

N-dimensional K-algebras and fractals without hypercomplex numbers

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Abstract

This paper is a continuation of our previous paper viXra:2512.0150, in which we expressed all vectors of the 3D Euclidean vector space with operations involving rotations and homotheties of the unit vector of the x-axis. We defined also a new vector multiplication and in this paper we will show that with the usual vector addition, we have a K-algebra. We will also study the 4D case and we will also express all vectors with simple rotations and homotheties of the unit vector of the x-axis about 3 planes, we will define a vector multiplication analogous to what we have seen in the 3D case, and we will show that we have again a K-algebra with the usual vector addition. We will construct also a 4D fractal set which contains the 3D fractal set (which contains the Mandelbrot set) seen in our previous paper and we will show several 4D projections of that set in the 3D space. We will show also that there are no quaternion numbers, because there are actually geometrical operations involving the unit vector of the x-axis. In that case we will use compositions of 2 rotations of the unit vector of the x-axis about different planes to express all the vectors. In that case we will have also a division algebra. Finally, we will show that for all dimensions, we have K-algebras if we express all vectors with operations involving homotheties and simple rotations of the unit vector of the first axis x_1 , with the usual vector addition and a generalization of the multiplication that we have seen in our previous papers for the 2D and 3D cases and for the 4D case in this paper. One advantage of the simple rotations, among other ones, is that they allow to construct interesting n-dimensional fractal sets linked to the Mandelbrot set.

1 Introduction

During the XIXth century some mathematicians were looking for a set of numbers with 3 dimensions analogous to complex numbers. Those numbers, and their operations, had to fulfill some important properties of complex numbers (associativity of the multiplication, etc.). Frobenius proved that this was not possible with 3 dimensions (see [Herstein1975], p. 368) and William Rowan Hamilton proposed the system of quaternions with 4 dimensions.

In a previous paper we have shown that there are no complex numbers nor negative numbers (see [Torres-HerediaA2025]) and we worked with geometrical operations involving rotations and homotheties of the first axis of the Euclidean vector space. We applied this principle in our previous paper (see [Torres-Heredia2025T]) for the 3D case. And this is not a problem because we are not looking for a system of numbers, but for geometrical operations in the 3D Euclidean vector

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space. Those operations don't have the same properties as the vector operations seen in the 2D case (associativity, etc.). We will see that those operations lead to interesting functions. And as in our previous paper (see [Torres-Heredia2025T]) we will show that there is also a fractal set in the 4D case. And this can be applied to other dimensions.

We will show also that there no quaternions numbers, by using the same principle: expressing all the vectors in the 4D Eclidean vector space with geometrical operations applied to the unit vector of the x-axis.

We will show also that we can apply those principles to the n-dimensional Eulidean vector space to get K-algebras.

The field K in this case is the set \mathbb{R} understood as the set of positions in the continuous line with a center and symmetries, with operations which are actually geometrical (see [Torres-HerediaA2025]).

2 The 3D k-Algebra

2.1 Showing that we have a K-algebra

As we have seen in our previous paper (see [Torres-Heredia2025T]), in this space we can perform additions of vectors with the usual rule:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a + d \\ b + e \\ c + f \end{pmatrix}$$

And each vector of the 3D space can be expressed as this:

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bR_z(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + cR_y(-90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

If we have a second vector, it can be expressed in the same way:

$$\begin{pmatrix} d \\ e \\ f \end{pmatrix} = d \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + eR_z(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + fR_y(-90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Then we rewrite the 2 vectors as this, where I_3 stands for the identity matrix of size 3:

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= aI_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bR_z(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + cR_y(-90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} d \\ e \\ f \end{pmatrix} &= dI_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + eR_z(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + fR_y(-90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Now, thanks to the rules of distribution of matrices, we get:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (aI_3 + bR_z(90^\circ) + cR_y(-90^\circ)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} d \\ e \\ f \end{pmatrix} = (dI_3 + eR_z(90^\circ) + fR_y(-90^\circ)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Now we can compose those two operations thanks to matrix multiplications, using the rules of matrix distribution:

$$(aI_3 + bR_z(90^\circ) + cR_y(-90^\circ))(dI_3 + eR_z(90^\circ) + fR_y(-90^\circ)) = aI_3dI_3 + aI_3eR_z(90^\circ) + aI_3fR_y(-90^\circ) + bR_z(90^\circ)dI_3 + bR_z(90^\circ)eR_z(90^\circ) + bR_z(90^\circ)fR_y(-90^\circ) + cR_y(-90^\circ)dI_3 + cR_y(-90^\circ)eR_z(90^\circ) + cR_y(-90^\circ)fR_y(-90^\circ)$$

And finally we get:

$$\begin{pmatrix} ad - be - cf \\ ae + bd + ce \\ af + cd + bf \end{pmatrix}$$

As we have seen, we have performed multiplications of matrix to get our result. For practical reasons, we can define a new multiplication between vectors, called $*$ in this framework:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} ad - be - cf \\ ae + bd + ce \\ af + cd + bf \end{pmatrix}$$

Now, we will show that this multiplication is bilinear (see [Schafer1966], p. 1).

First we will show the right distribution. We must to calculate:

$$\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right) * \begin{pmatrix} g \\ h \\ i \end{pmatrix}$$

With the multiplication programmed, the Maxima Algebra System gives us:

$$[(-f - c)i - (e + b)h + (d + a)g, (f + c)h + (d + a)h + (e + b)g, (e + b)i + (d + a)i + (f + c)g]$$

And if we calculate:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} g \\ h \\ i \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} * \begin{pmatrix} g \\ h \\ i \end{pmatrix}$$

the Maxima Algebra System gives us:

$$[-fi - ci - eh - bh + dg + ag, fh + dh + ch + ah + eg + bg, ei + di + bi + ai + fg + cg]$$

So

$$\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right) * \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} g \\ h \\ i \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} * \begin{pmatrix} g \\ h \\ i \end{pmatrix}$$

Now we will show the left distribution. We must to calculate:

$$\begin{pmatrix} g \\ h \\ i \end{pmatrix} * \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right)$$

