Formulae for the nth derivative via infinite matrix operations

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Abstract

We investigate how the sequence $f^{(n)}(0)$ acts for a specific class of functions: exponentiated polynomials, $e^{p(x)}$, of which we first look at $e{-}x^2$. This leads us into an *infinite dimensional matrix*, which can be analysed via tools from Lie algebra. To generalise this to all polynomials $p(x)$, we define a correspondence between the space of derivatives of $e^{p(x)}$, $\mathcal{F}$, and a general vector space of polynomials, $\mathcal{T}$; we find that the derivative in $\mathcal{F}$ also corresponds to an operator in $\mathcal{T}$. We then utilise Zassenhaus' formula to find how this operator iterates, hence giving us a general formula for the nth derivative of $e^{p(x)}$.

1 Preface

This was originally a question I came up with during Year 12 - I was studying coefficients of Taylor series, namely $f(x) = e^{-x^2}$. Centered at $x = 0$, we find

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} + O(x^{10})$$

though I was most interested in the coefficient sequence,

$$f^{(n)}(0) = 1, 0, -2, 0, 12, 0, -120, 0, 1680, \ldots$$

and how it arises. I was also interested in other similar sequences; like how Bernoulli numbers pop out of the series expansion of tan.

During this summer I returned to this question and found a more elegant way to view it, through linear algebra and 'infinite' matrices. After posting on MSE [1], I was guided to similar matrices which pop up in Lie algebras. I was then able to show the above sequence has even terms generated by $(-1)^k(2k)!/k!$.

Though this result is relatively trivial, and can be found immediately by manipulations on the series expansion of e^x - I was not entirely convinced that this would work, specifically whether term by term manipulation was well found, as I hadn't studied analysis at that point. Regardless, I was more interested in the recurrence that arises from taking repeated derivatives.

In this paper, I find a formula for the nth derivatives for functions of the form $e^{p(x)}$, p being any polynomial. The proof involves using the Zassenhaus formula with certain operators in a vector space of functions. I end up with a special case of Faa di Bruno's formula.

2 The motivation of matrices

The main idea is that one can look at a function and its derivatives as elements of a certain vector space, with a certain basis. The derivative can then be seen as a linear transformation between these vectors.

We begin with a motivating example; derivatives of cos. Repeated derivatives follow the pattern

$$\cos \to -\sin \to -\cos \to \sin$$

and so if we consider a *basis* for these derivatives, say $\{e_1, e_2\} = \{\cos, \sin\}$, then the derivative can be viewed as a certain matrix, namely

$$M_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

To use this, $\cos \leftrightarrow e_1$ and then $\frac{\mathrm{d}}{\mathrm{d}x} \cos x \leftrightarrow Me_1 = -e_2 \leftrightarrow -\sin$. Using $M^k e_1$ yields the kth derivative of cos, and e_2 can be used instead to find derivatives of sin. The question of finding the nth derivative now reduces to finding the nth power of M. We can diagonalise to find

$$M_1 = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i & \\ & -i \end{bmatrix} \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{1}$$

$$M_1{}^n = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i^n & \\ & i^{-n} \end{bmatrix} \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \cosh \log i^n & i \sinh \log i^n \\ -i \sinh \log i^n & \cosh \log i^n \end{bmatrix} \tag{3}$$

in our $\{e_1, e_2\}$ basis. We can then recover

$$\cos^{(n)}(a) = \cosh \log i^n \cos a - i \sinh \log i^n \sin a$$
$$\sin^{(n)}(a) = i \sinh \log i^n \cos a + \cosh \log i^n \sin a.$$

This form looks slightly strange, but applying some trigonometric identities, noting $i^n = in\pi/2$, and $\arctan(i \tanh x) = ix$, we find

$$\cos^{(n)}(a) = \cos(a + n\pi/2)$$
$$\sin^{(n)}(a) = \sin(a + n\pi/2)$$

which may be more familiar.

