

## Pivot Vector Direction Cosine Matrix

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### Abstract

This work derives the Direction Cosine Matrix using the Pivot Vector attitude parameters. The resulting matrix elements are simpler than those of other attitude representations, since they involve only vector dot products.

### 1. Introduction

Euler's Rotation Theorem (Goldstein, 1950) states that a rigid body with a single point held fixed and free to rotate in any manner can be brought from orientation P to orientation Q by a rotation about a fixed axis that passes through the fixed point. Euler's Rotation Theorem is the basis for the Axis-Angle parameterization of attitude transformation, which is the starting point for many other attitude parameterizations. The Rodrigues Rotation Equation (Shuster, 1993) enables a vector to be rotated when given the angle and axis about which the rotation takes place. The equation is used to derive a related Direction Cosine Matrix, DCM, (Shuster, 1993), which simplifies the process of combining rotations. A Euler Rotation Sequence about x, y, z axes can be combined by multiplying the DCMs of the respective single axis rotations to obtain the total DCM for the combined transformation.

Pivot Vector parameterization of attitude was developed elucidate the geometry of rotational transformations, while simplifying the mathematics (Patera, 2017). Pivot Vector Methodology, PVM, represents each rotation as a pair of Pivot Vectors, PVs, that define a geocentric arc on a unit sphere. Combining rotations amounts to linking the associated geocentric arcs to form a single geocentric arc representing the combined rotation, thus clarifying the geometry. The Pivot Vector Rotation Equation was derived in an earlier work (Patera, 2025).

The Quaternion Rotation Equation (Kuipers, 1999) is equivalent to the Rodrigues Rotation Equation and has an associated DCM (Kuipers, 1999). The geometry related to the Quaternion Rotation Equation was clarified by using PVM (Patera, 2025).

The goal of this paper is to derive the Pivot Vector Direction Cosine Matrix.

Section 2 presents a short review of PVM, since it may not be familiar to most readers. Section 3 summarizes the derivation of the PV Rotation Equation. Section 4 presents the derivation of the PV DCM. Section 5 contains the conclusion.

### 2. Review of PVM

Pivot Vectors (PVs) were defined and found to be very useful in parameterizing rigid body attitude transformations in 3-dimensions. Instead of a rotation about a rotation axis, PVM uses a pair of PVs in the rotation plane to produce a slewing transformation in the associated 2-dimensional rotation plane. Each PV extends from the center of a unit sphere to its equatorial rotation plane. The PVs define the orientation of the rotation plane, which in turn defines the direction of the rotation axis. A 180 degree rotation about the first PV followed by a 180 degree rotation about the second PV results in a rotation about the polar axis equal to twice the angular separation of the PVs. The PVM is defined by the following characteristics:

- A) The inverse transformation is found by interchanging the first and second PV in the pair.
- B) Reversing the direction of rotation of one or both of the PVs does not change the result, since reversing the direction is equivalent to rotating by -180 degrees instead of 180 degrees. Because a 180 degree rotation is equivalent to a -180 degree rotation, the transformations are equivalent.

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- C) If the angle between the PVs is initially larger than 90 degrees, it can be reduced to less than 90 degrees by reversing the direction of one of the PVs in the PV pair.
- D) PV pairs can be clocked in the rotation plane and still produce the same rotation result, as long as the separation angle between the PVs remains constant.
- E) Two rotational PV pairs in the same rotational plane can be linked by aligning the second PV of the first pair to the first PV of the second pair. Since the aligned PVs rotate by the sum of the two 180 degree rotations, which is 360 degrees, they effectively cancel. Thus, the combined rotation is defined by the first PV of the first pair and the second PV of the second pair.
- F) The PV pairs in the same rotational plane commute, so the order can be changed without changing the results. This is equivalent to saying that rotations about the same rotation axis commute. Thus, several PV pairs in the same rotation plane can be combined in any order, while still producing the same result.
- G) Each pair on a spherical surface is restricted to its associated rotational plane.
- H) PV pairs on a spherical surface do not commute unless they reside in the same rotational plane.
- I) The PV pairs of two distinct rotational planes can be linked at either of the two intersection points of the respective equatorial planes to form a new rotational plane defined by the 1<sup>st</sup> PV of the 1<sup>st</sup> pair and the 2<sup>nd</sup> PV of the 2<sup>nd</sup> pair. Note that these pairs do not commute due to the curvature of the sphere.
- J) Multiple PV pairs can be combined by repeatedly linking the pairs in sequence until a final single PV pair is obtained.

