{T}{M_p} - \tfrac{2\tilde{\alpha}}{M_p\Lambda_3^3}T_{\mu\nu}\partial^\mu\partial^\nu\pi \tag{19}$$

where $f(\Box\pi)$ is a nonlinear term reading [14]:

$$f(\Box\pi) = 3\tilde{\alpha}\pi^{\mu\nu}X_{\mu\nu}^{(1)} + \tfrac{4}{\Lambda_3^3}\left[\tfrac{\tilde{\alpha}^2}{2} + \tfrac{\tilde{\beta}}{3}\right]\pi^{\mu\nu}X_{\mu\nu}^{(2)} + \tfrac{5\tilde{\alpha}\tilde{\beta}}{\Lambda_3^6}\pi^{\mu\nu}X_{\mu\nu}^{(3)} + \tfrac{\tilde{\beta}}{2\Lambda_3^3}\partial_\alpha\partial_\beta\tilde{h}^{\mu\nu}\epsilon_\mu{}^{\alpha\rho\gamma}\epsilon_\nu{}^{\beta\sigma\delta}\pi_{\rho\sigma}\pi_{\gamma\delta} \tag{20}$$

we recall that we assumed the normalization $\tilde{h}_{\alpha\beta} = \hat{h}_{\mu\nu} + \eta_{\mu\nu}\pi + \tilde{\alpha}\frac{\partial_\mu\pi\partial_\nu\pi}{\Lambda_3^3}$, with ϵ the totally antisymmetric Levi-Civita tensor and $\pi^{\mu\nu} = \partial^\mu\partial^\nu\pi$.

From there, it is easy to infer that because of the presence of the nonlinear term $f(\Box\pi)$ in its equation of motion; the scalar mode π start becoming strongly coupled around the scale Λ_3 since from this scale, the nonlinear part $f(\Box\pi)$ of the equation start dominating over the linear part $3\Box\pi$, that will prevent linear perturbative approximation of the model.

In other words, it is the fact that Λ_3 is the scale around which the nonlinear part $f(\Box\pi)$ start of being significant that makes Λ_3 the strong coupling scale of dRGT massive gravity.

Thus the better way to raise up the strong coupling scale of dRGT massive gravity would be to succeed in raising up the scale from which the nonlinear part $f(\Box\pi)$ start of being significant without spoiling the general ghost-free and local character of dRGT massive gravity; a quest that will end here, since we are going to see in the next section how an appropriate expression for $\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$ permits to our model (1-3) to yield a dRGT massive gravity capable of overcoming both strong coupling problem and vDVZ discontinuity.

3. A dRGT Gravity Free from Strong Coupling Problem and vDVZ Discontinuity

To get a dRGT massive gravity capable of overcoming both strong coupling problem and vDVZ discontinuity from our recent model (1-3); we have to assume appropriate expression for the free tensor $\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$ of for example the kind:

$$\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A) = \tfrac{1}{k(\Phi)}\left[\mathcal{K}^l{}_{\mu\nu} + m^2\ \mathscr{U}_{\mu\nu}(\Phi) - m^2\mathcal{U}_{\mu\nu}\right] - A_\mu A_\nu \tag{21}$$

where $k(\Phi)$ is the function given by (6); $\mathcal{U}_{\mu\nu}$ the tensor given by (4); $\mathscr{U}_{\mu\nu}(\Phi)$ the tensor appearing in (1); A_μ the auxiliary vector field; φ_ρ the Stückelberg scalars fields, while $\mathcal{K}^l{}_{\mu\nu}$ is a tensor reading for example:

$$\mathcal{K}^l{}_{\mu\nu} = \tfrac{\alpha_0 \kappa}{12}\epsilon_\mu{}^{\alpha\rho\sigma}\epsilon_\nu{}^\beta{}_{\rho\sigma}\left[3\delta_{\alpha\beta} + \partial_\alpha\varphi_\beta - \delta_{\alpha\beta}\partial^\rho\varphi_\rho\right] \tag{22}$$

where α_0 is a free constant and $\kappa = \tfrac{1}{M_p^2}$ with M_p the Planck Mass.

Of course before going further, we can point out that the choice of this expression (22) for the tensor $\mathcal{K}^l{}_{\mu\nu}$ is not so arbitrary; indeed as we will see later, to succeed in our aim to raise up the strong coupling scale of dRGT massive gravity without giving rise to any new problems, we need such kind of divergence-free symmetric tensor $\mathcal{K}^l{}_{\mu\nu}$.

Indeed it is clear that in the case of (21); our original/general model (1-3) will simplify as:

$$G_{\mu\nu} = \kappa T_{\mu\nu} - m^2\mathcal{U}_{\mu\nu} + \mathcal{K}^l{}_{\mu\nu} \tag{23}$$

$$\nabla_\mu\left[\frac{\partial\mathcal{U}(g,\varphi^\rho)}{\partial(\partial_\mu\varphi^\rho)}\right] = \nabla^\mu\mathcal{U}_{\mu\nu} = 0 \tag{24}$$

$$\nabla^\mu\mathcal{K}^l{}_{\mu\nu} = 0 \tag{25}$$

which is nothing than a dRGT massive gravity with additional source of gravity $\mathcal{K}^l{}_{\mu\nu}$ beside the standard matter energy-momentum tensor $T_{\mu\nu}$.