With the multiplication programmed, the Maxima Algebra System gives us:

$$[-(f+c)i - (e+b)h + (d+a)g, (e+b)i + (d+a)h + (e+b)g, (d+a)i + (f+c)h + (f+c)g]$$

And if we calculate:

$$\begin{pmatrix} g \\ h \\ i \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} g \\ h \\ i \end{pmatrix} * \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

the Maxima Algebra System gives us:

$$[-fi - ci - eh - bh + dg + ag, ei + bi + dh + ah + eg + bg, di + ai + fh + ch + fg + cg]$$

So

$$\begin{pmatrix} g \\ h \\ i \end{pmatrix} * \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right) = \begin{pmatrix} g \\ h \\ i \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} g \\ h \\ i \end{pmatrix} * \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

We will show now that there is a compatibility with scalars. We calculate firstly:

$$\lambda \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right)$$

The Maxima Algebra System gives us:

$$[(-cf - be + ad)\lambda, (ce + ae + bd)\lambda, (bf + af + cd)\lambda]$$

We calculate now:

$$\left(\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) * \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

The Maxima Algebra System gives us:

$$[-cf\lambda - be\lambda + ad\lambda, ce\lambda + ae\lambda + bd\lambda, bf\lambda + af\lambda + cd\lambda]$$

And now we calculate:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} * (\lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix})$$

We get:

$$[-cf\lambda - be\lambda + ad\lambda, ce\lambda + ae\lambda + bd\lambda, bf\lambda + af\lambda + cd\lambda]$$

So:

$$\lambda \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right) = (\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}) * \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} * (\lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix})$$

Now, we will show that this multiplication is not commutative in general. For example, if we have the two vectors $\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, we get:

$$\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} * \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 20 \\ -9 \\ -11 \end{pmatrix}$$

and:

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} * \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 20 \\ -10 \\ -10 \end{pmatrix}$$

We can show that if the 2 vectors are in the plane OXY, the operation is commutative:

$$\begin{aligned} \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} * \begin{pmatrix} d \\ e \\ 0 \end{pmatrix} &= \begin{pmatrix} ad - be - 0 \\ ae + bd + 0e \\ a0 + 0d + b0 \end{pmatrix} \\ &= \begin{pmatrix} ad - be \\ ae + bd \\ 0 \end{pmatrix} \end{aligned}$$

This result is the same as that we got with the vector multiplication of the 2D case. That multiplication is analogous to what has been done with the multiplication of complex numbers, which is commutative.

Now, if the 2 vectors are in the plane OXZ, the operation is also commutative, for the same reasons:

$$\begin{aligned} \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} * \begin{pmatrix} d \\ 0 \\ f \end{pmatrix} &= \begin{pmatrix} ad - 0 - cf \\ a0 + 0d + c0 \\ af + cd + 0f \end{pmatrix} \\ &= \begin{pmatrix} ad - cf \\ 0 \\ af + cf \end{pmatrix} \end{aligned}$$

Now, we will show that this multiplication is not associative in general. For example, if we have the three vectors $\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$, we get:

$$\begin{aligned} \left(\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} * \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right) * \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} &= \begin{pmatrix} 20 \\ -9 \\ -11 \end{pmatrix} * \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 26 \\ -72 \\ -22 \end{pmatrix} \end{aligned}$$

and:

$$\begin{aligned} \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} * \left(\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} * \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \right) &= \begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} * \begin{pmatrix} 4 \\ -4 \\ 10 \end{pmatrix} \\ &= \begin{pmatrix} 20 \\ 12 \\ -66 \end{pmatrix} \end{aligned}$$

Anyway, this fact was proved by Frobenius, as we have seen in the Introduction.

If the three vectors are in the plane OXY (or in OXZ), the multiplication will be associative for the same reason seen above.

Now, we will show that there is an identity element for the multiplication, which is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$:

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} a1 - b0 - c0 \\ a0 + b1 + c0 \\ a0 + c1 + b0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1a - 0b - 0c \\ 1b + 0a + 0b \\ 1c + 0a + 0c \end{pmatrix} \\ = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We will show now that is the only one. Firstly, we must to solve the following equation:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

So we have:

$$\begin{pmatrix} ax - by - cz \\ ay + bx + cy \\ az + cx + bz \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

So we must to solve the following linear system:

$$\begin{cases} ax - by - cz = a \\ bx + ay + cy = b \\ cx + az + bz = c \end{cases}$$

The Maxima Algebra System gives us:

$$[x = 1, y = 0, z = 0]$$

Secondly, we must to solve the following equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

So we have:

$$\begin{pmatrix} xa - yb - zc \\ xb + ya + zb \\ xc + za + yc \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

So we must to solve the following linear system:

$$\begin{cases} ax - by - cz = a \\ bx + ay + bz = b \\ cx + cy + az = c \end{cases}$$

The Maxima Algebra System gives us:

$$[x = 1, y = 0, z = 0]$$

So, there is an identity element for the multiplication, which is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

So, we have a non-associative, non-commutative unital K-algebra.

2.2 The functions and their representations

With the operations that we have defined, we can construct functions. For example:

$$f(\vec{x}) = \vec{x}^2$$

where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. So,

$$\begin{aligned} f(\vec{x}) &= \vec{x} * \vec{x} \\ &= \begin{pmatrix} x_1^2 - x_2^2 - x_3^2 \\ 2x_1x_2 + x_2x_3 \\ 2x_1x_3 + x_2x_3 \end{pmatrix} \end{aligned}$$

So this is function defined from \mathbb{E}^3 to \mathbb{E}^3 , $f : \mathbb{E}^3 \rightarrow \mathbb{E}^3$.

If we have the unit sphere centered at the origin as a subset, its image will be another set of points (see the figure 1), which is a surface. Obviously, we represent only the terminal points of each vector.