In summary, PV pairs on a spherical surface parameterize 3-dimensional attitude transformations.

### 3. PV Rotation Equation

This derivation was presented in an earlier work (Patera, 2025) and is reproduced here for convenience. One can rotate a vector,  $\mathbf{V}$ , by  $\theta$  degrees about rotation axis  $\mathbf{R}$  using the PV pair  $\mathbf{a}$ ,  $\mathbf{b}$ , separated by  $\theta/2$  degrees assuming  $\mathbf{V}$  is not aligned with  $\mathbf{R}$ . If  $\mathbf{V}$  is aligned with  $\mathbf{R}$ , then  $\mathbf{V}$  does not change direction due to the rotation. Let  $\mathbf{b}$  be a PV normal to the  $\mathbf{V}$ ,  $\mathbf{R}$  plane, as defined by eq. (26).

$$\mathbf{b} = \frac{\mathbf{R} \times \mathbf{V}}{|\mathbf{R} \times \mathbf{V}|} \quad (1)$$

Pivot Vector  $\mathbf{a}$  can be constructed from  $\mathbf{R}$  and  $\mathbf{b}$ , as shown in eq. (2), based on (Patera, 2025).

$$\mathbf{a} = \cos\left(\frac{\theta}{2}\right) \mathbf{b} + \sin\left(\frac{\theta}{2}\right) \left(\frac{\mathbf{R} \times \mathbf{b}}{|\mathbf{R} \times \mathbf{b}|}\right) \quad (2)$$

The PV pair  $\mathbf{a}$ ,  $\mathbf{b}$  rotates an arbitrary vector,  $\mathbf{V}$ , about axis  $\mathbf{R}$  by  $\theta$  degrees. The component of  $\mathbf{V}$  normal to vector  $\mathbf{a}$  is given by subtracting  $\mathbf{V}$ 's parallel component from  $\mathbf{V}$ , as shown in eq. (3), where  $\mathbf{h}$  is the normal component.

$$\mathbf{h} = \mathbf{V} - (\mathbf{V} \cdot \mathbf{a}) \mathbf{a} \quad (3)$$

The rotation of  $\mathbf{V}$  about  $\mathbf{a}$  by 180 degrees is equivalent to subtracting  $2 \mathbf{h}$  from  $\mathbf{V}$  and using eq. (3), as given in eq. (4). Fig. 1 illustrates how a 180 degree rotation of  $\mathbf{V}$  about  $\mathbf{a}$  can be accomplished by subtracting  $2\mathbf{h}$  from  $\mathbf{V}$ . (N.B.  $\mathbf{V}$  also rotates about its own axis by 180 degrees, as does  $\mathbf{a}$  and  $\mathbf{b}$ )

$$\mathbf{V}' = \mathbf{V} - 2 \mathbf{h} = 2 (\mathbf{V} \cdot \mathbf{a}) \mathbf{a} - \mathbf{V} \quad (4)$$

The same 180 degree rotation about  $\mathbf{a}$  relocates  $\mathbf{b}$  to  $\mathbf{b}'$ , as shown in eq. (5).

$$\mathbf{b}' = 2 (\mathbf{b} \cdot \mathbf{a}) \mathbf{a} - \mathbf{b} \quad (5)$$

Now  $\mathbf{V}'$  is rotated about  $\mathbf{b}'$  by 180 degrees using eq. (4) with  $\mathbf{b}'$  replacing  $\mathbf{a}$  and  $\mathbf{V}'$  replacing  $\mathbf{V}$ , as given in eq. (6).

$$\mathbf{V}'' = 2(\mathbf{V}' \cdot \mathbf{b}') \mathbf{b}' - \mathbf{V}' \quad (6)$$

Using eq. (4) and eq. (5) in eq. (6), one obtains eq. (7), after simplification. Eq. (7) is the PV rotation equation.

$$\mathbf{V}'' = [4(\mathbf{V} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{a}) - 2(\mathbf{V} \cdot \mathbf{a})] \mathbf{a} - 2(\mathbf{V} \cdot \mathbf{b}) \mathbf{b} + \mathbf{V} \quad (7)$$