Indeed, in the case of (23-25), neither the standard equation of motion (8) of the dRGT Stückelberg scalars fields φ^ρ undergoes a change since (8) and (24) are the same, nor the equation of motion and/or energy momentum tensor $T_{\mu\nu}$ of standard matter fields undergo a change since $\mathcal{K}^l{}_{\mu\nu}$ involves no standard matter fields; instead the sole difference between our model (23-25) and standard dRGT massive gravity (7-8) is the contribution in our model, of $\mathcal{K}^l{}_{\mu\nu}$ as additional source of gravity beside the standard matter energy-momentum tensor $T_{\mu\nu}$; which, as we will see, is the simplest way of getting a dRGT massive gravity capable of overcoming both strong coupling problem and vDVZ discontinuity.

Also, it is important to stress/recall that (23-25) is a gravity theory which can be derived from variational principle.

Indeed, if the standard dRGT massive gravity (7-8) can be derived from the action (9); our improved dRGT massive (23-25) can for example be derived from the following action:

$$S = \frac{M_P^2}{2} \int \left[R + 2\kappa L_M + m^2 \mathcal{U}(g, \varphi^\rho) - L_{\chi_0} - L_{\chi_1} - L_{\chi_2} - L_{ek} \right] \sqrt{-g} d^4 x \tag{26}$$

where L_{χ_0}, L_{χ_1}, L_{χ_2} and L_{ek} are additional lagrangians respectively reading:

$$L_{\chi_0} = \omega_0 \left[\partial_\mu \chi_0 \partial^\mu \chi_0 + 1 \right] \tag{27}$$

$$L_{\chi_1} = \omega_1 \left[\partial_\mu \chi_1 \partial^\mu \chi_1 + 1 \right] + \varepsilon_1 \left[\omega_1 - m^2 \right] - m^2 \partial_\mu \chi_1 \partial^\mu \chi_1 \tag{28}$$

$$L_{\chi_2} = \omega_2 \left[\partial_\mu \chi_2 \partial^\mu \chi_2 + 1 \right] + \varepsilon_2 \left[\omega_2 + m^2 \right] + m^2 \partial_\mu \chi_2 \partial^\mu \chi_2 \tag{29}$$

$$L_{ek} = \frac{\chi_2}{m} \left[\omega_0 \partial_\mu \chi_0 \partial_\nu \chi_0 \partial^t \varphi^\mu \partial_t \varphi^\nu - \mathcal{K}^\lambda{}_{\mu\nu} \partial^t \varphi^\mu \partial_t \varphi^\nu \right]^2 \tag{30}$$

in which ω_0, χ_0, ω_1, χ_1, ε_1, ω_2, χ_2 and ε_2 are scalar auxiliary variable, with $m > 0$ a constant; φ_ρ the Stückelberg scalars fields of standard dRGT massive gravity; $\mathcal{K}^\lambda{}_{\mu\nu}$ the tensor given by (22) and ∂_t the partial time-derivative.

Indeed, as we explicitly show in the Appendix A; applying variational principle to the action (26) permits to get our improved dRGT massive gravity (23-25).

Also, we can as done in the Appendix B verify by performing hamiltonian analysis, that (23-25) is a Lorentz-invariant massive gravity as ghost free as the standard dRGT massive gravity (7-8); likewise, it is easy to verify that (23-25) is a local gravity theory, more exactly a gravity theory as local as the standard dRGT massive gravity (7-8).

In short, (23-25) is "simply" a dRGT massive gravity with a "safe" additional source of gravity $\mathcal{K}^\lambda{}_{\mu\nu}$ beside the standard matter energy-momentum tensor $T_{\mu\nu}$; however as we are going to see, contrary to the standard dRGT massive gravity (7-8), our model (23-25) is a ghost-free Lorentz-invariant massive gravity capable of overcoming both the long-standing strong coupling problem as well as vDVZ discontinuity.

To prove it, we can start by noting that in the decoupling limit; the action (26) of our model (23-25) would read:

$$S = \int \left[-\frac{1}{2} \tilde{h}^{\mu\nu} \mathcal{E}_{\mu\nu}^{\alpha\beta} \tilde{h}_{\alpha\beta} + \frac{3}{2} \pi \Box \pi - \frac{\tilde{\alpha}}{\Lambda_3^3} X_{\mu\nu}^{(1)} \partial^\mu \pi \partial^\nu \pi - \frac{1}{\Lambda_3^6} \left(\frac{\tilde{\alpha}^2}{2} + \frac{\tilde{\beta}}{3} \right) X_{\mu\nu}^{(2)} \partial^\mu \pi \partial^\nu \pi \right.$$
$$\left. - \frac{\tilde{\beta}}{\Lambda_3^6} \left(\frac{\tilde{\alpha}}{\Lambda_3^3} \partial^\mu \pi \partial^\nu \pi - \tilde{h}^{\mu\nu} \right) X_{\mu\nu}^{(3)} + \frac{1}{M_p} \left(\mathcal{T}^\lambda{}_{\mu\nu} \tilde{h}^{\mu\nu} - \frac{\tilde{\alpha}}{\Lambda_3^3} \mathcal{T}^\lambda{}_{\mu\nu} \partial^\mu \pi \partial^\nu \pi - \pi \mathcal{T}^\lambda \right) \right] \sqrt{-g} d^4 x \tag{31}$$

