Arrangement of Identical Touching Circles on a Spherical Surface Associated with Platonic Solids

Harish Chandra Rajpoot

M.M.M. University of Technology, Gorakhpur-273010 (UP), India 20 Aug 2015

1. Introduction

In this paper, the identical circles, each having a flat radius r, touching one another and covering a whole spherical surface with a certain radius R associated with platonic solids [1] will be analysed in details to find out

1. Relation between R and r
2. Radius of each circle in form of a great circle arc $(\text{arc}\, r)$ and,
3. Total surface area and its percentage $(\%)$ covered by all the circles on the whole spherical surface.

For such cases, each of the identical circles is assumed to be inscribed by each of the congruent regular polygonal (flat) faces of a platonic solid and spherical surface to be concentric with that platonic solid. Thus each of identical circles touches other circles exactly at the mid-points of edges of each face of the platonic solid. Then by joining the mid-point of one of the edges (also the point of tangency) to the centre of platonic solid (also the centre of the spherical surface), a right triangle, having its hypotenuse R and orthogonal sides (i.e. legs) r and h, is thus obtained. Where,

$$h = \text{normal distance of each (flat) face from the centre of the platonic solid}$$

$$= \text{normal distance of each plane (flat) circle from the centre of the spherical surface}$$

The values of important parameters such as normal distance (h) of each face from the centre of platonic solids will be directly adopted from previously derived results of all five platonic solids [2] by the author to derive the mathematical relations of R and r in all the cases to be discussed and analysed here. All five cases will be discussed and analysed in details in the order corresponding to all five platonic solids.

2. Four identical circles, each having a flat radius r, touching one another at 6 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius R : (Analogous to a regular tetrahedron)

In this case**,** let's assume that each of four identical circles, with a flat radius r, is inscribed by each of four congruent equilateral triangular faces of a regular tetrahedron with an edge length a such that regular tetrahedron is concentric with the spherical surface having the centre O & a radius R. In this case, all 6 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 6 edges of a regular tetrahedron. Now, consider one of the four identical circles with the centre C on the flat face & a flat radius r, touching three other circles at the points A, B, and D (lying on the spherical surface as well as on the edges of regular tetrahedron) and is inscribed by an equilateral triangular face of regular tetrahedron with an edge length a (as shown in the figure 1).

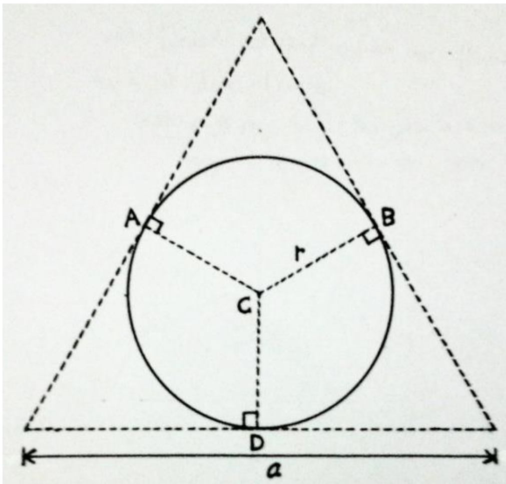

Figure 1: One of 4 identical circles, with the centre C on the flat face & a flat radius r, is touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of face) and is inscribed by an equilateral triangular face of a regular tetrahedron (concentric with the spherical surface) with an edge length a.

We know that the inscribed radius r & the edge length a of a regular n-gon are related by following formula [2,3],

$$r = \frac{a}{2}\cot\frac{\pi}{n} \quad \Rightarrow \; a = 2r\tan\frac{\pi}{n}$$

Hence, by setting $n = 3$ for an equilateral triangular face, we get

$$a = 2r\tan\frac{\pi}{3} \;\Rightarrow\; \boldsymbol{a = 2r\sqrt{3}}$$

Now, we have

$$\mathrm{h} = \text{normal distance of each face from the centre of the regular tetrahedron with edge length } a$$

$$= \text{normal distance of plane (flat) circle with centre C from the centre O of spherical surface}$$

$$\Rightarrow h = OC = \frac{a}{2\sqrt{6}} \qquad \text{(from the table of platonic solids)}$$

$$= \frac{(2r\sqrt{3})}{2\sqrt{6}} = \frac{r}{\sqrt{2}} \qquad \text{(setting the value of edge length } a\text{)}$$

$$\Rightarrow \boldsymbol{h = OC = \frac{r}{\sqrt{2}}}$$

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of regular tetrahedron) to the centre C of the plane (flat) circle & join any of the points A, B & D of tangency of the plane circle say point A (i.e. mid-point of one of the edges of regular tetrahedron) to the centre O of the spherical surface (i.e. the centre of regular tetrahedron). Thus, we obtain a right ΔOCA (as shown in the figure 2).

