Solid angle subtended by a beam with rectangular profile given horizontal & vertical beam angles

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1. Introduction

In this paper, general analytical expressions are derived for calculating the solid angle subtended by a beam with a rectangular cross-sectional profile emitted from a uniformly radiating point source. The formulation considers two commonly used beam descriptions. In the first case, the solid angle is expressed in terms of the horizontal and vertical beam angles θ_H and θ_V respectively measured in two mutually orthogonal central planes (horizontal and vertical) passing through the point source. In the second case, the solid angle is determined using the lateral beam angles α & β defined between adjacent lateral beam rays passing through the vertices of the rectangular beam profile and originating from the point source, as illustrated in Fig. 1. The derivations are based on the standard analytical formula for the solid angle subtended by a rectangular planar surface [1,2,3]. Both beam representations are systematically analysed and discussed, and the relationship between the two sets of angular parameters is established. The proposed expressions provide a convenient and generalized framework for accurately evaluating the solid angle of rectangular beams, which is relevant to applications in optics, radiometry, illumination engineering, and beam characterization.

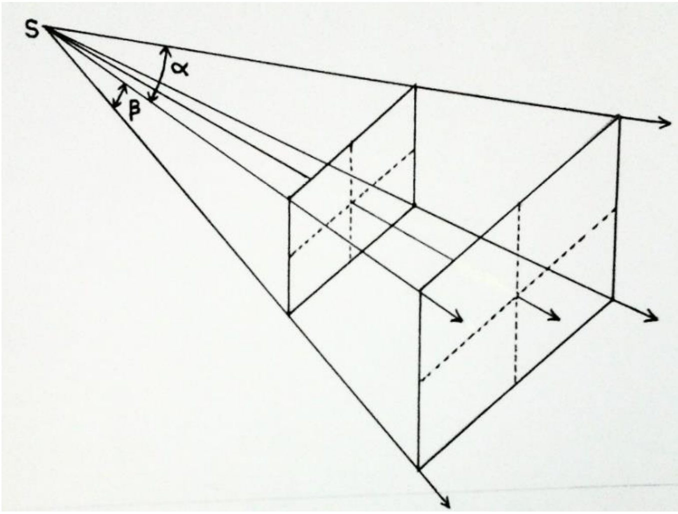

Figure 1: A beam with rectangular profile is emitted by a uniform point-source S. The angles θ_H & θ_V are the horizontal & vertical beam angles (i.e. angles subtended by the centralised dotted lines at the point-source S) and α & β are the lateral beam angles (i.e. angles between the lateral beam rays).

2. Solid angle subtended by the beam with rectangular profile given horizontal and vertical beam angles θ_H & θ_V

Let there be a beam with rectangular profile radiated from a uniform point-source S such that θ_H & θ_V are (plane) angles subtended by the beam in the centralised horizontal & vertical planes respectively passing through the point-source S. Now, consider an imaginary rectangular plane ABCD with centre O, length l & width b at a normal distance $OS = h$ from the uniform point-source S (as shown in the figure 2).

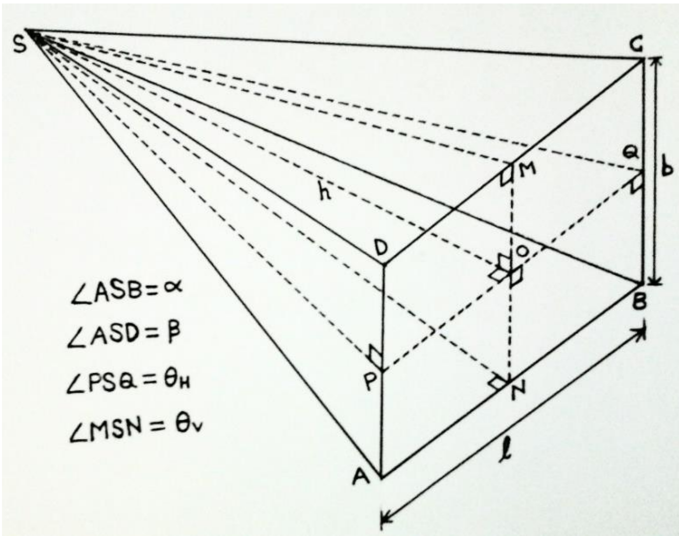

Figure 2: An imaginary rectangular plane ABCD, having centre O, length l & width b lying at a normal distance $OS = h$ from uniform point-source S, is representing the rectangular profile of the beam.