3 The 4D k-algebras

3.1 The 4D space of all the images of the horizontal unit vector

A vector in the 4D Euclidean vector space can be expressed as this, where $R_{xy}(90^\circ)$ is a rotation matrix about the xy plane, $R_{xz}(90^\circ)$ is a rotation matrix about the xz plane and $R_{xw}(90^\circ)$ is a rotation matrix about the xw plane (see [WikipediaR] and [WikipediaP]):

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &= a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + bR_{xy}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + cR_{xz}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + dR_{xw}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

The matrices are:

Image of the unit sphere centered at the origin

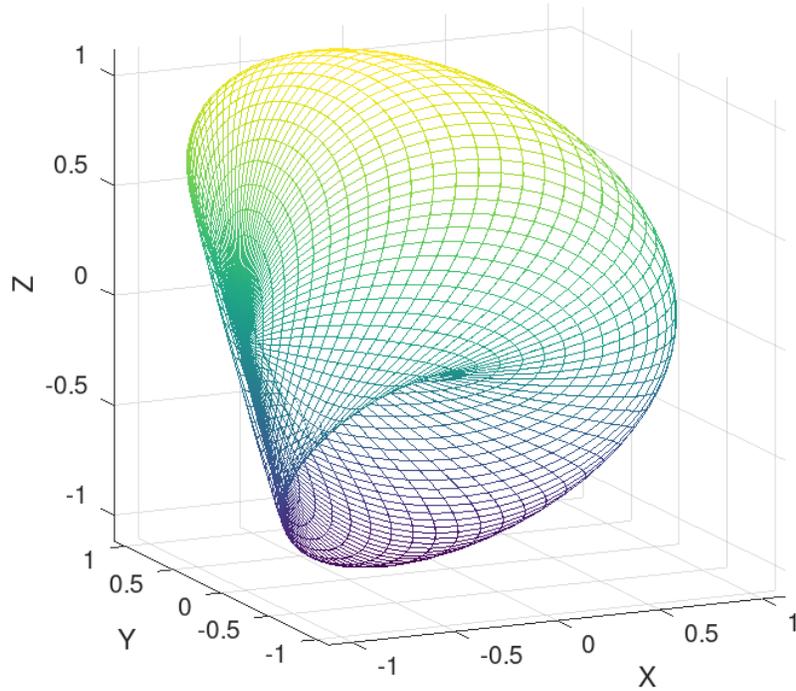


Figure 1: The image of the unit sphere by the function f

$$R_{xy}(90^\circ) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{xz}(90^\circ) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{xw}(90^\circ) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

3.2 The operations with the mathematical objects and the resulted vectors of these operations

3.2.1 The addition

In this space, which is \mathbb{E}^4 , we can perform additions of vectors with the usual rule:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} a + e \\ b + f \\ c + g \\ d + h \end{pmatrix}$$

3.2.2 The multiplication

As we have seen above, each vector of the 4D space can be expressed as this:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + bR_{xy}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + cR_{xz}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + dR_{xw}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

If we have a second vector, it can be expressed in the same way:

$$\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = e \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + fR_{xy}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + gR_{xz}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + hR_{xw}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then, thanks to the rules of distribution of matrices, we rewrite the 2 vectors as this, where I_4 stands for the identity matrix of size 4:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (aI_4 + bR_{xy}(90^\circ) + cR_{xz}(90^\circ) + dR_{xw}(90^\circ)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = (eI_4 + fR_{xy}(90^\circ) + gR_{xz}(90^\circ) + hR_{xw}(90^\circ)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now we can compose those two operations thanks to matrix multiplications, using the rules of matrix distribution. By programming this matrix multiplication with the Maxima Algebra System, we get:

$$\begin{pmatrix} -dh - cg - bf + ae \\ (d + c + a)f + be \\ (d + b + a)g + ce \\ (c + b + a)h + de \end{pmatrix}$$

Now, we can define a new multiplication between vectors, called $*$ in this framework:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} ae - bf - cg - dh \\ af + be + cf + df \\ ag + bg + ce + dg \\ ah + bh + ch + de \end{pmatrix}$$

3.3 Proving that we have a K-algebra

Now, we will show that this multiplication is bilinear.

First we will show the right distribution. We must to calculate:

$$\left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right) * \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix}$$

With the multiplication programmed, the Maxima Algebra System gives us:

$$\begin{pmatrix} (-h-d)p + (-g-c)o - (f+b)n + (e+a)m \\ (h+d)n + (g+c)n + (e+a)n + (f+b)m \\ (h+d)o + (f+b)o + (e+a)o + (g+c)m \\ (g+f+e+c+b+a)p + (h+d)m \end{pmatrix}$$

And if we calculate:

$$\begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix} * \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} * \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix}$$

the Maxima Algebra System gives us:

$$\begin{pmatrix} -hp - dp - go - co - fn - bn + em + am \\ hn + gn + en + dn + cn + an + fm + bm \\ ho + fo + eo + do + bo + ao + gm + cm \\ (g+f+e)p + (c+b+a)p + hm + dm \end{pmatrix}$$

So

$$\left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right) * \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix} * \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} * \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix}$$

Now we will show the left distribution. We must to calculate:

$$\begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} * \left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right)$$

The Maxima Algebra System gives us:

$$\begin{pmatrix} -(h+d)p - (g+c)o - (f+b)n + (e+a)m \\ (f+b)p + (f+b)o + (e+a)n + (f+b)m \\ (g+c)p + (e+a)o + (g+c)n + (g+c)m \\ (e+a)p + (h+d)(o+n+m) \end{pmatrix}$$

And if we calculate:

$$\begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

the Maxima Algebra System gives us:

$$\begin{pmatrix} -hp - dp - go - co - fn - bn + em + am \\ fp + bp + fo + bo + en + an + fm + bm \\ gp + cp + eo + ao + gn + cn + gm + cm \\ ep + ap + h(o + n + m) + d(o + n + m) \end{pmatrix}$$

So

$$\begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} * \left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right) = \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix} * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

We will show now that there is a compatibility with scalars. We calculate firstly:

$$\lambda \left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right)$$

The Maxima Algebra System gives us:

$$\begin{pmatrix} (-dh - cg - bf + ae) \lambda \\ (df + cf + af + be) \lambda \\ (dg + bg + ag + ce) \lambda \\ ((c + b + a)h + de) \lambda \end{pmatrix}$$

We calculate now:

$$\left(\lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right) * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

The Maxima Algebra System gives us:

$$\begin{pmatrix} -dh\lambda - cg\lambda - bf\lambda + ae\lambda \\ df\lambda + cf\lambda + af\lambda + be\lambda \\ dg\lambda + bg\lambda + ag\lambda + ce\lambda \\ h(c\lambda + b\lambda + a\lambda) + de\lambda \end{pmatrix}$$

And now we calculate:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \left(\lambda \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} \right)$$

We get:

$$\begin{pmatrix} -dh\lambda - cg\lambda - bf\lambda + ae\lambda \\ df\lambda + cf\lambda + af\lambda + be\lambda \\ dg\lambda + bg\lambda + ag\lambda + ce\lambda \\ (c + b + a)h\lambda + de\lambda \end{pmatrix}$$

So:

$$\lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = (\lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}) * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * (\lambda \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix})$$

So the multiplication is bilinear.