Applying Pythagoras Theorem in right ΔOCA as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = \left(\frac{r}{\sqrt{2}}\right)^2 + r^2 = \frac{3}{2}r^2 \qquad \Rightarrow r^2 = \frac{2}{3}R^2$$

$$\therefore \text{ Flat radius of each circle, } \boldsymbol{r = R\sqrt{\frac{2}{3}} \approx 0.81649658\,R} \qquad \ldots\ldots(1)$$

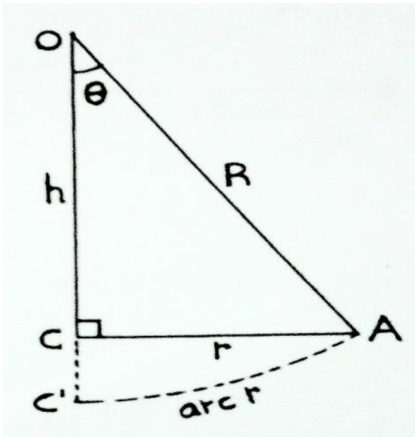

Figure 2: One of 4 identical circles has its centre C on the flat face of tetrahedron & its flat radius r while its centre C' & a radius $\mathrm{arc}\, r$ as a great circle arc on the spherical surface.

2.1. Arc radius ($\mathrm{arc}\, r$) of each of 4 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA (Fig. 2),

$$\sin\theta = \frac{CA}{OA} = \frac{r}{R} = \sqrt{\frac{2}{3}} \;\Rightarrow\; \theta = \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

$$\Rightarrow \theta = \frac{\mathrm{arc}\, AC'}{R} \Rightarrow \text{ arc radius } = \mathrm{acr}\, AC' = R\theta = R\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

$$\therefore \text{ Radius of each circle as a great circle arc, } \boldsymbol{\mathrm{arc}\, r = R\sin^{-1}\left(\sqrt{\frac{2}{3}}\right) \approx 0.955316618\,R} \qquad \ldots\ldots(2)$$

2.2. Total area (A_s) covered by the four identical circles on the spherical surface: In order to calculate the area covered by each of the four identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (see the figure 2 above) using formula of solid angle of a right cone concentric with a sphere [4] as follows

Solid angle subtended by each circle at the centre of the spherical surface, $\omega = 2\pi(1 - cos\theta)$

$$\Rightarrow \ \omega = 2\pi\left(1 - \frac{1}{\sqrt{3}}\right) \qquad \left(\text{setting the value of} \cos\theta = \frac{1}{\sqrt{3}} \text{ from the figure 2 above}\right)$$

Hence, the total surface area covered by all 4 identical circles on the sphere, is given as

$$A_s = (\text{no. of circles}) \times (\text{solid angle } (\omega) \text{ subtended by each circle}) \times (R^2) = 4\left(2\pi\left(1 - \frac{1}{\sqrt{3}}\right)\right)R^2$$

$$= 8\pi R^2\left(1 - \frac{1}{\sqrt{3}}\right)$$

$$\therefore \ \boldsymbol{A_s = 8\pi R^2\left(1 - \frac{1}{\sqrt{3}}\right) \approx 10.62234631\, R^2} \qquad \dots \dots (3)$$

Hence, the percentage of total surface area covered by all 4 identical circles on the sphere, is given as

$$\%\text{ of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100$$

$$= \frac{8\pi R^2\left(1 - \frac{1}{\sqrt{3}}\right)}{4\pi R^2} \times 100 = 200\left(1 - \frac{1}{\sqrt{3}}\right)\ \%$$

$$\therefore \ \%\text{ of total surface area covered} = \boldsymbol{200\left(1 - \frac{1}{\sqrt{3}}\right)}\ \% \approx \boldsymbol{84.53}\ \% \qquad \dots \dots (4)$$

Key point-1: 4 identical circles, touching one another at 6 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately $84.53\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $15.47\ \%$ of total surface area is left uncovered by the circles.

3. Six identical circles, each having a flat radius r, touching one another at 12 different points (i.e. each one touches four other circles) on the whole spherical surface with a radius R (Analogous to a regular hexahedron/cube)

In this case**,** let's assume that each of 6 identical circles, with a flat radius r, is inscribed by each of 6 congruent square faces of a cube (regular hexahedron) with an edge length a such that cube is concentric with the spherical surface having the centre O & a radius R. In this case, all 12 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 12 edges of a regular hexahedron i.e. cube. Now, consider one of 6 identical circles with the centre C on the flat face & a flat radius r, touching four other circles at the points A, B, D & E (lying on the spherical surface as well as on the edges of the cube) and is inscribed by a square face of the cube with an edge length a (see the figure 3 below).

We know that the inscribed radius r & the edge length a of a regular n-gon are related as follows

$$r = \frac{a}{2}\cot\frac{\pi}{n} \quad \Rightarrow a = 2r\tan\frac{\pi}{n}$$

Hence, by setting $n = 4$ for a square face, we get

$$a = 2r\tan\frac{\pi}{4} \ \ \Rightarrow \ \boldsymbol{a = 2r}$$

Now, we have

$\text{h} = \text{normal distance of each face from the centre of the cube with edge length } a$

$= \text{normal distance of plane (flat) circle with centre C from the centre O of the spherical surface}$

$$\Rightarrow\ h = OC = \frac{a}{2} \qquad \text{(from the table of platonic solids)}$$

$$= \frac{(2r)}{2} = r \qquad \text{(setting the value of edge length } a\text{)}$$

$$\Rightarrow\ \boldsymbol{h = OC = r}$$

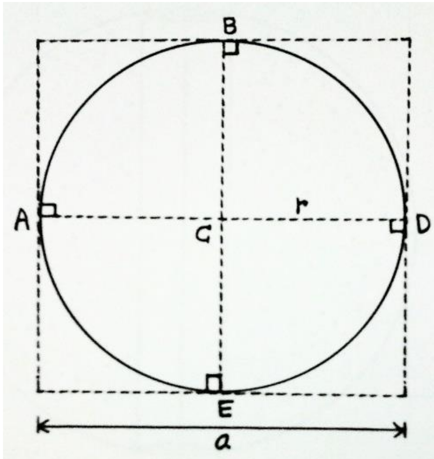

Figure 3: One of 6 identical circles, with the centre C on the flat face & a flat radius r, is touching four other circles at the points A, B, D & E (lying on the spherical surface as well as on the edges of face) and is inscribed by a square face of a cube (concentric with the spherical surface) with an edge length a.