where the tensor $\mathcal{T}^\lambda{}_{\mu\nu} = T_{\mu\nu} + M_p^2 \mathcal{K}^\lambda{}_{\mu\nu}$ denotes the conserved source of gravity and $\mathcal{T}^\lambda = g^{\mu\nu} \mathcal{T}^\lambda{}_{\mu\nu}$ its trace.

This comes from the fact that the additional lagrangians L_{χ_0}, L_{χ_1}, L_{χ_2} and L_{ek} not

contribute to the equation of motion (24) of the Stückelberg scalars φ_ρ since they only yield the additional source of gravity $\mathcal{K}^l_{\mu\nu}$; what means that, in the decoupling limit where we can decompose the Stückelberg fields φ^ρ as follows $\varphi^\rho = x^\rho - \frac{v^\rho}{mM_p} - \frac{\partial^\rho \pi(x)}{\Lambda_3^3}$ while assuming the transverse condition $\partial_\rho v^\rho = 0$; the additional lagrangians $L_{\chi 0}$, $L_{\chi 1}$, $L_{\chi 2}$ and L_{ek} contribute neither to the equation of motion of the scalar mode π nor to the equation of motion of the vector mode v^ρ appart as a source of gravity through $\mathcal{K}^l_{\mu\nu}$; whence in the decoupling limit the action (26) can be simplified as (31) which considering $\mathcal{T}^l_{\mu\nu} = T_{\mu\nu} + M_p^2 \mathcal{K}^l_{\mu\nu}$ will rewrite as follow:

$$
\begin{aligned}
S = \int \Big[&-\tfrac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} + \tfrac{3}{2}\pi\Box\pi - \tfrac{\tilde{\alpha}}{\Lambda_3^3}X^{(1)}_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \tfrac{1}{\Lambda_3^6}\Big(\tfrac{\tilde{\alpha}^2}{2} + \tfrac{\tilde{\beta}}{3}\Big)X^{(2)}_{\mu\nu}\partial^\mu\pi\partial^\nu\pi \\
&-\tfrac{\tilde{\beta}}{\Lambda_3^6}\Big(\tfrac{\tilde{\alpha}}{\Lambda_3^3}\partial^\mu\pi\partial^\nu\pi - \tilde{h}^{\mu\nu}\Big)X^{(3)}_{\mu\nu} + \tfrac{1}{M_p}\Big(T_{\mu\nu}\tilde{h}^{\mu\nu} - \tfrac{\tilde{\alpha}}{\Lambda_3^3}T_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \pi T\Big) \\
&+ M_p\Big(\mathcal{K}^l_{\mu\nu}\tilde{h}^{\mu\nu} - \tfrac{\tilde{\alpha}}{\Lambda_3^3}\mathcal{K}^l_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \pi\mathcal{K}^l\Big)\Big]\sqrt{-g}d^4x
\end{aligned}
\tag{32}
$$

Now since in the decoupling limit $\varphi^\rho = x^\rho - \frac{v^\rho}{mM_p} - \frac{\partial^\rho \pi(x)}{\Lambda_3^3}$, where where can assume the transverse condition $\partial_\rho v^\rho = 0$, we will get:

$$
\partial_\rho\varphi^\rho = 4 - \tfrac{\Box\pi}{\Lambda_3^3}
\tag{33}
$$

$$
\partial_\alpha\varphi_\beta = \delta_{\alpha\beta} - \tfrac{\partial_\alpha\partial_\beta\pi}{\Lambda_3^3}
\tag{34}
$$

$$
\mathcal{K}^l_{\mu\nu} = \tfrac{\alpha_0}{12M_p^2\Lambda_3^3}\epsilon_\mu{}^{\alpha\rho\sigma}\epsilon_\nu{}^\beta{}_{\rho\sigma}\Big[\delta_{\alpha\beta}\Box\pi - \partial_\alpha\partial_\beta\pi\Big]
\tag{35}
$$

$$
\mathcal{K}^l = g^{\mu\nu}\mathcal{K}^l_{\mu\nu} = -\tfrac{3}{2}\tfrac{\alpha_0}{M_p^2\Lambda_3^3}\Box\pi
\tag{36}
$$

then we can rewrite (32) as follow:

$$
\begin{aligned}
S = \int \Big[&-\tfrac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} + \tfrac{3}{2}\Big(1 + \tfrac{\alpha_0}{M_p\Lambda_3^3}\Big)\pi\Box\pi - \tfrac{\tilde{\alpha}}{\Lambda_3^3}X^{(1)}_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \tfrac{1}{\Lambda_3^6}\Big(\tfrac{\tilde{\alpha}^2}{2} + \tfrac{\tilde{\beta}}{3}\Big)X^{(2)}_{\mu\nu}\partial^\mu\pi\partial^\nu\pi \\
&-\tfrac{\tilde{\beta}}{\Lambda_3^6}\Big(\tfrac{\tilde{\alpha}}{\Lambda_3^3}\partial^\mu\pi\partial^\nu\pi - \tilde{h}^{\mu\nu}\Big)X^{(3)}_{\mu\nu} + \tfrac{1}{M_p}\Big(T_{\mu\nu}\tilde{h}^{\mu\nu} - \tfrac{\tilde{\alpha}}{\Lambda_3^3}T_{\mu\nu}\partial^\mu\pi\partial^\nu\pi - \pi T\Big) \\
&+ M_p\Big(\mathcal{K}^l_{\mu\nu}\tilde{h}^{\mu\nu} - \tfrac{\tilde{\alpha}}{\Lambda_3^3}\mathcal{K}^l_{\mu\nu}\partial^\mu\pi\partial^\nu\pi\Big)\Big]\sqrt{-g}d^4x
\end{aligned}
\tag{37}
$$

i.e. all happen as if the scalar mode π gains amongst other things the additional independent kinetic term $\frac{3}{2}\frac{\alpha_0}{m^2}\pi\Box\pi$; that would permit to raise up its strong coupling scale. Indeed, from (37) we will by considering (35) get for the helicity-2 mode $\hat{h}_{\mu\nu}$ and the helicity-0 mode π, the equations of motion respectively reading:

$$
\mathcal{E}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} - \tfrac{\tilde{\beta}}{\Lambda_3^6}X^{(3)}_{\mu\nu} = \tfrac{T_{\mu\nu}}{M_p} + M_p\mathcal{K}^l_{\mu\nu}
\tag{38}
$$

$$3\Box\pi + \frac{1}{\Lambda_k^3}f(\Box\pi) = \frac{m^2 T}{\Lambda_k^3} - \frac{2\tilde{\alpha}m^2}{\Lambda_k^3\Lambda_3^3}T_{\mu\nu}\partial^\mu\partial^\nu\pi - \frac{2\tilde{\alpha}}{\Lambda_k^3}\mathcal{K}^\iota_{\mu\nu}\partial^\mu\partial^\nu\pi \tag{39}$$

where $f(\Box\pi)$ is the nonlinear term given by (20), while the constant $\Lambda_k^3 = \Lambda_3^3\left[1 + \frac{\alpha_0}{M_p\Lambda_3^3}\right]$ is the one which sets the strong coupling scale of the theory, since in the case of the above (39), the scale around which the nonlinear part $f(\Box\pi)$ start dominating over the linear part $3\Box\pi$ is $\Lambda_k = \left[m^2 M_p + \frac{\alpha_0}{M_p}\right]^{\frac{1}{3}}$; a scale which in principle could be as huge as we want through the free constant α_0.

For example in the case of $\alpha_0 = M_p^2\left[M_p^2 - m^2\right]$, we will have $\Lambda_k = M_p$ i.e. the strong coupling scale will be Planck scale; whence (23-25) is an "improved" dRGT massive gravity capable of escaping to the strong coupling problem; this is for example more explicit in case $\tilde{\alpha} = 0$ where we will have:

$$3\Box\pi + \frac{1}{\Lambda_k^3}\left[\frac{4\tilde{\beta}}{3\Lambda_3^3}\pi^{\mu\nu}X_{\mu\nu}^{(2)} + \frac{\tilde{\beta}}{2\Lambda_3^3}\partial_\alpha\partial_\beta\tilde{h}^{\mu\nu}\epsilon_\mu^{\alpha\rho\gamma}\epsilon_\nu^{\beta\sigma\delta}\pi_{\rho\sigma}\pi_{\gamma\delta}\right] = \frac{m^2 T}{\Lambda_k^3} \tag{40}$$

Before concluding; it is interesting to notice that, our model (23-25) is also capable of escaping to vDVZ discontinuity [8, 9, 12].

This comes from the fact that the vDVZ discontinuity originates from the fact that the scalar mode $\pi(x)$ not decouples from matter (principally from the trace of matter energy-momentum tensor) even in the limit where the mass m of the graviton goes towards zero; which would lead to measurable departure from general relativity no matter the smallness of the mass m of the graviton.

Of course as we know, there are two possible way of solving the vDVZ problem, either have a very tiny coupling factor proportional to the mass of the graviton [8, 9, 12] or invoke the nonlinear Vainshtein mehanism [14].

For example in standard dRGT massive gravity, it is the Vainshtein mehanism which is invoked, the corollary being unfortunately to have the low strong coupling scale $\Lambda_3 = \left[m^2 M_p\right]^{\frac{1}{3}}$.

Fortunately things go better in the case of our model (23-25), because in this latter framework, the coupling factor $\frac{m^2}{\Lambda_k^3}$ between π and matter T is proportional to the graviton mass m.

This means that in our framework (23-25), the scalar mode π decouples from matter in the limit where the mass m of the graviton goes towards zero; what naturally expels the vDVZ discontinuity; whence in summary our model (23-25) is a model capable of overcoming both strong coupling problem and vDVZ discontinuity.