With hope, we move on to study repeated derivatives of e^{-x^2}.

$$\begin{aligned}
f(x) &= e^{-x^2} &&= e^{-x^2} \\
f'(x) &= -2xe^{-x^2} &&= -2xe^{-x^2} \\
f''(x) &= 4x^2e^{-x^2} - 2\textcolor{red}{e^{-x^2}} &&= 4x^2e^{-x^2} - 2f \\
&\vdots
\end{aligned}$$

where we see a sort-of recurrence highlighted in red; f'' contains f in a way. The infinite basis $\{e_i\} = \{(-2x)^i e^{-x^2}\}_{i=0}^{\infty}$ allows us to proceed like the above, and the operator of differentiation can then be seen as the matrix

$$M = \begin{bmatrix} & -2 & & & \\ 1 & & -4 & & \\ & 1 & & -6 & \\ & & 1 & & -8 \\ & & & 1 & \ddots \end{bmatrix}$$

which has infinite size/width. Unlike the above, diagonalising this seems hopeless - via SciPy, I was able to verify that no eigenvalues/vectors of this converge, so it is unlikely a diagonalisation even exists.

3 Lie Algebras and e^{-x^2}

After much deliberation, I posted on MSE [1] in hopes that there was another way to diagonalise, or a separate method entirely to find powers of this matrix. I was referred to *Weyl algebras*, a type of Lie algebra.

A Lie algebra is a vector space equipped with a binary product called the Lie bracket, which is bilinear, alternating and satisfies the Jacobi identity,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \;=\; 0.$$

The Weyl algebra is a certain Lie algebra that is the key to finding powers of the above matrix. Specifically, the Weyl algebra of one variable (say over the ring $\mathbb{R}[t]$) is defined from a ring generated by two operators: multiplication by t and differentiation with respect to t. We can view these operators and the underlying ring in a more pertinent way - as the *infinite* matrices;

$$\mathbf{x} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \end{bmatrix} \qquad \partial = \begin{bmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 \\ & & & \ddots \end{bmatrix}$$

acting on the vector space of polynomials, with respect to basis $\{t^i\}_{i=0}^{\infty}$. These are non-commutative operators, read right-to-left.

So, we have a vector space, but to make this a Lie algebra, we need to add in a binary operator; the Lie bracket. We define the commutator as $[\partial, \mathbf{x}] = \partial\,\mathbf{x} - \mathbf{x}\,\partial$. Applying this to a dummy polynomial $f = \sum c_i t^i$, we see

$$\begin{aligned}\partial\,\mathbf{x}\,(f) &= \sum (i+1)\, c_i t^i \\ &= \sum i c_i t^i + f \\ &= \mathbf{x}\,\partial\,(f) + f\end{aligned}$$

so $\partial\mathbf{x}(f) - \mathbf{x}\partial(f) = f \implies [\partial, \mathbf{x}] = 1$. The ring equipped with this Lie bracket forms a Lie algebra.

The main result we will use is the *Zassenhaus formula*,

$$e^{s(X+Y)} = e^{sX}\ e^{sY}\ e^{-\frac{s^2}{2}[X,Y]}\ e^{\frac{s^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])}\ \cdots$$

which refers to non-commutative X, Y in a Lie algebra.

With all the preliminary work on Lie algebras covered, we move on to the main question. We will deal with two vector spaces; one named $\mathcal{F}$, which contains derivatives of e^{-x^2} and has basis $\{(-2x)^i e^{-x^2}\}$; and one is the Weyl algebra as defined above, over the vector space $\mathcal{T}$ of polynomials in t, with basis $\{t^i\}$. We consider a *map/correspondence* between these vector spaces, so $(-2x)^i e^{-x^2} \leftrightarrow t^i$. The operator $\mathbf{x}$ in $\mathcal{T}$ corresponds to multiplying by $(-2x)$ in $\mathcal{F}$, while the operator ∂ in $\mathcal{T}$ corresponds to multiplying by $i/(-2x)$ in $\mathcal{F}$.