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of the cube) to the centre C of the plane (flat) circle & join any of the points A, B, D & E of tangency of the plane circle say point A (i.e. mid-point of one of the edges of cube) to the centre O of the spherical surface (i.e. the centre of cube). Thus, we obtain a right ΔOCA (as shown in the figure 4 below).

Applying Pythagoras Theorem in right ΔOCA (Fig. 4) as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = (r)^2 + r^2 = 2r^2 \qquad \Rightarrow r^2 = \frac{1}{2}R^2$$

$$\therefore\ \text{Flat radius of each circle, } \boldsymbol{r = \frac{R}{\sqrt{2}} \approx 0.707106781\,R} \qquad \ldots\ldots(5)$$

3.1. Arc radius ($\mathbf{arc}\,\boldsymbol{r}$) of each of 6 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA (Fig. 4),

$$\sin\theta = \frac{CA}{OA} = \frac{r}{R} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\Rightarrow\ \theta = \frac{\text{arc } AC'}{R} \Rightarrow \text{arc radius } = \text{acr } AC' = R\theta = \frac{\pi}{4}R$$

$$\therefore\ \text{Radius of each circle as a great circle arc, } \boldsymbol{\mathbf{arc}\,r = \frac{\pi R}{4} \approx 0.785398163\,R} \quad \ldots(6)$$

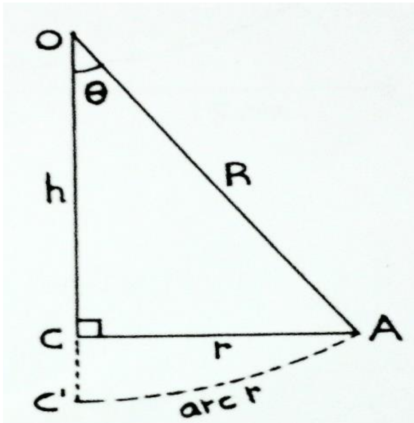

Figure 4: One of 6 identical circles has its centre C on the flat face of a cube & its flat radius r while its centre C' & a radius $\mathrm{arc}\,r$ as a great circle arc on the spherical surface.

3.2. Total area (A_s) covered by 6 identical circles on the spherical surface: In order to calculate the area covered by each of the six identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (See the figure 4 above) using formula of solid angle of a right cone concentric with a sphere as follows

$$\text{Solid angle subtended by each circle at the centre of the spherical surface,} \ \omega = 2\pi(1 - \cos\theta)$$

$$\Rightarrow\ \omega = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right) \qquad \left(\text{setting the value of } \cos\theta = \frac{1}{\sqrt{2}} \text{ from the figure 4 above}\right)$$

Hence, the total surface area covered by all 6 identical circles on the sphere, is given as

$$A_s = (\text{no. of circles}) \times (\text{solid angle }(\omega)\text{ subtended by each circle}) \times (R^2) = 6\left(2\pi\left(1-\frac{1}{\sqrt{2}}\right)\right)R^2$$

$$= 12\pi R^2\left(1-\frac{1}{\sqrt{2}}\right)$$

$$\therefore\ \boldsymbol{A_s = 12\pi R^2\left(1-\frac{1}{\sqrt{2}}\right) \approx 11.04181421\,R^2} \qquad \dots\dots(7)$$

Hence, the percentage of total surface area covered by all 6 identical circles on the sphere, is given as

$$\%\text{ of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100$$

$$= \frac{12\pi R^2\left(1-\frac{1}{\sqrt{2}}\right)}{4\pi R^2} \times 100 = 300\left(1-\frac{1}{\sqrt{2}}\right)\%$$

$$\therefore\ \%\text{ of total surface area covered} = \boldsymbol{300\left(1-\frac{1}{\sqrt{2}}\right)}\% \approx \boldsymbol{87.87}\,\% \qquad \dots\dots(8)$$

Key point 2: 6 identical circles, touching one another at 12 different points (i.e. each one touches four other circles) on a whole spherical surface, always cover up approximately 87.87 % of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 12.13 % of total surface area is left uncovered by the circles.

4. Eight identical circles, each having a flat radius r, touching one another at 12 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius R (Analogous to a regular octahedron)

In this case**,** let's assume that each of 8 identical circles, with a flat radius r, is inscribed by each of 8 congruent equilateral triangular faces of a regular octahedron with an edge length a such that regular octahedron is concentric with the spherical surface having the centre O & a radius R. In this case, all 12 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 12 edges of a regular octahedron. Now, consider one of 8 identical circles with the centre C on the flat face & a flat radius r, touching three other circles at the points A, B, and D (lying on the spherical surface as well as on the edges of regular octahedron) and is inscribed by an equilateral triangular face of regular octahedron with an edge length a (as shown in the figure 5).