It is worth to mention that if instead of the expression (30) we consider for L_{ek} the expression reading:

$$\begin{aligned}L_{ek} = \frac{\chi_2}{m}\Big[&\omega_0\partial_\mu\chi_0\partial_\nu\chi_0\partial^t\varphi^\mu\partial_t\varphi^\nu - \mathcal{K}^\iota_{\mu\nu}\partial^t\varphi^\mu\partial_t\varphi^\nu - m^2\mathcal{U}_{\mu\nu}\partial^t\varphi^\mu\partial_t\varphi^\nu \\ &+m^2\,\mathcal{Y}_{\mu\nu}(\Phi)\partial^t\varphi^\mu\partial_t\varphi^\nu - k(\Phi)A_\mu A_\nu\partial^t\varphi^\mu\partial_t\varphi^\nu\Big]^2\end{aligned} \tag{41}$$

where $k(\Phi)$ is the function given by (6); $\mathcal{U}_{\mu\nu}$ the tensor given by (4); $\mathcal{Y}_{\mu\nu}(\Phi)$ the tensor appearing in (1); A_μ the auxiliary vector field and $\mathcal{K}^\lambda_{\mu\nu}$ the tensor given by (22); then we will instead of the model $(23 - 25)$ get the one reading:

$$G_{\mu\nu} = \kappa T_{\mu\nu} + \mathcal{K}^\lambda_{\mu\nu} - m^2 \; \mathcal{Y}_{\mu\nu}(\Phi) + k(\Phi)A_\mu A_\nu \tag{42}$$

$$\nabla_\mu \left[\frac{\partial \mathcal{U}(g,\varphi^\rho)}{\partial(\partial_\mu \varphi^\rho)} \right] = \nabla^\mu \mathcal{U}_{\mu\nu} = 0 \tag{43}$$

$$\nabla^\mu \left[\mathcal{K}^\lambda_{\mu\nu} - m^2 \; \mathcal{Y}_{\mu\nu}(\Phi) + k(\Phi)A_\mu A_\nu \right] = 0 \tag{44}$$

which is nothing than a gravity theory which reduces to the massive gravity (23-25) after electroweak phase transition while being a massless gravity theory before electroweak phase transition in the spirit of our original model (1-3) which finally is well and truly a framework permitting, for appropriate $\mathcal{S}_{\mu\nu}(g, \Psi, \varphi^\rho, A)$ like (21), to get a dRGT massive gravity like (23-25) quite capable of overcoming both strong coupling problem and vDVZ discontinuity.

4. Disscusion

It is very interesting to notice that our pure four-dimensional model (23-25) is, in its ability to solve strong coupling problem and vDVZ discontinuity, roughly similar to the embedded model [12] of Gabadadge, Older and Pirtskhalava.

This comes from the fact that in the case of our framework (23-25); the lagrangian of the scalar mode π gains amongst other things and in a safe way, the additional independent kinetic term $\frac{3}{2}\frac{\alpha_0}{M_p\Lambda_3^3}\pi\Box\pi$ beside the standard one $\frac{3}{2}\pi\Box\pi$.

Indeed, the addition of the independent kinetic term $\frac{3}{2}\frac{\alpha_0}{M_p\Lambda_3^3}\pi\Box\pi$ is made in a way that left unchanged the fundamental equation (8) or (24) of the four Sückelberg fieldds φ^ρ, equation which guarantees that φ^ρ propagate only three degrees of freedom [15]; what permits to preserve (see Appendix B) the ghost-free and local characeter of (23-25) while ensuring its ability to solve both strong coupling problem and vDVZ discontinuity.

Let's note that for the future, it would be interesting to explore the possibility for (23-25) or equivalently for the action (26) to help in quantum gravity research due to the fact that it offers a way to improve the UV behavior of dRGT gravity; it would also be interesting to explore the phenomenology of (23-25) in order to for example constrain the values of α_0 with experimental data [16, 17]; of course we could also think to explore expressions for $\mathcal{K}^\lambda_{\mu\nu}$ phenomenologically better than the one of (22).

Be that as it may, we have seen how in principle, a safe four-dimensional dRGT massive gravity capable of overcoming the strong coupling problem and vDVZ discontinuity can be achieved from our original general model (1-3) or from an action like (26), and to my knowledge, (23-25) is to date the first pure four-dimensional "improved" dRGT gravity

handling both the long-standing strong coupling problem and vDVZ discontinuity of massive gravity, what constitutes a real breakthrough, especially that the model (23-25) is rather simple considering the challenge of solving the strong coupling problem and vDVZ discontinuity in massive gravity.