We can see that the matrix M above corresponds to the operator $\mathbf{x} - 2\partial$, and so powers of M correspond to powers of $\mathbf{x} - 2\partial$. But we can explicitly find the latter by exploiting exponential generating functions! Consider

$$F(s) := e^{(\mathbf{x}-2\partial)s} = \sum_{n\geq 0} (\mathbf{x} - 2\partial)^n \frac{s^n}{n!}$$

for some commuting dummy variable s. We can expand F using the Zassenhaus formula and find;

$$
\begin{aligned}
F(s) = e^{(\mathbf{x}-2\partial)s} &= e^{\mathbf{x}s} e^{-2\partial s} e^{\frac{-s^2}{2}[\mathbf{x},-2\partial]} \\
&= e^{\mathbf{x}s} e^{-2\partial s} e^{-s^2} \\
&= \sum_{i\geq 0} \frac{(\mathbf{x}s)^i}{i!} \sum_{j\geq 0} \frac{(-2\partial s)^j}{j!} \sum_{k\geq 0} \frac{\left(-s^2\right)^k}{k!} \\
&= \sum_{n\geq 0} \sum_{i+j+2k=n} \frac{(\mathbf{x}s)^i (-2\partial s)^j \left(-s^2\right)^k}{i!j!k!} \\
&= \sum_{n\geq 0} \sum_{i+j+2k=n} \frac{\mathbf{x}^i (-2\partial)^j (-1)^k s^n}{i!j!k!}.
\end{aligned}
$$

Equating these expansions of F we can recover

$$
(\mathbf{x}-2\partial)^n = \sum_{i+j+2k=n} \frac{n!}{i!j!k!} \mathbf{x}^i (-2\partial)^j (-1)^k.
$$

Now we can bring this back into our original vector space $\mathcal{F}$, since $f^{(n)} \leftrightarrow M^n e_0 \leftrightarrow (\mathbf{x}-2\partial)^n(1)$. Note that $\partial(1) = 0$, so $j \neq 0 \implies$ the summand is 0; hence we may simplify the sum. We recover the result

$$
\frac{\mathrm{d}}{\mathrm{d}x^n} e^{-x^2} = \sum_{i+2k=n} \frac{n!\,(-1)^k}{i!k!} (-2x)^i e^{-x^2}
$$

4 Corollaries

For the original sequence, we would want to find $f^{(n)}(0)$. For the summand to be non-zero, we need $i = 0$, hence

$$
\left.\frac{\mathrm{d}}{\mathrm{d}x^{2k}} e^{-x^2}\right|_{x=0} = \frac{(2k)!\,(-1)^k}{k!}
$$

and 0 for odd order derivatives.

One could then use this to get the Taylor series of e^{-x^2};

$$
e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}
$$

but I view that as using a jackhammer to crack an egg.

A more interesting result comes from 'inverting' M. Via SciPy, I was able to invert truncations of M, which seemed to converge to another infinite matrix.

The following is an inverse of the 10×10 truncation.

$$\begin{bmatrix} 0 & 1 & 0 & 4 & 0 & 32 & 0 & 384 & 0 & 6144 \\ -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 8 & 0 & 96 & 0 & 1536 \\ 0.08 & 0 & -0.17 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 12 & 0 & 192 \\ 0.01 & 0 & 0.02 & 0 & -0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 0 & 0.01 & 0 & 0.07 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.06 & 0 \end{bmatrix}$$

As the ith column of M^1 corresponded to the first derivative of $(-2x)^i e^{-x^2}$, I expected the ith column of M^{-1} to correspond to the *integral* of $(-2x)^i e^{-x^2}$. One can see that this does hold true for the even-indexed columns.

Looking at the first column more closely, I found the entries were

$$-\frac{1}{2}, -\frac{1}{12}, -\frac{1}{120}, -\frac{1}{1680} \cdots \sim -\frac{n!}{(2n)!}$$

and it seems that

$$\int e^{-x^2}\,\mathrm{d}x = \sum_{n=1}^{\infty} \frac{n!}{(2n)!} (2x)^{2n-1} e^{-x^2}$$

by rewriting in our basis. In fact, this seems to converge 'faster' than the usual Taylor expansion. I do not have a proof of this equality but I believe the constancy theorem could be used; if f is the sum, then $f'(x) = e^{-x^2}$ implies $\frac{\mathrm{d}}{\mathrm{d}x} f'(x) e^{x^2} = 0$ - but the resulting sum is not trivially zero.

5 An algorithm for Zassenhaus

I hope to extend this method to other functions - however it is likely that further terms in the Zassenhaus formula would be needed. The above example was nice in the fact that only the first commutator was non-zero; all of the others being 0. With heavy reference to [2], I was able to program this.