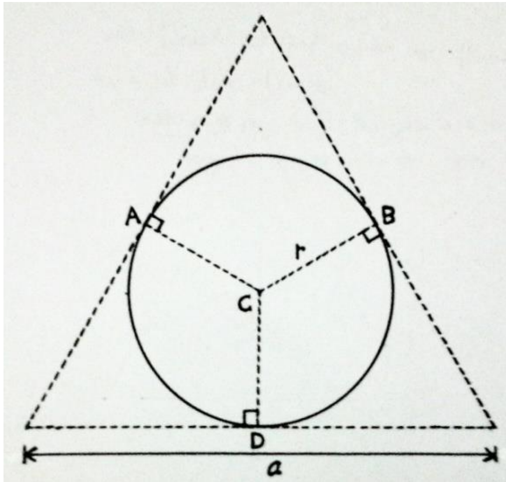

Figure 5: One of 8 identical circles, with the centre C on the flat face & a flat radius r, is touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of face) and is inscribed by an equilateral triangular face of a regular octahedron (concentric with the spherical surface) with an edge length a.

We know that the inscribed radius r & the edge length a of a regular n-gon are related as follows

$$r = \frac{a}{2}\cot\frac{\pi}{n} \quad \Rightarrow\ a = 2r\tan\frac{\pi}{n}$$

Hence, by setting $n = 3$ for an equilateral triangular face, we get

$$a = 2r\tan\frac{\pi}{3} \ \Rightarrow\ \boldsymbol{a = 2r\sqrt{3}}$$

Now, we have

$h =$ normal distance of each face from the centre of the regular octahedron with edge length a

$=$ normal distance of plane (flat) circle with centre C from the centre O of the spherical surface

$$\Rightarrow\ h = OC = \frac{a}{\sqrt{6}} \qquad \text{(from the table of platonic solids)}$$

$$= \frac{(2r\sqrt{3})}{\sqrt{6}} = r\sqrt{2} \qquad \text{(setting the value of edge length } a\text{)}$$

$$\Rightarrow\ \boldsymbol{h = OC = r\sqrt{2}}$$

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of regular octahedron) to the centre C of the plane (flat) circle & join any of the points A, B & D of tangency of the plane circle say point A (i.e. mid-point of one of the edges of regular octahedron) to the centre O of the spherical surface (i.e. the centre of regular octahedron). Thus, we obtain a right ΔOCA (as shown in the figure 6).

Applying Pythagoras Theorem in right ΔOCA (Fig. 6) as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = \left(r\sqrt{2}\right)^2 + r^2 = 3r^2 \qquad \Rightarrow r^2 = \frac{1}{3}R^2$$

$$\therefore\ \text{Flat radius of each circle,}\ \boldsymbol{r = \frac{R}{\sqrt{3}} \approx 0.577350269\, R} \qquad \ldots\ldots(9)$$

4.1. Arc radius ($\mathbf{arc}\, r$) of each of 8 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA (Fig. 6),

$$\sin\theta = \frac{CA}{OA} = \frac{r}{R} = \frac{1}{\sqrt{3}} \ \Rightarrow\ \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow\ \theta = \frac{\text{arc}\, AC'}{R} \Rightarrow \text{arc radius } = \text{acr}\, AC' = R\theta = R\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

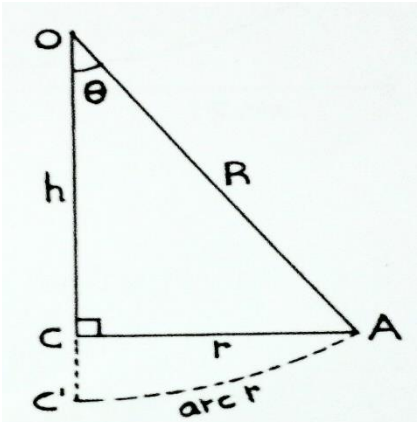

Figure 6: One of 8 identical circles has its centre C on the flat face of octahedron & its flat radius r while its centre C' & a radius $\mathbf{arc}\, r$ as a great circle arc on the spherical surface.

$$\therefore\ \text{Radius of each circle as a great circle arc,}\ \boldsymbol{\text{arc}\, r = R\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.615479708\, R} \qquad \ldots\ldots(10)$$

4.2. Total area (A_s) covered by 8 identical circles on the spherical surface: In order to calculate the area covered by each of the eight identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (see the figure 6 above) using formula of solid angle of a right cone concentric with a sphere as follows

$$\text{Solid angle subtended by each circle at the centre of the spherical surface,}\ \omega = 2\pi(1 - cos\theta)$$

$$\Rightarrow\ \omega = 2\pi\left(1 - \sqrt{\frac{2}{3}}\right) \qquad \left(\text{setting the value of}\cos\theta = \sqrt{\frac{2}{3}}\ \text{from the figure 6 above}\right)$$

Hence, the total surface area covered by all 8 identical circles on the sphere, is given as

$$A_s = (\text{no. of circles}) \times (\text{solid angle } (\omega) \text{ subtended by each circle}) \times (R^2) = 8\left(2\pi\left(1-\sqrt{\frac{2}{3}}\right)\right)R^2$$

$$= 16\pi R^2\left(1-\sqrt{\frac{2}{3}}\right)$$

$$\therefore\ \boldsymbol{A_s = 16\pi R^2\left(1-\sqrt{\frac{2}{3}}\right) \approx 9.223887892\ R^2} \qquad \dots\dots(11)$$

Hence, the percentage of total surface area covered by all 8 identical circles on the sphere, is given as

$$\%\text{ of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100$$

$$= \frac{16\pi R^2\left(1-\sqrt{\frac{2}{3}}\right)}{4\pi R^2} \times 100 = 400\left(1-\sqrt{\frac{2}{3}}\right)\%$$

$$\therefore\ \%\text{ of total surface area covered} = \boldsymbol{400\left(1-\sqrt{\frac{2}{3}}\right)}\% \approx \boldsymbol{73.4}\ \% \qquad \dots\dots(12)$$

Key point-3: 8 identical circles, touching one another at 12 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately 73.4 % of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 26.6 % of total surface area is left uncovered by the circles.