Acknowledgments

Appendix A

As one can verify:

a) varying the action (26) with respect to $\delta g^{\mu\nu}$ would yield:

$$G_{\mu\nu} = \kappa T_{\mu\nu} - m^2 \mathcal{U}_{\mu\nu} + \frac{1}{\sqrt{-g}} \left[\frac{\delta(\sqrt{-g}L_{\chi_0})}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}L_{\chi_1})}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}L_{\chi_2})}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}L_{ek})}{\delta g^{\mu\nu}} \right] \qquad (A1)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_M)}{\delta g^{\mu\nu}}$ denotes the energy-momentum tensor of standard matter and $\mathcal{U}_{\mu\nu}$ the tensor given by (4), while:

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_{\chi_0})}{\delta g^{\mu\nu}} = \omega_0 \partial_\mu \chi_0 \partial_\nu \chi_0 - \frac{\omega_0}{2} g_{\mu\nu} \left[\partial_\sigma \chi_0 \partial^\sigma \chi_0 + 1 \right] \qquad (A2)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_{\chi_1})}{\delta g^{\mu\nu}} = [\omega_1 - m^2] \left[\partial_\mu \chi_1 \partial_\nu \chi_1 - \frac{1}{2} g_{\mu\nu} \partial_\sigma \chi_1 \partial^\sigma \chi_1 \right] - \frac{\omega_1}{2} g_{\mu\nu} \qquad (A3)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_{\chi_2})}{\delta g^{\mu\nu}} = [\omega_2 + m^2] \left[\partial_\mu \chi_2 \partial_\nu \chi_2 - \frac{1}{2} g_{\mu\nu} \partial_\sigma \chi_2 \partial^\sigma \chi_2 \right] - \frac{\omega_2}{2} g_{\mu\nu} \qquad (A4)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_{ek})}{\delta g^{\mu\nu}} = \frac{\chi_2}{m} \left[2 \frac{\delta \mathcal{E}_k}{\delta g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \mathcal{E}_k \right] \mathcal{E}_k \qquad (A5)$$

where for notation simplification we introduced the term $\mathcal{E}_k$ reading:

$$\mathcal{E}_k = \left[\omega_0 \partial_\mu \chi_0 \partial_\nu \chi_0 - \mathcal{K}^\lambda{}_{\mu\nu} \right] \partial^t \varphi^\mu \partial_t \varphi^\nu \qquad (A6)$$

b) varying (26) with respect to $\delta \varphi^\rho$; $\delta \omega_0$ and $\delta \chi_0$ would respectively yield:

$$m^2 \nabla_\mu \left[\frac{\partial \mathcal{U}(g, \varphi^\rho)}{\partial(\partial_\mu \varphi^\rho)} \right] + 2 \nabla^\mu \left[\frac{\chi_2}{m} \mathcal{E}_k \frac{\partial \mathcal{E}_k}{\partial(\partial^\mu \varphi^\rho)} \right] = 0 \qquad (A7)$$

$$\partial_\mu \chi_0 \partial^\mu \chi_0 + 1 + 2 \frac{\chi_2}{m} \mathcal{E}_k \frac{\partial \mathcal{E}_k}{\partial \omega_0} = 0 \qquad (A8)$$

$$\nabla^\mu \left[2\omega_0 \partial_\mu \chi_0 + 2\tfrac{\chi_2}{m} \mathcal{E}_k \tfrac{\partial \mathcal{E}_k}{\partial(\partial^\mu \chi_0)} \right] = 0 \tag{A9}$$

c) varying (26) with respect to $\delta\omega_1$, $\delta\varepsilon_1$ and $\delta\chi_1$ would respectively yield:

$$\partial_\mu \chi_1 \partial^\mu \chi_1 + 1 + \varepsilon_1 = 0 \tag{A10}$$

$$\omega_1 - m^2 = 0 \tag{A11}$$

$$\nabla^\mu \left[2(\omega_1 - m^2) \partial_\mu \chi_1 \right] = 0 \tag{A12}$$

d) varying (26) with respect to $\delta\omega_2$, $\delta\varepsilon_2$, $\delta\chi_2$ and δA_μ would respectively yield:

$$\partial_\mu \chi_2 \partial^\mu \chi_2 - 1 + \varepsilon_2 = 0 \tag{A13}$$

$$\omega_2 + m^2 = 0 \tag{A14}$$

$$\nabla^\mu \left[2(\omega_2 + m^2) \partial_\mu \chi_2 \right] - \tfrac{1}{m} \left[\mathcal{E}_k \right]^2 = 0 \tag{A15}$$

$$2\tfrac{\chi_2}{m} \mathcal{E}_k \tfrac{\partial \mathcal{E}_k}{\partial A_\mu} = 0 \tag{A16}$$

From there, it is clear that because of (A14); the equation (A15) would enforce:

$$\tfrac{1}{m} \left[\mathcal{E}_k \right]^2 = 0 \tag{A17}$$

Now since m is a constant, then one infers by considering (A6) that the above (A17) enforces in its turn:

$$\mathcal{E}_k = \left[\omega_0 \partial_\mu \chi_0 \partial_\nu \chi_0 - \mathcal{K}^\imath{}_{\mu\nu} \right] \partial^t \varphi^\mu \partial_t \varphi^\nu = 0 \tag{A18}$$

Likewise since $\partial^t \varphi^\mu \partial_t \varphi^\nu$ is not always zero; then one infers that the above (A18) enforces:

$$\omega_0 \partial_\mu \chi_0 \partial_\nu \chi_0 - \mathcal{K}^\imath{}_{\mu\nu} = 0 \tag{A19}$$