I first recall what this paper states about the Zassenhaus formula.

The Zassenhaus formula is

$$e^{s(X+Y)} = e^{sX} e^{sY} \prod_{i=2}^{\infty} e^{s^i C_i}$$

where C_i are polynomials of commutators, namely

- $C_2 = -\frac{1}{2}[X, Y]$
- $C_3 = \frac{1}{6}(2[Y, [X, Y]] + [X, [X, Y]])$
- $C_4 = -\frac{1}{24}(3[Y, [Y, [X, Y]]] + 3[Y, [X, [X, Y]]] + [X, [X, [X, Y]]])$
- for further terms, $C_{n+1} = \frac{1}{n+1} f_{\lfloor n/2 \rfloor, n}$

where

$$f_{n,k} = \sum_{j=0}^{\lfloor k/n \rfloor - 1} \frac{(-1)^j}{j!} \mathrm{ad}_{C_n}^j (f_{n-1, k-nj})$$

$$f_{1,k} = \sum_{j=1}^{k} \frac{(-1)^k}{j!\,(k-j)!} \mathrm{ad}_Y^{k-j} \left(\mathrm{ad}_X^j (Y)\right)$$

in which the adjoint operator, ad is defined as

- $\mathrm{ad}_A (B) = [A, B]$
- $\mathrm{ad}_A^j (B) = \left[A, \mathrm{ad}_A^{j-1}(B)\right]$ i.e. repeated commutations
- $\mathrm{ad}_A^0 (B) = B$.

The paper describes an algorithm to find further Zassenhaus terms;

- find $f_{1,k}$ by the above
- define $C_n = 1/n\ f_{1,n-1}$ for $n = 2, 3, 4$
- define $f_{n,k}$ for $k \geq n \geq 2$ by the above
- define $C_n = 1/n\ f_{\lfloor (n-1)/2 \rfloor, n-1}$ for $n \geq 5$.

The paper also contains a Mathematica program able to compute C_n. I've tweaked this to be able to substitute operators directly;

```
<< NCAlgebra`
<< NCGBX`
Clear[Comm, gb, ad, ff, cc, d, x, ex, exv];

SetNonCommutative[x,d]
SetMonomialOrder[x,d] (* x to the left, d to the right *)
NCSetOutput[NonCommutativeMultiply -> True] (* pretty output *)

(* commutation relations *)
gb = NCMakeGB[{
   d ** x - x ** d - 1}, 20];
```

```
(* define commutator *)
Comm[A_, B_] := A ** B - B ** A;


(* formula from first paper *)
ad[a_, 0, b_]:= b;
ad[a_, j_Integer, b_]:= exv[Comm[a, ad[a, j-1, b]]];

ff[1, k_]:= ff[1, k] = Sum[((-1)^k/(j! (k-j)!))
                        ad[Y, k-j, ad[X, j, Y]], {j, 1, k}];

cc[2] = (1/2) ff[1, 1];

ff[p_, k_]:= ff[p, k] = Sum[((-1)^j/j!) ad[cc[p], j, ff[p-1,
             k - p j]], {j, 0, IntegerPart[k/p] - 1}];

cc[p_Integer]:= cc[p] = Expand[(1/p) ff[IntegerPart[(p-1)/2],
                         p-1]];

(* maybe useful *)
ex[a_] := NCReplaceRepeated[a, gb] // NCExpand
exv[a_] := ex[ex[ex[ex[ex[a]]]]]
```

This defines the non-commutative operators `x` and `d` and their commutator. This now outputs C_n with the commutator values - this can also be modified easily for other operators.

$$e^{s(X+Y)} = e^{sX} e^{sY} \prod_{i=2}^{\infty} e^{s^i C_i}$$

By expressing these as generating functions, one can isolate powers of s and see

$$(X+Y)^n = \sum_{\alpha_0 + \sum_{i=1}^{n} i\alpha_i = n} \frac{n! X^{\alpha_0} Y^{\alpha_1}}{\alpha_0! \alpha_1!} \prod_{i=2}^{\infty} \frac{C_i^{\alpha_i}}{\alpha_i!}$$

just from expanding the Cauchy product on the right.