5. Twelve identical circles, each having a flat radius r, touching one another at 30 different points (i.e. each one touches five other circles) on the whole spherical surface with a radius R (Analogous to a regular dodecahedron)

In this case, let's assume that each of 12 identical circles, with a flat radius r, is inscribed by each of 12 congruent regular pentagonal faces of a regular dodecahedron with an edge length a such that regular dodecahedron is concentric with the spherical surface having the centre O & a radius R. In this case, all 30 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 30 edges of a regular dodecahedron. Now, consider one of the 12 identical circles with the centre C on the flat face & a flat radius r, touching five other circles at the points A, B, D, E, and F (lying on the spherical surface as well as on the edges of the dodecahedron) and is inscribed by a regular pentagonal face of the dodecahedron with an edge length a (see the figure 7 below).

We know that the inscribed radius r & the edge length a of a regular n-gon are related as follows

$$r = \frac{a}{2}\cot\frac{\pi}{n} \quad \Rightarrow\ a = 2r\tan\frac{\pi}{n}$$

Hence, by setting $n = 4$ for a square face, we get

$$a = 2r\tan\frac{\pi}{5} \quad \Rightarrow\ \boldsymbol{a = 2r\sqrt{5-2\sqrt{5}}}$$

Now, we have

$h =$ normal distance of each face from the centre of the dodecahedron with edge length a

$=$ normal distance of plane (flat) circle with centre C from the centre O of the spherical surface

$$\Rightarrow\ h = OC = \frac{(3+\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}} \qquad \text{(from the table of platonic solids)}$$

$$= \frac{(3+\sqrt{5})\left(2r\sqrt{5-2\sqrt{5}}\right)}{2\sqrt{10-2\sqrt{5}}} \qquad \text{(setting the value of edge length } a\text{)}$$

$$= r\sqrt{\frac{(5-2\sqrt{5})(3+\sqrt{5})^2}{10-2\sqrt{5}}} = r\sqrt{\frac{(5-2\sqrt{5})(7+3\sqrt{5})}{5-\sqrt{5}}} = r\sqrt{\frac{(5+\sqrt{5})(5+\sqrt{5})}{25-5}}$$

$$= \frac{(5+\sqrt{5})r}{2\sqrt{5}} = \frac{(1+\sqrt{5})r}{2}$$

$$\Rightarrow\ \boldsymbol{h = OC = \frac{(1+\sqrt{5})r}{2}}$$

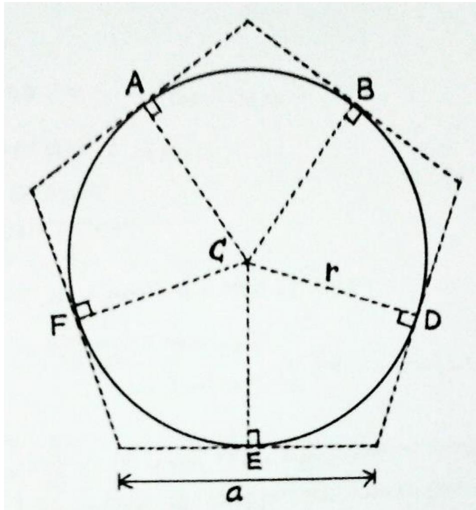

Figure 7: One of 12 identical circles, with the centre C on the flat face & a flat radius r, is touching five other circles at the points A, B, D, E & F (lying on the spherical surface as well as on the edges of face) and is inscribed by a regular pentagonal face of a dodecahedron (concentric with the spherical surface) with an edge length a.

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of the regular dodecahedron) to the centre C of the plane (flat) circle & join any of the points A, B, D, E & F of tangency of the plane circle say point A (i.e. mid-point of one of the edges of dodecahedron) to the centre O of the spherical surface (i.e. the centre of dodecahedron). Thus, we obtain a right ΔOCA (as shown in the figure 8 below).

Applying Pythagoras Theorem in right ΔOCA (Fig. 8) as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = \left(\frac{(1+\sqrt{5})r}{2}\right)^2 + r^2 = r^2\left(\frac{6+2\sqrt{5}+4}{4}\right) = r^2\left(\frac{5+\sqrt{5}}{2}\right)$$

$$\Rightarrow r^2 = R^2\left(\frac{2}{5+\sqrt{5}}\right) = R^2\left(\frac{2(5-\sqrt{5})}{(5+\sqrt{5})(5-\sqrt{5})}\right) = R^2\left(\frac{2(5-\sqrt{5})}{20}\right) = R^2\frac{(5-\sqrt{5})}{10}$$

$$\therefore \text{ Flat radius of each circle, } \boldsymbol{r = R\sqrt{\frac{5-\sqrt{5}}{10}} \approx 0.525731112\,R} \qquad \ldots\ldots(13)$$