Clearly, because of the constraint (A18); the equations (A7), (A8), (A9) and (A16) would respectively simplify as:

$$m^2 \nabla_\mu \left[\tfrac{\partial \mathcal{U}(g,\varphi^\rho)}{\partial(\partial_\mu \varphi^\rho)} \right] = 0 \tag{A20}$$

$$\partial_\mu \chi_0 \partial^\mu \chi_0 + 1 = 0 \tag{A21}$$

$$\nabla^\mu \left[2\omega_0 \partial_\mu \chi_0 \right] = 0 \tag{A22}$$

$$0 = 0 \tag{A23}$$

In the same way, because of (A11), (A14), (A18) and (A21); the equations (A2), (A3), (A4) and (A5) would simplify as:

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_{\chi_0})}{\delta g^{\mu\nu}} = \omega_0 \partial_\mu \chi_0 \partial_\nu \chi_0 \tag{A24}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_{\chi_1})}{\delta g^{\mu\nu}} = -\frac{1}{2} m^2 g_{\mu\nu} \tag{A25}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_{\chi_2})}{\delta g^{\mu\nu}} = +\frac{1}{2} m^2 g_{\mu\nu} \tag{A26}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_{ek})}{\delta g^{\mu\nu}} = 0 \tag{A27}$$

whence the gravitational equation (A1) would simplify as:

$$G_{\mu\nu} = \kappa T_{\mu\nu} - m^2 \mathcal{U}_{\mu\nu} + \omega_0 \partial_\mu \chi_0 \partial_\nu \chi_0 \tag{A28}$$

which because of (A19) will finally rewrite as:

$$G_{\mu\nu} = \kappa T_{\mu\nu} - m^2 \mathcal{U}_{\mu\nu} + \mathcal{K}^l_{\ \mu\nu} \tag{A29}$$

while the equation of motion of the Stückelberg scalar fields φ^ρ remains (A20).
It is also worth to mention that applying the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ to (A29) would impose:

$$\nabla^\mu \mathcal{K}^l_{\ \mu\nu} = 0 \tag{A30}$$

whence applying the variational principle to the action (26) would yield the "improved" dRGT massive gravity (23-25).

Appendix B

To verify that the theory (23-25) is as ghost-free as the standard dRGT gravity (7-8); we can perform the Hamiltonian analysis of the model.

Indeed as well known, in the ADM decomposition, by assuming [18, 19, 20]:

$$ds^2 = -N^2 dt^2 + \gamma_{ij}\Big[dx^i + N^i dt\Big]\Big[dx^j + N^j dt\Big] \tag{B1}$$

$$\mathcal{R}^0 = \sqrt{\gamma}\Big[R(\gamma) + \tfrac{1}{\gamma}(\tfrac{1}{2}p^2 - p_{ij}p^{ij})\Big] \tag{B2}$$

$$\mathcal{R}_i = 2\sqrt{\gamma}\nabla\Big[\tfrac{p_{ij}}{\sqrt{\gamma}}\Big] \tag{B3}$$

where $N = \frac{1}{\sqrt{-g^{00}}}$ is the lapse function; $N_i = g_{0i}$ the shift function; $\gamma_{ij} = g_{ij}$ the three-dimensional space metric; p^{ij} is the conjugate momentum of γ^{ij}; $R(\gamma)$ the 3D Ricci scalar; γ the determinant of γ_{ij}; $p = p^{ij}\gamma_{ij}$.

we can write the lagrangian of the dRGT gravity (9) as follows [18, 19, 20]:

$$\mathcal{L}_{dRGT} = p^{ij}\partial_t\gamma_{ij} + N\mathcal{R}^0 + N^i\mathcal{R}_i + m^2 N\sqrt{\gamma}\,\mathcal{U}(\gamma, N, N^i, f) \tag{B4}$$

while the one of our model (26) would read:

$$\mathcal{L}_{dRGT^+} = p^{ij}\partial_t\gamma_{ij} + N\mathcal{R}^0 + N^i\mathcal{R}_i + m^2 N\sqrt{\gamma}\,\mathcal{U}(\gamma, N, N^i, f) - L_{\chi_0} - L_{\chi_1} - L_{\chi_2} - L_{ek} \tag{B5}$$

where $m^2\mathcal{U}(\gamma, N, N^i, f)$ denotes the potential of dRGT gravity with L_{χ_0}, L_{χ_1}, L_{χ_2} and L_{ek} the lagrangian given by (27), (28), (29) and (30) which in ADM decomposition will respetively read:

$$L_{\chi_0} = \omega_0\Big[-\tfrac{1}{N^2}(\dot{\chi}_0 - N^i\partial_i\chi_0) + \gamma^{ij}\partial_i\chi_0\partial_j\chi_0 + 1\Big] \tag{B6}$$

$$L_{\chi_1} = \Big[\omega_1 - m^2\Big]\Big[-\tfrac{1}{N^2}(\dot{\chi}_1 - N^i\partial_i\chi_1) + \gamma^{ij}\partial_i\chi_1\partial_j\chi_1\Big] + \varepsilon_1\Big[\omega_1 - m^2\Big] + \omega_1 \tag{B7}$$