I was able to characterise tan and sec's derivative matrix similarly to e^{-x^2}, which gave $\partial + \mathbf{x}^2\partial$ and $\partial + \mathbf{x}\partial\mathbf{x}$ respectively. However, the Zassenhaus commutators C_n do not terminate computationally. Instead for the nth derivative of these, one would need to compute up to C_n in the Zassenhaus formula.

6 A proof for $e^{p(x)}$

While tan's commutators do not terminate, functions of the form $e^{p(x)}$ for p a polynomial seem to always terminate after $\deg p$ commutators. I've managed to prove this, and hence find a formula for the nth derivative of $e^{p(x)}$.

For a polynomial $p(x)$ of degree n and the function $e^{p(x)}$, we will use the basis $\{p'^{i_1}\, p''^{i_2}\, p'''^{i_3} \dots p^{(n)i_n} e^p\}$, where $i_1, \dots, i_n$ are indices in $\mathbb{Z}^{\geq 0}$. This space is $\mathcal{F}$. It will correspond to the basis $\{t_1^{i_1} t_2^{i_2} \dots t_n^{i_n}\}$ of the vector space $\mathcal{T}$, polynomials of n variables. We will also have the operators;

- $\hat{\mathbf{x}}_k$, multiplication by t_k
- ∂_k, differentiation with respect to t_k

and in $\mathcal{F}$, these look like;

- $\hat{\mathbf{x}}_k$, multiplication by $p^{(k)}$ (note that $p^{(n)}$ is a constant)
- ∂_k, multiplication by $i_k/p^{(k)}$; if $i_k = 0$ then this is the zero map
- $\partial_k e^p = 0$.

This gives $\mathcal{T}$ the structure of the Weyl algebra of n variables - we also have the commutator, $[\partial_i, \hat{\mathbf{x}}_j] = \delta_{ij}$.

We can now move on to looking at $e^{p(x)}$. First, we look at derivatives of e^p for degree 2 (and 3), from earlier;

- if p is a quadratic, $\frac{\mathrm{d}}{\mathrm{d}x} p'^i e^p = i p'^{i-1} p'' + p'^{i+1} e^p$, so the matrix in $\mathcal{T}$ is $[\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2 \partial_1]$ (as we saw with e^{-x^2})
- if p is a cubic, $\frac{\mathrm{d}}{\mathrm{d}x} p'^i p''^j = i p'^{i-1} p''^{j+1} + j p'^i p''^{j-1} p''' + p'^{i+1} p''^j$ so the matrix in $\mathcal{T}$ is $[\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2 \partial_1 + \hat{\mathbf{x}}_3 \partial_2]$

and via induction we can spot that for a degree n polynomial, the derivative operator in $\mathcal{T}$ is

$$\left[\sum_{j=1}^{n} \hat{\mathbf{x}}_j \partial_{j-1} \right].$$

We now need to find powers of this operator. Label the summand A_j and the partial sums T_n. We proceed by expanding the generating function

$$\exp\left(s \sum_{j=1}^{n} \hat{\mathbf{x}}_j \partial_{j-1} \right)$$

using the Zassenhaus formula, $e^{s(X+Y)} = e^{sX} e^{sY} \prod_{k=2}^{\infty} e^{s^k C_k}$. Via the program in chapter 5, I was able to conjecture a form for C_k. I have proved this conjecture, but I've relegated the proof to appendix A - it only consists of a few inductions, none of which are particularly enlightening. We find the result

$$C_k = \frac{1}{k!}\hat{\mathbf{x}}_n \partial_{n-k-1}.$$

Finally, we proceed to the generating function. We consider inducting on $T_n = A_n T_{n-1}$;

$$\exp(sT_n) = \exp(sT_{n-1}) \prod_{j=1}^{n} \exp\left(\frac{s^i}{i!}\hat{\mathbf{x}}_n \partial_{n-j}\right)$$