5.1. Arc radius ($\mathbf{arc}\,r$) of each of 12 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA (Fig. 8),

$$\sin\theta = \frac{CA}{OA} = \frac{r}{R} = \sqrt{\frac{5-\sqrt{5}}{10}} \ \Rightarrow\ \theta = \sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)$$

$$\Rightarrow\ \theta = \frac{\text{arc } AC'}{R} \Rightarrow \text{ arc radius } = \text{acr } AC' = R\theta = R\sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)$$

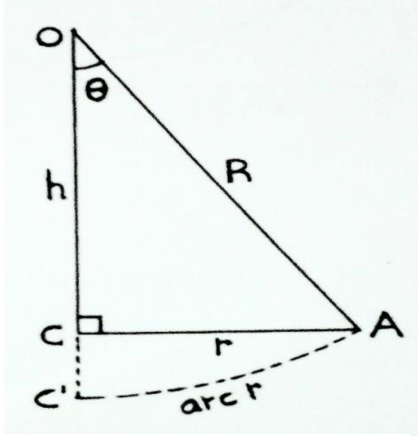

Figure 8: One of 12 identical circles has its centre C on the flat face of a regular dodecahedron & its flat radius r while its centre C' & a radius arc r as a great circle arc on the spherical surface.

$$\therefore\ \text{Radius of each circle as a great circle arc,}\ \ \mathbf{arc}\ \boldsymbol{r} = \boldsymbol{R}\ \mathbf{sin}^{-1}\left(\sqrt{\frac{\mathbf{5}-\sqrt{\mathbf{5}}}{\mathbf{10}}}\right)$$

$$\approx \mathbf{0.553574358}\ \boldsymbol{R} \qquad \ldots\ldots (14)$$

5.2. Total area (A_s) covered by 12 identical circles on the spherical surface: In order to calculate the area covered by each of 12 identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (see the figure 8) using formula of solid angle of a right cone concentric with a sphere as follows

$$\text{Solid angle subtended by each circle at the centre of the spherical surface,}$$
$$\omega = 2\pi(1 - cos\theta)$$

$$\Rightarrow\ \omega = 2\pi\left(1 - \sqrt{\frac{5+\sqrt{5}}{10}}\right) \quad \left(\text{setting the value of} \cos\theta = \sqrt{\frac{5+\sqrt{5}}{10}} \text{ from Fig. 8}\right)$$

Hence, the total surface area covered by all 12 identical circles on the sphere, is given as

$$A_s = (\text{no. of circles}) \times (\text{solid angle } (\omega) \text{ subtended by each circle}) \times (R^2)$$

$$= 12\left(2\pi\left(1 - \sqrt{\frac{5+\sqrt{5}}{10}}\right)\right)R^2 = 24\pi R^2\left(1 - \sqrt{\frac{5+\sqrt{5}}{10}}\right)$$

$$\therefore\ \boldsymbol{A_s} = \mathbf{24}\boldsymbol{\pi R^2}\left(\mathbf{1} - \sqrt{\frac{\mathbf{5}+\sqrt{\mathbf{5}}}{\mathbf{10}}}\right) \approx \mathbf{11.26066376}\ \boldsymbol{R^2} \qquad \ldots\ldots (15)$$

Hence, the percentage of total surface area covered by all 12 identical circles on the sphere, is given as

$$\%\ \text{of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100$$

$$= \frac{24\pi R^2\left(1 - \sqrt{\frac{5+\sqrt{5}}{10}}\right)}{4\pi R^2} \times 100 = 600\left(1 - \sqrt{\frac{5+\sqrt{5}}{10}}\right)\%$$

$$\therefore\ \%\ \text{of total surface area covered} = \mathbf{600}\left(\mathbf{1} - \sqrt{\frac{\mathbf{5}+\sqrt{\mathbf{5}}}{\mathbf{10}}}\right)\% \approx \mathbf{89.61}\ \% \qquad \ldots\ldots (16)$$

Key point-4: 12 identical circles, touching one another at 30 different points (i.e. each one touches five other circles) on a whole spherical surface, always cover up approximately 89.61 % of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 10.39 % of total surface area is left uncovered by the circles.

6. Twenty identical circles, each having a flat radius r, touching one another at 30 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius R (Analogous to a regular icosahedron)

In this case**,** let's assume that each of 20 identical circles, with a flat radius r, is inscribed by each of 20 congruent equilateral triangular faces of a regular icosahedron with an edge length a such that regular icosahedron is concentric with the spherical surface having the centre O & a radius R. In this case, all 30 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 30 edges of a regular icosahedron. Now, consider one of the 12 identical circles with the centre C on the flat face & a flat radius r, touching three other circles at the points A, B, and D (lying on the spherical surface as well as on the edges of regular icosahedron) and is inscribed by an equilateral triangular face of regular icosahedron with an edge length a (as shown in the figure 9).