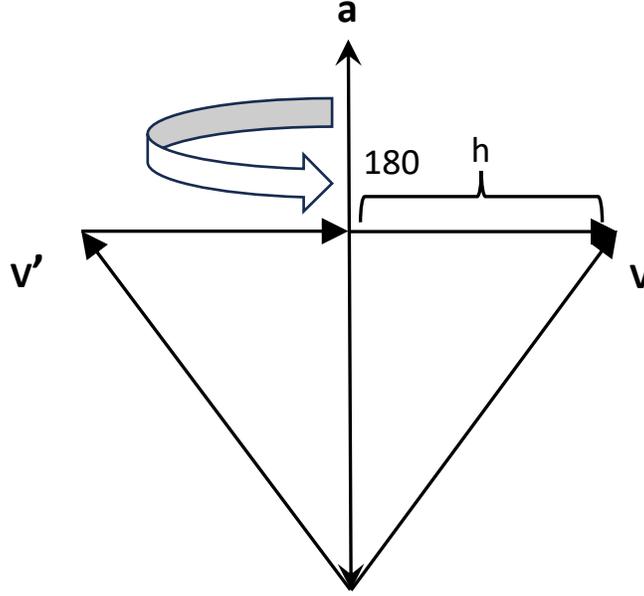


Fig. 1. Rotating  $\mathbf{V}$  about  $\mathbf{a}$  by 180 degrees is equivalent to subtracting  $2\mathbf{h}$  from  $\mathbf{V}$ .

Since the vector,  $\mathbf{V}$ , is arbitrary, the rotation generated by a rotation of  $\theta$  about axis  $\mathbf{R}$  is given by eq. (7) with  $\mathbf{a}$  and  $\mathbf{b}$  defined in eq. (2) and eq. (1). Eq. (7) shows that the only change to  $\mathbf{V}$  occurs in the plane containing  $\mathbf{a}$  and  $\mathbf{b}$ , which indicates that the rotation occurs in a 2-dimensional plane. The component of  $\mathbf{V}$  normal to the plane containing  $\mathbf{a}$  and  $\mathbf{b}$  is not changed by the rotation.

#### 4. PV Direction Cosine Matrix

The DCM associated with the PV rotation equation given by eq. (7) can be obtained by considering eq. (8), where  $\mathbf{U}$  is the desired DCM.

$$\mathbf{V}'' = \mathbf{U} \mathbf{V} = \sum_1^3 \mathbf{U}_{i,1} \mathbf{V}_1 + \mathbf{U}_{i,2} \mathbf{V}_2 + \mathbf{U}_{i,3} \mathbf{V}_3 \quad (8)$$

The elements of  $\mathbf{U}$  can be obtained by considering the x, y, z components of  $\mathbf{V}''$  and  $\mathbf{V}$  in eq. (8) and relating them to the corresponding components in eq. (7). The result yields elements of  $\mathbf{U}$ , as given in eq. (9) and eq. (10).

$$\mathbf{C} = 4(\mathbf{a} \cdot \mathbf{b}) \quad (9)$$

$$\mathbf{U} = \begin{bmatrix} C \mathbf{b}_x \mathbf{a}_x - 2(\mathbf{b}_x^2 + \mathbf{a}_x^2) + 1 & C \mathbf{b}_y \mathbf{a}_x - 2(\mathbf{b}_y \mathbf{b}_x + \mathbf{a}_y \mathbf{a}_x) & C \mathbf{b}_z \mathbf{a}_x - 2(\mathbf{b}_z \mathbf{b}_x + \mathbf{a}_z \mathbf{a}_x) \\ C \mathbf{b}_x \mathbf{a}_y - 2(\mathbf{b}_y \mathbf{b}_x + \mathbf{a}_y \mathbf{a}_x) & C \mathbf{b}_y \mathbf{a}_y - 2(\mathbf{b}_y^2 + \mathbf{a}_y^2) + 1 & C \mathbf{b}_z \mathbf{a}_y - 2(\mathbf{b}_z \mathbf{b}_y + \mathbf{a}_z \mathbf{a}_y) \\ C \mathbf{b}_x \mathbf{a}_z - 2(\mathbf{b}_z \mathbf{b}_x + \mathbf{a}_z \mathbf{a}_x) & C \mathbf{b}_y \mathbf{a}_z - 2(\mathbf{b}_y \mathbf{b}_z + \mathbf{a}_y \mathbf{a}_z) & C \mathbf{b}_z \mathbf{a}_z - 2(\mathbf{b}_z^2 + \mathbf{a}_z^2) + 1 \end{bmatrix} \quad (10)$$