This can be expanded into

$$\begin{aligned}
\exp\left(s\sum_{j=1}^{n} \hat{\mathbf{x}}_j \partial_{j-1}\right) &= \prod_{j=1}^{1} \exp\left(\frac{s^i}{i!}\hat{\mathbf{x}}_n \partial_{n-j}\right) \cdots \prod_{j=1}^{n} \exp\left(\frac{s^i}{i!}\hat{\mathbf{x}}_n \partial_{n-j}\right) \\
&= \prod_{k=1}^{n}\prod_{j=1}^{k} \exp\left(\frac{s^j}{j!}\hat{\mathbf{x}}_k \partial_{k-j}\right) \\
&= \prod_{k=1}^{n}\prod_{j=1}^{k} \sum_{m_{k,j}=0}^{\infty} \left(\frac{s^j}{j!}\hat{\mathbf{x}}_k \partial_{k-j}\right)^{m_{kj}} \Big/ m_{k,j}! \\
&= \sum_{\left(\sum_{\substack{j=1\dots k \\ k=1\dots n \\ m_{k,j}\in\mathbb{Z}\geq 0}} j m_{k,j} = \zeta\right)} \prod_{m_{k,j}} \frac{(\hat{\mathbf{x}}_k \partial_{k-j})^{m_{k,j}}}{m_{k,j}! j!^{m_{k,j}}} s^{\zeta}
\end{aligned}$$

and isolating powers of s on both sides we find

$$\left(\sum_{j=1}^{n} \hat{\mathbf{x}}_j \partial_{j-1}\right)^{\zeta} = \sum_{\left(\sum_{\substack{j=1\dots k \\ k=1\dots n \\ m_{k,j}\in\mathbb{Z}\geq 0}} j m_{k,j} = \zeta\right)} \prod_{m_{k,j}} \frac{(\hat{\mathbf{x}}_k \partial_{k-j})^{m_{k,j}}}{m_{k,j}! j!^{m_{k,j}}} \zeta!$$

which is our final formula in $\mathcal{T}$!

7 Corollaries

Translating this back into $\mathcal{F}$, this means

$$M^{\zeta} = \sum_{\left(\sum_{\substack{j=1\dots k \\ k=1\dots n \\ m_{k,j}\in\mathbb{Z}\geq 0}} j m_{k,j} = \zeta\right)} \prod_{m_{k,j}} \frac{\left(p^{(k)} i_{k-j} / p^{(k-j)}\right)^{m_{k,j}}}{m_{k,j}! j!^{m_{k,j}}} \zeta!$$

For example, in the $n = 2$ case,

$$M^{\zeta} = \sum_{i+j+2k=\zeta} \zeta! \frac{(p')^i (p''\partial)^j (p'')^k}{i!j!k!2!^k}$$

which is in fact the same formula we found in section 3.

If we now apply this to e^p, we note that only i_0 is non-zero, so for any $k \neq j$ terms, the summand is zero due to the ∂. So, specifying $m_{k,k} =: M_k$ are non-zero, the sum collapses into

$$\frac{\mathrm{d}^{\zeta}}{\mathrm{d}x^{\zeta}} e^p = M^{\zeta}(e^p) = e^p \cdot \sum_{\left(\sum_{\substack{k=1\dots n \\ M_k \in \mathbb{Z}_{\geq 0}}} kM_k = \zeta\right)} \prod_{M_k} \frac{\left(p^{(k)}\right)^{M_k}}{M_k! k!^{M_k}} \zeta!.$$

In the special case (akin to section 3) where $p = Ax^n$, then at $x = 0$, the only non-zero derivative is $p^{(n)} = n!A$. Hence the product inside this sum is 0 unless the only non-zero M_k is M_n. We recover that

$$\begin{aligned}
\left.\frac{\mathrm{d}^{\zeta}}{\mathrm{d}x^{\zeta}} e^{Ax^n}\right|_{x=0} &= \sum_{\left(\sum_{\substack{k=1\dots n \\ M_k \in \mathbb{Z}_{\geq 0}}} kM_k = \zeta\right)} \prod_{M_k} \frac{\left(p^{(k)}\right)^{M_k}}{M_k! k!^{M_k}} \zeta! \\
&= \sum_{\left(\sum_{M_n \in \mathbb{Z}_{\geq 0}} nM_n = \zeta\right)} \frac{A^{M_n}}{M_n!} \zeta!
\end{aligned}$$