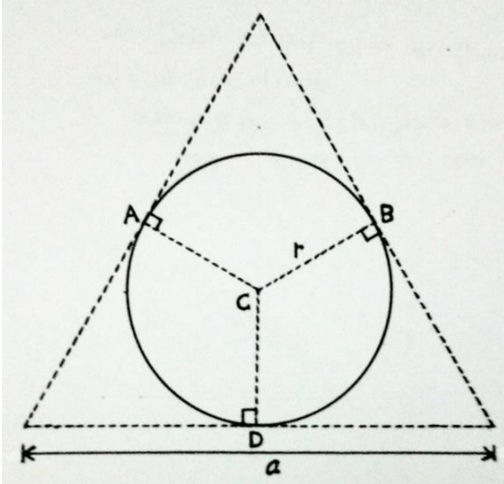

Figure 9: One of 20 identical circles, with the centre C on the flat face & a flat radius r, is touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of face) and is inscribed by an equilateral triangular face of a regular icosahedron (concentric with the spherical surface) with an edge length a.

We know that the inscribed radius r & the edge length a of a regular n-gon are related as follows

$$r = \frac{a}{2}\cot\frac{\pi}{n} \quad \Rightarrow a = 2r\tan\frac{\pi}{n}$$

Hence, by setting $n = 3$ for an equilateral triangular face, we get

$$a = 2r\tan\frac{\pi}{3} \quad \Rightarrow \boldsymbol{a = 2r\sqrt{3}}$$

Now, we have

$$h = \text{normal distance of each face from the centre of the regular icosahedron with edge length } a$$

$$= \text{normal distance of plane (flat) circle with centre C from the centre O of the spherical surface}$$

$$\Rightarrow h = OC = \frac{a(3+\sqrt{5})}{4\sqrt{3}} \qquad \text{(from the table of platonic solids)}$$

$$= \frac{(3+\sqrt{5})(2r\sqrt{3})}{4\sqrt{3}} = \frac{(3+\sqrt{5})r}{2} \qquad \text{(setting the value of edge length } a\text{)}$$

$$\Rightarrow \boldsymbol{h = OC = \frac{(3+\sqrt{5})r}{2}}$$

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of regular icosahedron) to the centre C of the plane (flat) circle & join any of the points A, B & D of tangency of the plane circle say point A (i.e. mid-point of one of the edges of regular icosahedron) to the centre O of the spherical surface (i.e. the centre of regular icosahedron). Thus, we obtain a right ΔOCA (as shown in the figure 10 below).

Applying Pythagoras Theorem in right ΔOCA (Fig. 10) as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = \left(\frac{(3+\sqrt{5})r}{2}\right)^2 + r^2 = r^2\left(\frac{14+2\sqrt{5}+4}{4}\right) = r^2\left(\frac{9+\sqrt{5}}{2}\right)$$

$$\Rightarrow r^2 = R^2\left(\frac{2}{9+\sqrt{5}}\right) = R^2\left(\frac{2(9-\sqrt{5})}{(9+\sqrt{5})(9-\sqrt{5})}\right) = R^2\left(\frac{2(9-\sqrt{5})}{(81-5)}\right)$$

$$= R^2\left(\frac{9-\sqrt{5}}{38}\right) \Rightarrow r = R\sqrt{\frac{9-\sqrt{5}}{38}}$$

$$\therefore \text{ Flat radius of each circle, } \boldsymbol{r = R\sqrt{\frac{9-\sqrt{5}}{38}} \approx 0.421898342\, R} \quad \ldots\ldots(17)$$

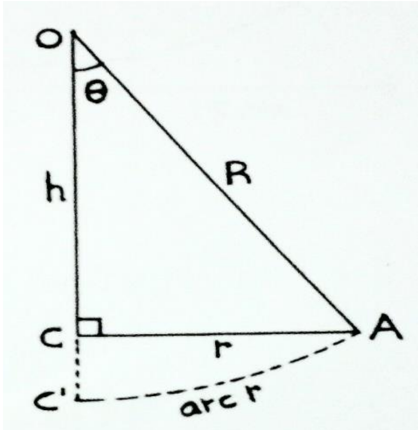

Figure 10: One of 20 identical circles has its centre C on the flat face of icosahedron & its flat radius r while its centre C' & a radius $\text{arc}\, r$ as a great circle arc on the spherical surface.

6.1. Arc radius ($\text{arc}\, r$) of each of 20 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA (Fig. 10),

$$\sin\theta = \frac{CA}{OA} = \frac{r}{R} = \sqrt{\frac{9-\sqrt{5}}{38}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\sqrt{\frac{9-\sqrt{5}}{38}}\right)$$

$$\Rightarrow \theta = \frac{\text{arc } AC'}{R} \Rightarrow \text{ arc radius } = \text{acr } AC' = R\theta = R\sin^{-1}\left(\sqrt{\frac{9-\sqrt{5}}{38}}\right)$$

$$\therefore \text{ Radius of each circle as a great circle arc, } \boldsymbol{\text{arc}\, r = R\sin^{-1}\left(\sqrt{\frac{9-\sqrt{5}}{38}}\right) \approx 0.435538116\, R} \quad \ldots\ldots(18)$$

6.1. Total area (A_s) covered by 20 identical circles on the spherical surface: In order to calculate the area covered by each of 20 identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (see the figure 10 above) using formula of solid angle of a right cone concentric with a sphere as follows

$$\text{Solid angle subtended by each circle at the centre of the spherical surface}, \omega = 2\pi(1 - cos\theta)$$

$$\Rightarrow \omega = 2\pi\left(1 - \sqrt{\frac{29+\sqrt{5}}{38}}\right) \qquad \left(\text{setting the value of} \cos\theta = \sqrt{\frac{29+\sqrt{5}}{38}} \text{ from the figure 10 above}\right)$$