Now, we will show that this multiplication is not commutative in general. For example, if we

have the two vectors $\begin{pmatrix} 1 \\ -2 \\ 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ 1 \\ -3 \end{pmatrix}$, we get:

$$\begin{pmatrix} 1 \\ -2 \\ 3 \\ 1 \end{pmatrix} * \begin{pmatrix} -2 \\ 3 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 19 \\ -6 \\ -8 \end{pmatrix}$$

and:

$$\begin{pmatrix} -2 \\ 3 \\ 1 \\ -3 \end{pmatrix} * \begin{pmatrix} 1 \\ -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ -5 \\ -1 \end{pmatrix}$$

There are cases in which the multiplication is commutative. We can show that if the 2 vectors are in the plane OXY, the operation is commutative, for the same reasons seen above, for the 3D case:

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} * \begin{pmatrix} e \\ f \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ae - bf \\ af + be \\ 0 \\ 0 \end{pmatrix}$$

Because of the theorem of Frobenius, this multiplication is not associative in general. If the three vectors are in the plane OXY, the multiplication will be associative for the same reasons seen above for the 3D case.

Now, we will check that there is an identity element for the multiplication, which is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. With

the Maxima Algebra System we get:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

We will show now that is the only one. Firstly, we must to solve the following equation:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

So we have:

$$\begin{pmatrix} -cz - by + ax - dw \\ dy + cy + ay + bx \\ dz + bz + az + cx \\ dx + (c + b + a)w \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

So we must to solve the following linear system:

$$\begin{cases} ax - by - cz - dw = a \\ bx + ay + cy + dy = b \\ cx + az + bz + dz = c \\ dx + (a + b + c)w = d \end{cases}$$

The wxMaxima Algebra System gives us:

$$[x = 1, y = 0, z = 0, w = 0]$$

Secondly, we must to solve the following equation:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} * \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

So we have:

$$\begin{pmatrix} -cz - by + ax - dw \\ bz + ay + bx + bw \\ az + cy + cx + cw \\ d(z + y + x) + aw \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

So we must to solve the following linear system:

$$\begin{cases} ax - by - cz - dw = a \\ bx + ay + bz + bw = b \\ cx + cy + az + cw = c \\ dx + dy + dz + aw = d \end{cases}$$

The wxMaxima Algebra System gives us:

$$[x = 1, y = 0, z = 0, w = 0]$$

So, there is an identity element for the multiplication, which is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

So, we have a non-associative, non-commutative unital K-algebra with 4 dimensions.

3.4 The 4D fractal set containing the Mandelbrot set

Now we can construct a 4D set which contains the 3D fractal set seen in our previous paper (see [Torres-Heredia2025T]) only with vectors and the new operations of this framework in \mathbb{E}^4 . We recall that the 3D fractal set seen contains the Mandelbrot set in the planes OXY and OXZ.

$$\text{Let } \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \text{ and } \vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We will perform an iteration of a quadratic map to get the vectors \vec{x}_k :

$$\vec{x}_k = \vec{x}_{k-1}^2 + \vec{c}$$

And we will see if the vector norm of \vec{x}_k remains bounded for all $k > 0$. If it is the case, we will add the vector \vec{c} to the fractal set.

We will show now more details about those calculations. Firstly, the square of a vector will give a simpler expression:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}^2 = \begin{pmatrix} -z^2 - y^2 + x^2 - w^2 \\ yz + 2xy + wy \\ yz + 2xz + wz \\ w(z + y + x) + wx \end{pmatrix}$$

Now, we are going to represent graphically a projection of a 4D form in the 3D space for different values of the 4th component. For that, we will adapt the "escape time" algorithm used for the Mandelbrot set (see [WikipediaP]).

So we must to fix the 4th component of the vector \vec{c} in order to have the representation of a form in the 3D space. For example, we can fix the component c_4 with a value w_0 .

After that, we test different values of the vector \vec{c} with the iterations. If the 4D norm of \vec{x}_k remains bounded for all $k > 0$, then we add \vec{c} to the 3D subset which is a part of the 4D fractal set. And we can represent graphically this subset in the 3D space with $c_4 = w_0$. Obviously, we represent only the terminal points of each vector \vec{c} .

This time we will generate the images with a C++ program which will represent little blue cubes with light effects.

Now, first of all, we get the 3D fractal set which contains the Mandelbrot set, when $w_0 = 0$, (see the figure 2). In our previous paper (see [Torres-Heredia2025T]), we have seen cross-sections of this form.