$$L_{\chi_2} = \Big[\omega_2 + m^2\Big]\Big[-\tfrac{1}{N^2}(\dot{\chi}_2 - N^i\partial_i\chi_2) + \gamma^{ij}\partial_i\chi_2\partial_j\chi_2\Big] + \varepsilon_2\Big[\omega_2 + m^2\Big] + \omega_2 \tag{B8}$$

$$L_{ek} = \tfrac{\chi_2}{2mN^4}\Big[(\omega_0\dot{\chi}_0 - \mathcal{K}^l{}_{tt})(\partial_t\varphi^t)^2 - (\omega_0\partial_i\chi_0\partial_j\chi_0 - \mathcal{K}^l{}_{ij})\partial^t\varphi^i\partial_t\varphi^j\Big]^2 \tag{B9}$$

with $\mathcal{K}^l{}_{tt}$ and $\mathcal{K}^l{}_{ij}$ respectively the time-time and space-space components of the tensor $\mathcal{K}^l{}_{\mu\nu}$ given by (22).

Now as we well know [18, 19, 20]; the lagrangian (B4) of standard dRGT gravity yield the Hamiltonian $\mathcal{H}_{dRGT}$ reading:

$$\mathcal{H}_{dRGT} = \int \left[N\mathcal{R}^0 + N^i\mathcal{R}_i + m^2 N\sqrt{\gamma}\,\mathcal{U}(\gamma, N, N^i, f) \right] d^3x \tag{B10}$$

which because of the special form of the dRGT potential $\mathcal{U}(\gamma, N, N^i, f)$, is proved [18, 19, 20] to be a ghost-free and local Hamiltonian guaranteeing the ghost-free and local character of standard dRGT gravity.

Now, to see that our model (23-25) is as ghost-free and local as the standard dRGT gravity (7-8); we can verify after some calculus or equivalently by considering the equations (A11), (A14), (A19) and (A21), that the Hamiltonian $\mathcal{H}_{dRGT+}$ of the action (B5) of our model (23-25) would read:

$$\mathcal{H}_{dRGT+} = \int \left[N\mathcal{R}^0 + N^i\mathcal{R}_i + m^2 N\sqrt{\gamma}\,\mathcal{U}(\gamma, N, N^i, f) + N\sqrt{\gamma}\,\mathcal{K}^l{}_{tt} \right] d^3x \tag{B11}$$

where $\mathcal{K}^l{}_{tt}$ is the time-time component of the tensor $\mathcal{K}^l{}_{\mu\nu}$ given by (22), more precisely we have:

$$\mathcal{K}^l{}_{tt} = -\frac{\alpha_0\kappa}{6}\left[9 - 3\partial^t\varphi_t - 2\partial^j\varphi_j \right] \tag{B12}$$

From there, we can because of the peculiar expression of $\mathcal{K}^l{}_{tt}$, verify that the Hamiltonian (B11) yield exactly the same equation of motion for the Stückelberg fields φ^σ than the standard Hamiltonian (B10) in conformity with the equations (8) and (24).

Furthermore since in the physical gauge (unitary gauge) $\varphi^\sigma = x^\sigma$, we have $\mathcal{K}^l{}_{tt} = 0$; it results that in the physical gauge (unitary gauge), the Hamiltonian (B11) reduces exactly to the ghost-free and local Hamiltonian (B10) of standard dRGT massive gravity, what means that our improved massive gravity (23-25) is as ghost-free and local as the standard dRGT gravity (7-8).

Let's also note that the one can show that (23-25) is a safe massive gravity which propagates five degrees of freedom by verifying that the equation of motion (24) for the four Stückelberg fields φ^σ imposes three degrees of freedom instead of four; whence the two degrees of freedom yielded by the metric plus the three degrees of freedom yielded by (24) ensure the five degrees of freedom of massive gravity.

Indeed recalling the equation of motion (24) of the Stückelberg fields φ^σ as:

$$\mathcal{E}_\nu = \nabla_\mu\left[\frac{\partial\mathcal{U}(g,\varphi^\rho)}{\partial(\partial_\mu\varphi^\rho)} \right] = \nabla^\mu\mathcal{U}_{\mu\nu} = 0 \tag{B13}$$

One can show that the Hessian matrix $\mathcal{A}_{\mu\nu}$ defined as [2, 15, 20]:

$$\mathcal{A}_{\mu\nu} = -\frac{\delta\mathcal{E}_\nu}{\delta\ddot{\varphi}^\mu} = \frac{\delta^2\mathcal{L}}{\delta\dot{\varphi}^\mu\dot{\varphi}^\nu} \tag{B14}$$

is not invertible [2, 15, 20] i.e.:

$$\det(\mathcal{A}_{\mu\nu}) = 0 \tag{B15}$$

which guarantees that the equation of motion $\mathcal{E}_\nu = \nabla^\mu \mathcal{U}_{\mu\nu} = 0$ for the four Stückelberg fields φ^σ propagates three degrees of freedom instead of four; making thus (23-25) a ghost free massive gravity.

5. References

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