On Uniqueness of Row Normal Form

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Abstract

In this note, it is shown that the Row Echelon Normal Form (RENF), also called the Reduced Row Echelon Form (RREF) of a matrix, is unique. The result is proved from first principles.

1 Introduction

Elementary row operations are:

1. Interchange pairs of rows.
2. Add a multiple of one row to another
3. Multiply a row by a non-zero constant.

Remark Observe that these operations are reversible (there are operations to undo the effect)

It is always possible to arrange row operations (say, using the Gauss-Siedel elimination process) so that the coefficient matrix has the following form:

1. All zero rows are the last rows (at the end)
2. The first non-zero entry in a non-zero row is 1– the *leading one* of the non-zero rows.
3. The leading one is the only non-zero entry in its column

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4. Leading one in row i is to the left of leading one in row j if $i < j$.

This form is called the **Row Echelon Normal Form** [5] or the Row Reduced Echelon Form [2, 6].

Columns containing a leading one are called *leading columns* or *pivot columns* and rest are called *free columns*. RENF can be obtained by a finite sequence of row operations.

The proof is either left as an exercise[5] or is not given [2, 6]. Surowski and Wang[7] introduce Hermite normal form to prove the result. Other proofs[1, 4, 3] are based on removing some columns. They use the fact that the elementary row operations do not change the solution.

In this note, the column removal proof is first rewritten so that the removal of columns appears more justified. And then, in the next section, a proof from the first principles is given.

The proof[1, 4] is basically as follows. Assume that a matrix A has two RENF, say S_1 and S_2. Assume j is the first column in which S_1 and S_2 differ. Let α_i be the position of leading one in i^{th}row of S_1, and let β_i be the corresponding position in S_2.

Assume that the matrix is a coefficient matrix of a set of linear equations. Let k be the smallest index for which $\alpha_k > j$. Fix variables $x_i = 0$ whenever $i < \alpha_k$ is a free column (and $i \neq j$). Also, fix all $x_i = 0$ for $i \geq \alpha_k$. Carry out the same row operations on the reduced coefficient matrix that were done to get S_1 and S_2.

Remark Basically, this amounts to keeping all pivot columns till α_{k-1} and dropping all free columns before column j. Columns $j + 1$ onwards are also dropped.

Fix the variable $x_j = -1$ and interpret these as a set of equations. The right-hand side of the i^{th}row will be a_{ij}. The left-hand side will be x_{α_i}. The solution of these will be $x_{\alpha_i} = a_{ij}$. But as the j^{th}column in S_1 and S_2 differ, the solutions will be different (or one may be consistent and the other inconsistent), a contradiction.

2 Uniqueness of RENF

Assume that a matrix A has two RENF, say S_1 and S_2. As S_1 and S_2 are obtained by elementary row operations from A, rows of S_1 (similarly S_2) are a linear combination of rows of A. As elementary row operations are reversible, rows of A are also linear combinations of rows of S_1; they are also linear combinations of rows of S_2. By reversing one set of row operations, we see that rows of S_1 are a linearly combination of rows of S_2 and conversely.

Let α_i be the position of leading one in i^{th}row of S_1 and let β_i be the corresponding position in S_2.

As rows of one RENF are a linear combination of the other RENF, if the j^{th}column is zero in one, it can not be non-zero in the other. Thus, the position of zero columns in both matrices matches.

If $\alpha_1 < \beta_1$, then the α_1 column of S_2 will be zero, but the corresponding column in S_1 will not. By a similar argument, $\alpha_1 \not> \beta_1$. Hence, $\alpha_1 = \beta_1$ (the position of the first non-zero columns in two matrices).

Assume that $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$. We know that $k \geq 1$.

Assume that $R_1, R_2, \ldots$ are rows of S_1. The linear combination $d_1R_1 + d_2R_2 + \ldots + d_rR_r$ will have value d_i in the ${\alpha_i}^{\text{th}}$column.

As $\beta_{k+1} > \beta_k$, the $(k+1)^{\text{th}}$row will have zero in the first β_k columns. Hence, rows $R_1, \ldots, R_k$ can not contribute in any linear combination of rows of S_1 which gives the $(k+1)^{\text{th}}$row of S_2. Or $(k+1)^{\text{th}}$row of S_2 is determined only by rows $R_{k+1}, R_{k+2}, \ldots$ of S_1.

By a similar argument, if $R'_1, R'_2, \ldots$ are rows of S_2, the $(k+1)^{\text{th}}$row of S_1 is determined only by rows $R'_{k+1}, R'_{k+2}, \ldots$ of S_2.

If $\alpha_{k+1} < \beta_{k+1}$, then all entries in the first α_{k+1} columns of S_2 from row $k+1$ onwards will be zero. Or there is no way we can get a non-zero entry (by linear combination of these rows) in location α_{k+1}. Thus, row $R_{\alpha_{k+1}}$ of S_1 can not be expressed as a linear combination of rows of S_2, a contradiction.

By a similar argument, $\alpha_{k+1} > \beta_{k+1}$ will result in a contradiction. Thus, $\alpha_{k+1} = \beta_{k+1}$ or *all pivot positions match.*

In each non-zero row of RENF, there is only one 1 in pivot columns. If we take any linear combination of two rows of RENF, we will get two non-zero entries corresponding to pivot columns. Thus, to get a row of S_2 from rows of S_1 (as a linear combination), we can take only one row of S_1, or the rows in S_1 and S_2 match. Or $S_1 = S_2$.

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