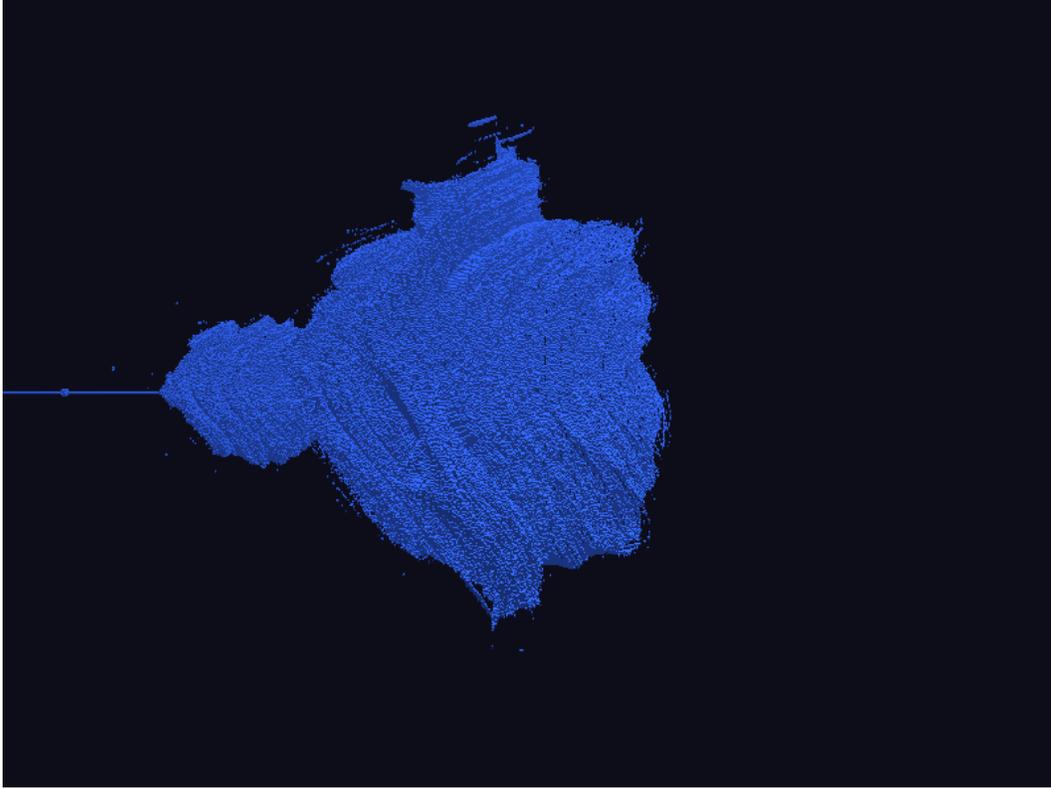


Figure 2: A 4D projection in the 3D space when $w=0$

We have other images of this 3D form from other positions of the camera, in front of the set, on the other side and behind the set (see the figure 3, the figure 4, and the figure 5).

And, when $w_0 = 0.5$, we get another 3D form, in front of the set, on the other side and behind the set (see the figure 6).

We have other images of this 3D form from other positions of the camera (see the figure 7, the figure 8 and the figure 9).

And we can continue as this to have other fractal sets with 5 dimensions, 6 dimensions, etc.

3.5 The vector meaning of the quaternions

We will show that there are no quaternion numbers but geometrical operations as we saw in our first paper for complex numbers (see [Torres-HerediaA2025]).

As we have seen before, in the Euclidean 4D space, we work with rotations about several planes. The 4 axes are x , y , z and w . This time we will work with some rotations which are not simple:

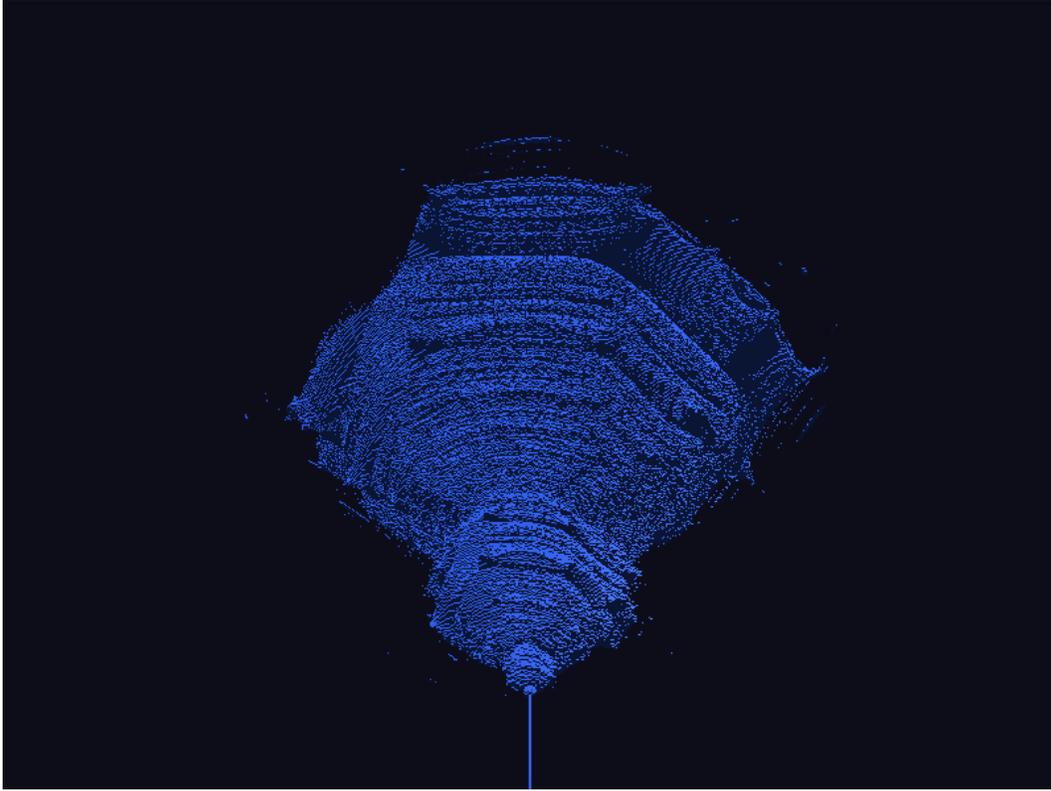


Figure 3: A 4D projection in the 3D space when $w=0$

$$R_{xz}(-90^\circ) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{yw}(-90^\circ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$R_{yz}(90^\circ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{zw}(90^\circ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Now, a vector in the 4D Euclidean vector space can be expressed as this (it has been checked with the Maxima Algebra System):