In this, we require M_n to be a multiple of ζ for the sum to have an index, so equivalently we can write

$$\left.\frac{\mathrm{d}^{\zeta}}{\mathrm{d}x^{\zeta}} e^{Ax^n}\right|_{x=0} = \begin{cases} \frac{\zeta!}{r!} A^r & \text{if } \zeta/n =: r \in \mathbb{N} \\ \\ 0 & \text{else .} \end{cases}$$

A corollary is that the taylor series for e^{Ax^n} at $x = 0$ is given by

$$e^{Ax^n} = \sum_{r=0}^{\infty} \frac{A^r}{r!} x^{nr}.$$

8 Comments

I only discovered during section 5 that all I had done was proven Faà di Bruno's formula for functions of the form $e^{p(x)}$ - using a much more complicated argument. However, I have hopes that this theory could be extended to other functions as well, and hopefully produce neat results. For reference, Faà di Bruno's formula states;

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{\sum_{i=1}^n i m_i = n} \frac{n!}{\prod m_i! \prod i!^{m_i}} \cdot f^{(m_1+\cdots+m_n)}(g(x)) \cdot \prod_{j=1}^{n} \left(g^{(j)}(x)\right)^{m_j}$$

which is usually proven via a combinatorial argument.

I have not studied much algebra so I expect details to be wrong in section 3 - the only substantial result I use is the Zassenhaus formula, which I believe I've applied correctly.

I didn't spend much time investigating integrals in section 4 as they didn't seem readily generalisable, but I may return to this.

One of my main questions which I cannot find an answer to concerns section 5 - is there a sufficient condition to see whether the Zassenhaus formula (or equivalently, adjoints of operators) will terminate? They terminated nicely for $e^{p(x)}$, forming the backbone of section 6, but they did not for tan or sec.

[[This work was mostly completed after I had finished my first year of undergrad, about 18 months ago, and is not written in formal language. I thought I would write this little comment (and the abstract) now, on 21/02/2026, to prepare it for viXra. If there are any comments, please email me.]]

References

[1] Qiaochu Yuan (2024), https://math.stackexchange.com/questions/4977773/taking-the-nth-power-of-an-infinite-dimensional-matrix

[2] Casas, Murua, Nadinic (2012), https://arxiv.org/pdf/1204.0389

9 Appendix A - Zassenhaus polynomials for the operator of $e^{p(x)}$

I've relegated this to an appendix as the proof is not particularly enlightening - it involves several inductions and algebra juggling, and is tedious to follow. Essentially, I follow the paper[2] discussed in chapter 5 - I compute the commutators, adjoints, and then the sums $f_{n,k}$, quantities C_k.

$$\exp\left(s\sum_{j=1}^{n}\hat{\mathbf{x}}_j\partial_{j-1}\right)$$

We will be using the Zassenhaus formula on this, hence we need to find commutators of the sum. Label the terms of this sum A_j and the partial sums T_n. In this algebra, the only non-zero commutator comes from $\mathbf{x}_n$ and ∂_n i.e. they have the same subscript. Hence it suffices to compute the commutator of T_{n-1} and A_n,

$$\begin{aligned}
[T_{n-1}, A_n] = \left[\sum_{j=1}^{n-1}\hat{\mathbf{x}}_j\partial_{j-1}, \hat{\mathbf{x}}_n\partial_{n-1}\right] = \sum_{j=1}^{n-1}\left[\hat{\mathbf{x}}_j\partial_{j-1}, \hat{\mathbf{x}}_n\partial_{n-1}\right] &= \left[\hat{\mathbf{x}}_{n-1}\partial_{n-2}, \hat{\mathbf{x}}_n\partial_{n-1}\right] \\
&= \hat{\mathbf{x}}_n\left[\hat{\mathbf{x}}_{n-1}, \partial_{n-1}\right]\partial_{n-2} \\
&= -\hat{\mathbf{x}}_n\partial_{n-2}
\end{aligned}$$

and so any further commutators with A_n vanish. If we consider the second adjoint,

$$\begin{aligned}
[T_{n-1}, [T_{n-1}, A_n]] = \sum_{j=1}^{n-1}\left[\hat{\mathbf{x}}_j\partial_{j-1}, -\hat{\mathbf{x}}_n\partial_{n-2}\right] &= -\left[\hat{\mathbf{x}}_{n-2}\partial_{n-3}, \hat{\mathbf{x}}_n\partial_{n-2}\right] \\
&= -\hat{\mathbf{x}}_n\left[\hat{\mathbf{x}}_{n-2}, \partial_{n-2}\right]\partial_{n-3} \\
&= \hat{\mathbf{x}}_n\partial_{n-3}
\end{aligned}$$