In right ΔSOP (Fig. 2),

$$\tan\angle OSP = \frac{OP}{OS} \quad \Rightarrow \tan\frac{\angle PSQ}{2} = \frac{OP}{OS}$$

$$\Rightarrow \tan\frac{\theta_H}{2} = \frac{\frac{l}{2}}{h} \quad \Rightarrow \quad \boldsymbol{l = 2h\tan\frac{\theta_H}{2}} \qquad \dots\dots(1)$$

Similarly, in right ΔSOM (Fig. 2),

$$\tan\angle OSM = \frac{OM}{OS} \quad \Rightarrow \tan\frac{\angle MSN}{2} = \frac{OM}{OS}$$

$$\Rightarrow \ \tan\frac{\theta_V}{2} = \frac{\frac{b}{2}}{h} \quad \Rightarrow \ \boldsymbol{b = 2h\tan\frac{\theta_V}{2}} \qquad \dots\dots\dots\dots(2)$$

Now, the solid angle (ω) subtended by the rectangular plane ABCD, having length l & width b lying at a normal distance $OS = h$, at the point-source S is given by the standard formula of rectangular plane [1,2] as follows

$$\boldsymbol{\omega = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2+4h^2)(b^2+4h^2)}}\right)} \qquad \dots\dots\dots\dots\dots(3)$$

Now, substituting the values of length l & width b from (1) & (2) into in the above expression (3), we obtain

$$\omega = 4\sin^{-1}\left(\frac{\left(2h\tan\frac{\theta_H}{2}\right)\left(2h\tan\frac{\theta_V}{2}\right)}{\sqrt{\left(\left(2h\tan\frac{\theta_H}{2}\right)^2+4h^2\right)\left(\left(2h\tan\frac{\theta_V}{2}\right)^2+4h^2\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{4h^2 tan\frac{\theta_H}{2} tan\frac{\theta_V}{2}}{4h^2\sqrt{\left(1+tan^2\frac{\theta_H}{2}\right)\left(\left(1+tan^2\frac{\theta_V}{2}\right)\right)}}\right) = 4\sin^{-1}\left(\frac{tan\frac{\theta_H}{2} tan\frac{\theta_V}{2}}{sec\frac{\theta_H}{2} sec\frac{\theta_V}{2}}\right) = 4\sin^{-1}\left(sin\frac{\theta_H}{2} sin\frac{\theta_V}{2}\right)$$

Hence, the solid angle (ω) subtended by a beam with rectangular profile at the uniform point-source given the horizontal & vertical (plane) beam angles θ_H & θ_V, is given as

$$\boldsymbol{\omega = 4\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)} \qquad (\forall \quad \theta_H, \theta_V \in [0,\pi]) \qquad \dots\dots\dots\dots\dots(4)$$

Note: If $\theta_H, \theta_V \geq \pi$ then apply the following formula to get solid angle subtended by the beam

$$\boldsymbol{\omega = 4\pi - 4\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)} \qquad (\forall \quad \theta_H, \theta_V \in [\pi, 2\pi]) \qquad \dots\dots\dots\dots\dots(5)$$

2.1. Surface area intercepted by the beam with a spherical surface

The surface area intercepted by the beam, having rectangular profile with horizontal & vertical beam angles θ_H & θ_V, with the spherical surface having a radius R & centre at the point-source is given as

$$A_S = (\text{solid angle}) \times (\text{radius of spherical surface})^2 = \omega R^2 = 4R^2\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)$$

$$\text{Spherical surface area intercepted by the beam,} \quad A_S = 4R^2\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)$$

Where, $\theta_H, \theta_V \in [0,\pi] \quad or \quad 0 \leq \theta_H, \theta_V \leq \pi$

$$\text{Spherical surface area intercepted by the beam,} \quad A_S = 4R^2\left(\pi - \sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)\right)$$

Where, $\theta_H, \theta_V \in [\pi, \ 2\pi] \quad or \quad \pi \leq \theta_H, \theta_V \leq 2\pi$

2.2. Beam with a circular profile equivalent to the beam with a rectangular profile emitted from the same uniform point-source

The beams with the circular & the rectangular profiles are said to be equivalent to each other if they subtend an equal solid angle at a given uniform point-source. Thus profile of the original beam is changed without any change in the total energy\luminous flux associated with the original beam. Now, let's consider a beam with circular profile subtending a cone angle θ_C & a solid angle ω at the uniform point-source S (as shown in the figure 3).