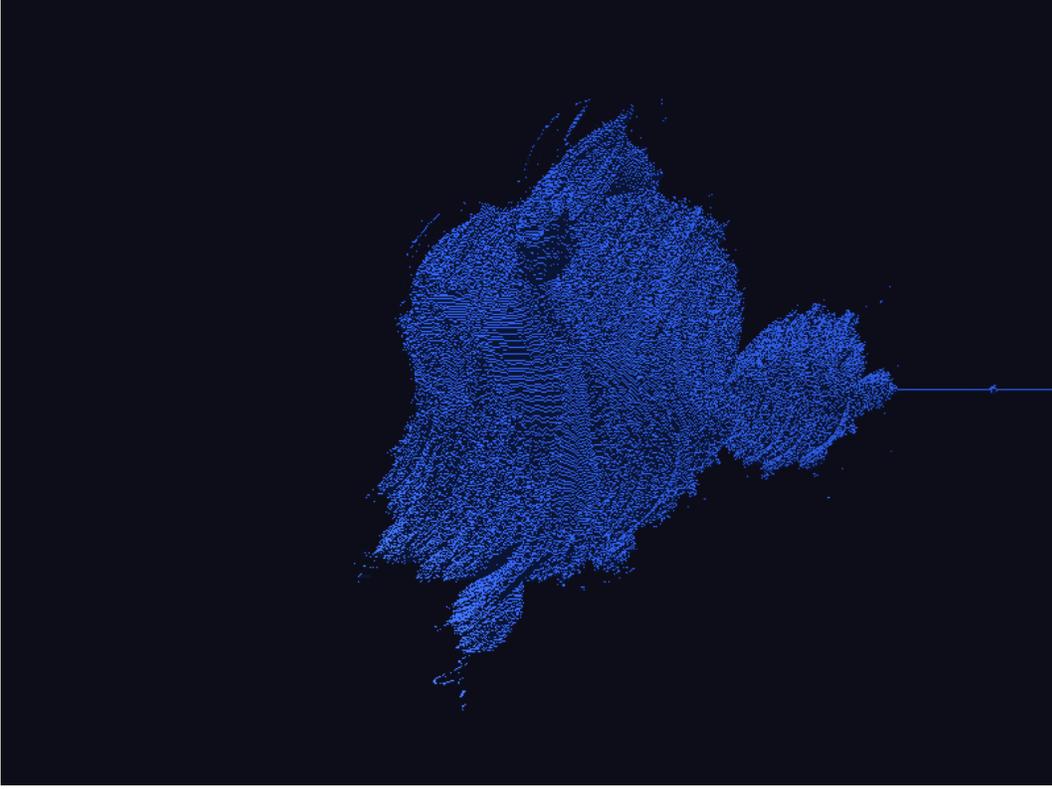


Figure 4: A 4D projection in the 3D space when $w=0$

$$\begin{aligned}
 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &= a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
 &= a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b R_{xy}(90^\circ) R_{zw}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c R_{xz}(90^\circ) R_{yw}(-90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + d R_{xw}(90^\circ) R_{yz}(90^\circ) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
 \end{aligned}$$

It can be written also as this:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (aI_4 + bR_{xy}(90^\circ)R_{zw}(90^\circ) + cR_{xz}(90^\circ)R_{yw}(-90^\circ) + dR_{xw}(90^\circ)R_{yz}(90^\circ)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

If we have a second vector, we will have:

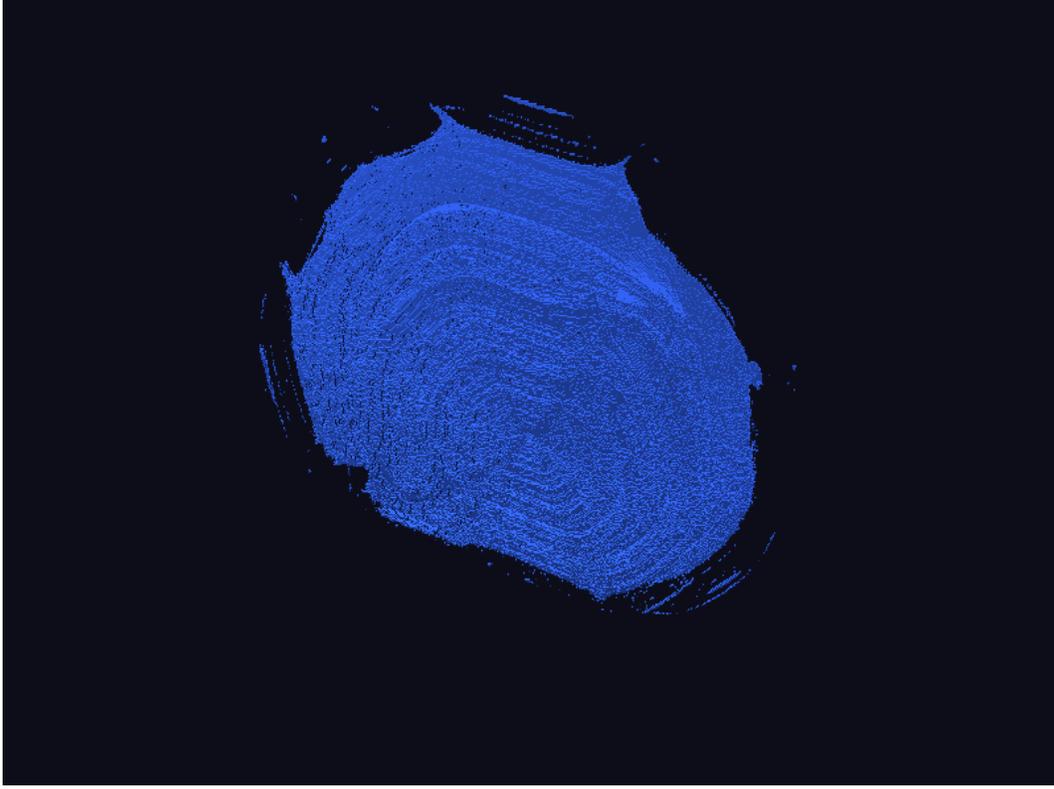


Figure 5: A 4D projection in the 3D space when $w=0$

$$\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = (eI_4 + fR_{xy}(90^\circ)R_{zw}(90^\circ) + gR_{xz}(90^\circ)R_{yw}(-90^\circ) + hR_{xw}(90^\circ)R_{yz}(90^\circ)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We can multiply those vectors as we have seen before:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \begin{pmatrix} d \\ e \\ f \\ g \end{pmatrix} = (aI_4 + bR_{xy}(90^\circ)R_{zw}(90^\circ) + cR_{xz}(90^\circ)R_{yw}(-90^\circ) + dR_{xw}(90^\circ)R_{yz}(90^\circ))(eI_4 + fR_{xy}(90^\circ)R_{zw}(90^\circ) + gR_{xz}(90^\circ)R_{yw}(-90^\circ) + hR_{xw}(90^\circ)R_{yz}(90^\circ)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And we get with the Maxima Algebra System:

$$\begin{pmatrix} -dh - cg - bf + ae \\ ch - dg + af + be \\ -bh + ag + df + ce \\ ah + bg - cf + de \end{pmatrix}$$