and this continues until we reach $\hat{\mathbf{x}}_n\partial_0 = \hat{\mathbf{x}}_n$, which commutes with everything. By induction, the general formula is

$$\mathrm{ad}^k_{T_{n-1}}\left(A_n\right) = \left(-1\right)^k\hat{\mathbf{x}}_n\partial_{n-k-1}.$$

We've characterised all of the commutators, so now we move on to look at the polynomials f mentioned in section 5. We find;

$$\begin{aligned}
f_{1,k} &= \sum_{j=1}^{k} \frac{(-1)^k}{j!\,(k-j)!} \mathrm{ad}_{A_n}^{k-j}\left(\mathrm{ad}_{T_{n-1}}^{j}(A_n)\right) \\
&= \sum_{j=1}^{k} \frac{(-1)^k}{j!\,(k-j)!} \mathrm{ad}_{A_n}^{k-j}\left((-1)^j\,\hat{\mathbf{x}}_n \partial_{n-j-1}\right) \\
&= \sum_{j=1}^{k} \frac{(-1)^k}{j!\,(k-j)!} \left((-1)^j\,\hat{\mathbf{x}}_n \partial_{n-j-1}\right)\delta_{kj} \\
&= \frac{1}{k!}\hat{\mathbf{x}}_n \partial_{n-k-1}
\end{aligned}$$

for $k = 0, \ldots, n-1$ and 0 otherwise. It falls out that

- $C_2 = \frac{1}{2} f_{11} = \frac{1}{2\times 1!}\hat{\mathbf{x}}_n \partial_{n-2}$
- $C_3 = \frac{1}{3} f_{12} = \frac{1}{3\times 2!}\hat{\mathbf{x}}_n \partial_{n-3}$
- $C_4 = \frac{1}{4} f_{13} = \frac{1}{4\times 3!}\hat{\mathbf{x}}_n \partial_{n-4}$

Further,

$$\begin{aligned}
f_{2,k} &= \sum_{j=0}^{\lfloor k/2 \rfloor -1} \frac{(-1)^j}{j!} \mathrm{ad}_{C_2}^{j}\left(f_{1,k-2j}\right) \\
&= \sum_{j=0}^{\lfloor k/2 \rfloor -1} \frac{(-1)^j}{j! 2^j\,(k-2j)!} \mathrm{ad}_{\hat{\mathbf{x}}_n \partial_{n-2}}^{j}\left(\hat{\mathbf{x}}_n \partial_{n-k+2j-1}\right) \\
&= \sum_{j=0}^{\lfloor k/2 \rfloor -1} \frac{(-1)^j}{j! 2^j\,(k-2j)!} \left(\mathrm{ad}_{\hat{\mathbf{x}}_n \partial_{n-2}}^{j}\left(\hat{\mathbf{x}}_n \partial_{n}\right)\delta_{n,n-k+2j-1} + \mathrm{ad}_{\hat{\mathbf{x}}_n \partial_{n-2}}^{0}\left(\hat{\mathbf{x}}_n \partial_{n-k-1}\right)\delta_{j,0}\right) \\
&= \frac{1}{k!}\hat{\mathbf{x}}_n \partial_{n-k-1}
\end{aligned}$$

so C_5, C_6 follow the same pattern as the previous C_i; since $C_n = 1/n\ f_{2,n-1}$ for $n = 5, 6$. An induction can be made to prove that this pattern holds for all C_i, since the only term that is non-zero in the sum is when $j = 0$. Hence

$$\begin{aligned}
f_{n,k} &= \sum_{j=0}^{\lfloor k/n \rfloor -1} \frac{(-1)^j}{j!} \mathrm{ad}_{C_n}^{j}\left(f_{n-1,k-nj}\right) \\
&= f_{n-1,k}
\end{aligned}$$

and so

$$C_k = \frac{1}{k!}\hat{\mathbf{x}}_n \partial_{n-k-1}.$$