Hence, the total surface area covered by all 20 identical circles on the sphere, is given as

$$A_s = (\text{no. of circles}) \times (\text{solid angle } (\omega) \text{ subtended by each circle}) \times (R^2)$$

$$= 20\left(2\pi\left(1 - \sqrt{\frac{29+\sqrt{5}}{38}}\right)\right)R^2 = 40\pi R^2\left(1 - \sqrt{\frac{29+\sqrt{5}}{38}}\right)$$

$$\therefore \boldsymbol{A_s = 40\pi R^2\left(1 - \sqrt{\frac{29+\sqrt{5}}{38}}\right) \approx 11.73156864\, R^2} \quad \ldots\ldots(19)$$

Hence, the percentage of total surface area covered by all 20 identical circles on the sphere, is given as

$$\% \text{ of total surface area covered} = \frac{\text{total surface area covered by all the circles}}{\text{total surface area of the sphere}} \times 100$$

$$= \frac{40\pi R^2\left(1-\sqrt{\frac{29+\sqrt{5}}{38}}\right)}{4\pi R^2} \times 100 = 1000\left(1-\sqrt{\frac{29+\sqrt{5}}{38}}\right)\%$$

$$\therefore \ \% \text{ of total surface area covered} = \mathbf{1000}\left(\mathbf{1}-\sqrt{\frac{\mathbf{29}+\sqrt{\mathbf{5}}}{\mathbf{38}}}\right)\% \approx \mathbf{93.36}\ \% \qquad \ldots\ldots(20)$$

Key point-5: 20 identical circles, touching one another at 30 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately 93.36 % of the total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately 6.64 % of total surface area is left uncovered by the circles.

Thus, 8 identical circles, touching one another at 12 different points (each one touches three other circles), cover up the minimum approximately 73.4 % while 20 identical circles, touching one another at 30 different points (each one touches three other circles), cover up the maximum approximately 93.36 % of the total surface area of a sphere.

Conclusions: Consider a configuration in which a finite number of identical circles are arranged in mutual contact over the entire surface of a sphere of radius R. Using elementary geometric relations, all important parameters characterizing this arrangement can be systematically determined. These parameters include the flat (planar) radius and the arc radius of each individual circle, as well as the total surface area of the sphere covered by the circles. The resulting expressions and tabulated values, presented below, provide a convenient and practical reference for analyzing such circle arrangements on spherical surfaces. The developed formulation is particularly useful for geometric construction, visualization, and quantitative analysis of identical circular elements distributed over a sphere, and it offers a straightforward framework for applications involving spherical packing, surface coverage, and the geometric modeling of polyhedral and related structures.

Total no. of identical circles touching one another on a whole spherical surface with a radius R	**No. of circles touching each circle**	**Total no. of points of tangency**	**Flat radius of each circle (i.e. radius of each plane (flat) circle)**	**Arc radius of each circle (i.e. radius of each circle as a great circle arc on spherical surface)**	**Total surface area covered by all the circles**	**Percentage (%) of total surface area covered by all the circles**
4	3	6	$R\sqrt{\frac{2}{3}}$ $\approx 0.81649658\ R$	$R\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$ $\approx 0.955316618\ R$	$8\pi R^2\left(1-\frac{1}{\sqrt{3}}\right)$ $\approx 10.62234631\ R^2$	84.53 %
6	4	12	$\frac{R}{\sqrt{2}}$ $\approx 0.707106781\ R$	$\frac{\pi R}{4}$ $\approx 0.785398163\ R$	$12\pi R^2\left(1-\frac{1}{\sqrt{2}}\right)$ $\approx 11.04181421\ R^2$	87.87 %
8	3	12	$\frac{R}{\sqrt{3}}$ $\approx 0.577350269\ R$	$R\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $\approx 0.615479708\ R$	$16\pi R^2\left(1-\sqrt{\frac{2}{3}}\right)$ $\approx 9.223887892\ R^2$	73.4 %

12	5	30	$R\sqrt{\frac{5-\sqrt{5}}{10}}$ $\approx 0.525731112\ R$	$R\sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)$ $\approx 0.553574358\ R$	$24\pi R^2\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right)$ $\approx 11.26066376\ R^2$	89.61 %
20	3	30	$R\sqrt{\frac{9-\sqrt{5}}{38}}$ $\approx 0.421898342\ R$	$R\sin^{-1}\left(\sqrt{\frac{9-\sqrt{5}}{38}}\right)$ $\approx 0.435538116\ R$	$40\pi R^2\left(1-\sqrt{\frac{29+\sqrt{5}}{38}}\right)$ $\approx 11.73156864\ R^2$	93.36 %

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

M.M.M. University of Technology, Gorakhpur-273010 (UP) India *20 Aug 2015*

Email: rajpootharishchandra@gmail.com, Author's Home Page: https://notionpress.com/author/HarishChandraRajpoot

References

[1] Grünbaum B. Regular polyhedra—old and new. Aequationes mathematicae. 1977 Feb;16(1):1-20.

[2] Rajpoot HC. HCR's or H. Rajpoot's Formula for Regular Polyhedron. 2014.

[3] Rajpoot HC. HCR's theorem for dihedral angles in a regular n-gonal right pyramid. 2015 Jul. doi:10.13140/RG.2.2.16488.74242.

[4] Rajpoot CH. Advanced geometry. 1st ed. Chennai: Notion Press Media Pvt. Ltd.; 2013 Apr 3. ISBN: 978-93-83808-15-1.