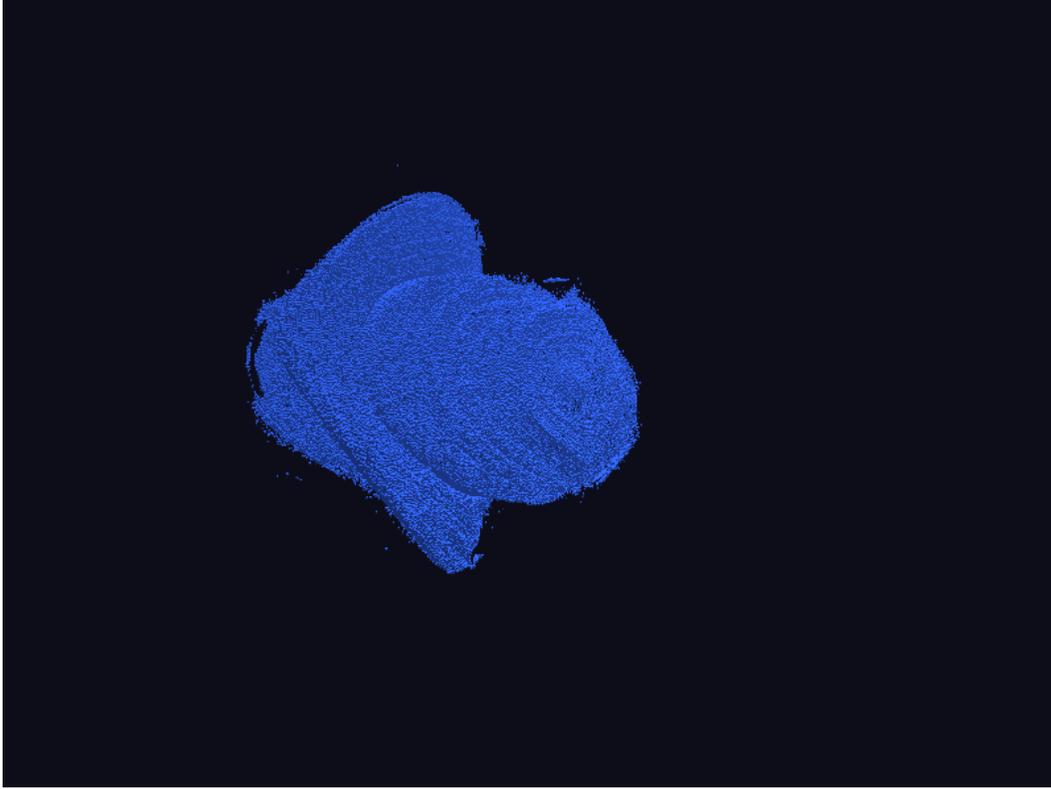


Figure 6: A 4D projection in the 3D space when $w=0.5$

And it corresponds to the multiplication of the quaternions $a + bi + cj + dk$ and $e + fi + gj + hk$ (see [Hanson2006], p.63). And with the addition and this multiplication we will have all the properties of quaternions.

So we have shown that there are no quaternion numbers but geometrical operations involving rotations and homotheties of the unit vector of the x-axis. And in this case we have a K-algebra which is also a division algebra and it is associative.

4 Proving that we have K-algebras in all dimensions

So, a vector in the Euclidean vector space of dimension $n \geq 4$, with axis x_1, x_2, \dots, x_n , can be expressed as this, where $R_{x_1 x_j}$ is a simple rotation matrix of 90° about the plane $x_1 x_j$:

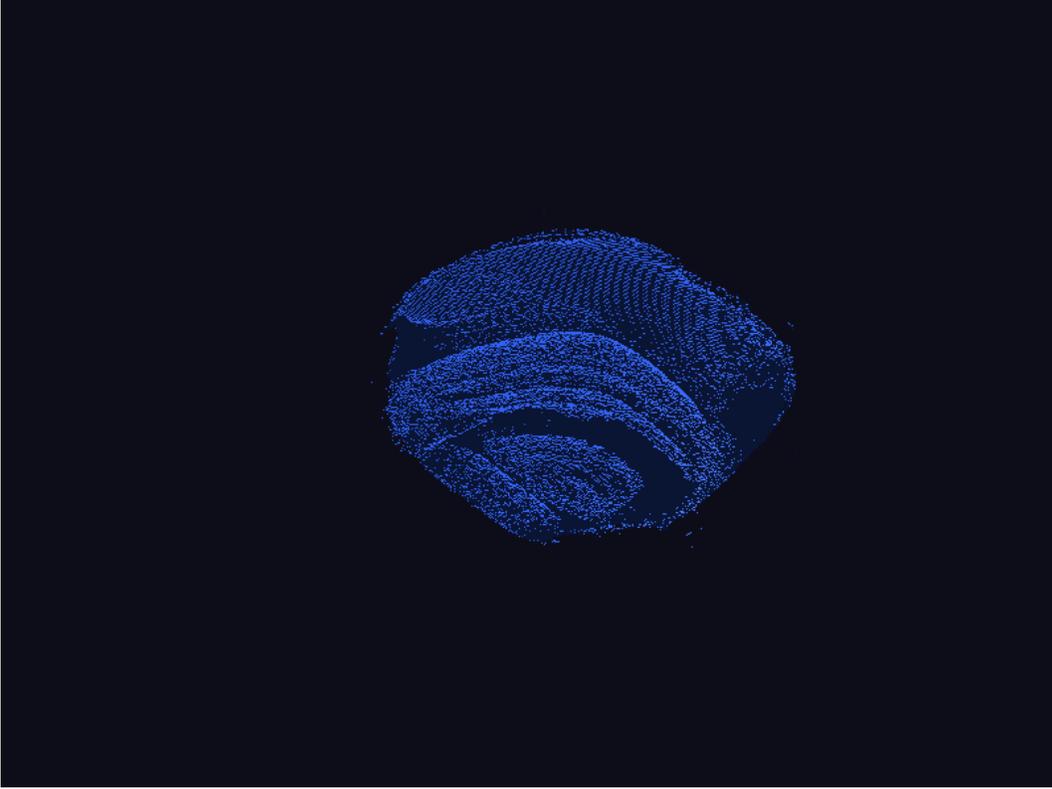


Figure 7: A 4D projection in the 3D space when $w=0.5$

$$\begin{aligned}
 \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} &= a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\
 &= a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 R_{x_1 x_2} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_n R_{x_1 x_n} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
 &= (a_1 I_n + a_2 R_{x_1 x_2} + \cdots + a_n R_{x_1 x_n}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
 &= (a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
 \end{aligned}$$

As we are using simple rotations, this proof doesn't apply to the vector operations analogous to



Figure 8: A 4D projection in the 3D space when $w=0.5$

what has been done with quaternions.

If we have two other vectors, we will have also:

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = (c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We recall that in dimension n , the multiplication is defined as this:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j}) (b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now, we will check the right distributivity of this multiplication by using the distributivity of the multiplication of matrices:

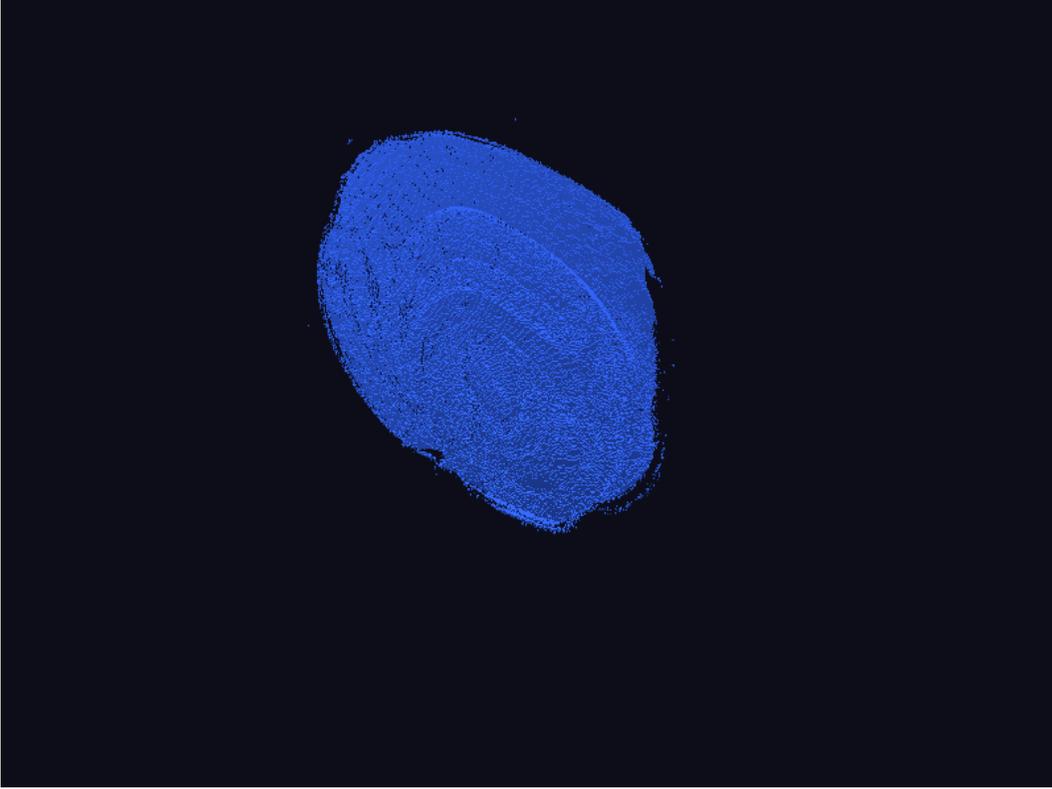


Figure 9: A 4D projection in the 3D space when $w=0.5$

$$\left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right) * \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \left((a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j}) + (b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) \right) (c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The we get:

$$\left((a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j}) (c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j}) + (b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) (c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j}) \right) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So:

$$(a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j})(c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} +$$

$$(b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j})(c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So, we have proven that:

$$\left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right) * \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} * \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} * \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Now, we will check the left distributivity of this multiplication:

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} * \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right) = (c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j}) \left((a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j}) + (b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) \right) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then we get:

$$\left((c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j})(a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j}) + \right.$$

$$\left. (c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j})(b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) \right) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So:

$$(c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j})(a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} +$$

$$(c_1 I_n + \sum_{j=1}^n c_j R_{x_1 x_j})(b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So, we have proven that:

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} * \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} * \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

We will show now that there is a compatibility with scalars. We calculate firstly:

$$\lambda \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right)$$

So we get:

$$\lambda \left((a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j})(b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

Now, because of the rules of multiplication between scalars and matrices, it is clear that this result is equal to:

$$(\lambda(a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j}))(b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$(a_1 I_n + \sum_{j=1}^n a_j R_{x_1 x_j})(\lambda(b_1 I_n + \sum_{j=1}^n b_j R_{x_1 x_j})) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So:

$$\lambda \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} * \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right) = \left(\lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \right) * \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} * \left(\lambda \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right)$$

And it is also clear, because of the definition of the multiplication involving matrices, that there is an identity element for the multiplication and for each dimension.

So, in each dimension there is a unital non-associative and non-commutative K-algebra with the usual vector addition and the multiplication defined here.

5 Conclusion

So we have seen that there are n-dimensional K-algebras with the operations defined in our previous papers and in this paper. We have shown that we can construct fractal sets in several dimensions which are linked to the Mandelbrot set. And we have shown that there are no quaternion numbers.

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