Eq. (10) can be written in compact form, as shown in eq. (11), where eqs. (12) & (13) define  $\delta_{ij}$ .

$$\mathbf{U}_{i,j} = C \mathbf{a}_i \mathbf{b}_j - 2(\mathbf{b}_i \mathbf{b}_j + \mathbf{a}_i \mathbf{a}_j) + \delta_{ij} \quad (11)$$

$$\delta_{ij} = 1, \text{ if } i = j \quad (12)$$

$$\delta_{ij} = 0, \text{ if } i \neq j \quad (13)$$

The rotation angle can be extracted from the DCM by evaluating the trace of  $\mathbf{U}$  given by eq. (14). Solving for the rotation angle yields eq. (15), which is equivalent to the standard equation associated with the Rodrigues DCM shown in eq(16).

$$\text{trace} = 4(\mathbf{a} \cdot \mathbf{b})^2 - 1 = 4 \cos^2\left(\frac{\theta}{2}\right) - 1 \quad (14)$$

$$\theta = 2 \cos^{-1}\left(\frac{\sqrt{\text{trace}+1}}{2}\right) \quad (15)$$

$$\theta = \cos^{-1}\left(\frac{\text{trace}-1}{2}\right) \quad (16)$$

The rotation axis associated with  $\mathbf{U}$  is found by considering the x, y, z components of the cross product, as shown in eqs. (17-19).

$$\mathbf{U}_{32} - \mathbf{U}_{23} = 4(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{a})_x \quad (17)$$

$$\mathbf{U}_{13} - \mathbf{U}_{31} = 4(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{a})_y \quad (18)$$

$$\mathbf{U}_{21} - \mathbf{U}_{12} = 4(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{a})_z \quad (19)$$

The rotation axis is given by eq. (20), where *mag* is defined in eq. (21). Since the DCM given by eq. (10) is equivalent to DCMs from other parameterizations, the resulting Euler axis and angle must also be in agreement with other parameterizations.

$$\mathbf{axis} = \frac{1}{\text{mag}} \begin{pmatrix} \mathbf{U}_{32} - \mathbf{U}_{23} \\ \mathbf{U}_{13} - \mathbf{U}_{31} \\ \mathbf{U}_{21} - \mathbf{U}_{12} \end{pmatrix} \quad (20)$$

$$\text{mag} = \sqrt{(\mathbf{U}_{32} - \mathbf{U}_{23})^2 + (\mathbf{U}_{13} - \mathbf{U}_{31})^2 + (\mathbf{U}_{21} - \mathbf{U}_{12})^2} \quad (21)$$

For comparison purposes, the Rodrigues DCM for a rotation of  $\theta$  degrees about the unit vector axis,  $\mathbf{e}$ , is given by eq. (22), where  $C = \cos(\theta) - 1$ ,  $S = \sin(\theta)$ . Eq. (22) is the axis-angle representation of a rotation of  $\theta$  about axis  $\mathbf{e}$ . Since eq. (22) contains trigonometric functions, it involves greater computational effort than does the DCM in eq. (10).

$$\mathbf{U}(\mathbf{e}, \theta) = \begin{pmatrix} 1 + (\mathbf{e}_z^2 + \mathbf{e}_y^2)C & -(\mathbf{e}_x \mathbf{e}_y C + \mathbf{e}_z S) & \mathbf{e}_y S - \mathbf{e}_x \mathbf{e}_z C \\ \mathbf{e}_z S - \mathbf{e}_x \mathbf{e}_y C & 1 + (\mathbf{e}_x^2 + \mathbf{e}_z^2)C & -(\mathbf{e}_z \mathbf{e}_y C + \mathbf{e}_x S) \\ -(\mathbf{e}_z \mathbf{e}_x C + \mathbf{e}_y S) & \mathbf{e}_x S - \mathbf{e}_y \mathbf{e}_z C & 1 + (\mathbf{e}_x^2 + \mathbf{e}_y^2)C \end{pmatrix} \quad (22)$$

## 5. Conclusion

This work summarized the Pivot Vector parameterization of vehicle attitude and the derivation of the Pivot Vector rotation equation. The associated DCM was derived along with a method to extract the

associated rotation axis and angle of rotation. The DCM was found to be very easy to compute, since it involves only dot products and not trigonometric functions.

### **References**

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