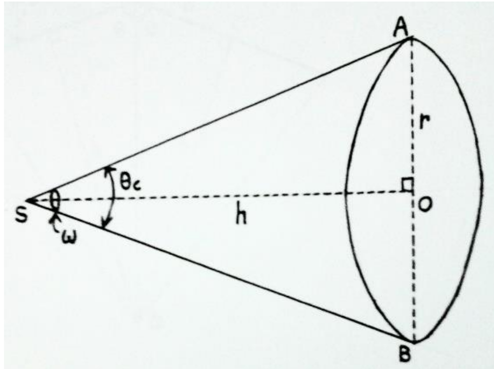

Figure 3: An imaginary circular plane, having centre O & a radius r lying at a normal distance $OS = h$ from the uniform point-source S, is representing the circular profile of the beam with a cone angle θ_C.

Now, consider an imaginary circular plane, with centre O & a radius r, at a normal distance h from the point source S.

In right ΔAOS (Fig. 3),

$$\cos\angle ASO = \frac{OS}{AS} \Rightarrow \cos\frac{\angle ASB}{2} = \frac{OS}{\sqrt{(OS)^2 + (OA)^2}}$$

$$\Rightarrow \frac{\boldsymbol{h}}{\sqrt{\boldsymbol{h}^2 + \boldsymbol{r}^2}} = \cos\frac{\boldsymbol{\theta_C}}{\mathbf{2}} \qquad \dots\dots\dots\dots\dots\dots(6)$$

Now, the solid angle (ω) subtended by the circular plane at the point-source S is by the standard formula of solid angle by a circular plane [3] as follows

$$\omega = 2\pi\left(1 - \frac{h}{\sqrt{h^2 + r^2}}\right) = 2\pi\left(1 - \cos\frac{\theta_C}{2}\right) \quad (\text{substituing the value from eq}(6))$$

Now, equating the values of solid angles subtended by the beams with circular & rectangular profiles at the given uniform point source S, as follows

$$2\pi\left(1 - \cos\frac{\theta_C}{2}\right) = 4\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)$$

$$\Rightarrow \cos\frac{\theta_C}{2} = 1 - \frac{2}{\pi}\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right) \Rightarrow \theta_C = 2\cos^{-1}\left\{\frac{\pi - 2\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)}{\pi}\right\}$$

Thus for a given uniform point-source, a beam having rectangular profile with horizontal & vertical beam angles θ_H & θ_V, is equivalent to a beam having circular profile with a cone angle θ_C given as

$$\boldsymbol{\theta_C} = \mathbf{2}\cos^{-1}\left\{\frac{\boldsymbol{\pi} - \mathbf{2}\sin^{-1}\left(\sin\frac{\boldsymbol{\theta_H}}{\mathbf{2}}\sin\frac{\boldsymbol{\theta_V}}{\mathbf{2}}\right)}{\boldsymbol{\pi}}\right\} \qquad (\forall \quad \boldsymbol{\theta_H}, \boldsymbol{\theta_V} \in [\mathbf{0}, \boldsymbol{\pi}]) \qquad \dots\dots\dots\dots(7)$$

Note: If $\pi \le \theta_H, \theta_V \le 2\pi$ then we have

$$2\pi\left(1 - \cos\frac{\theta_C}{2}\right) = 4\pi - 4\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)$$

$$\Rightarrow \cos\frac{\theta_C}{2} = 2 - \frac{2}{\pi}\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right) \Rightarrow \theta_C = 2\cos^{-1}\left\{\frac{2\pi - 2\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)}{\pi}\right\}$$

$$\boldsymbol{\theta_C = 2\cos^{-1}\left\{\frac{2\pi - 2\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)}{\pi}\right\}} \qquad (\forall \quad \theta_H, \theta_V \in [\pi, 2\pi]) \qquad \dots\dots\dots(8)$$

2. Solid angle subtended by the beam with rectangular profile given lateral beam angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ Let there be a beam with rectangular profile emitted from a uniform point-source S such that **$\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ are the lateral beam angles** measured between the adjacent lateral beam rays (passing through the vertices of rectangular profile) originated from the point-source S. Now, consider an imaginary rectangular plane ABCD with centre O, length l & width b at a normal distance $OS = h$ from the uniform point-source S (as shown in the figure 2 above).

In right ΔANS (see fig-2 above)

$$\tan\angle ASN = \frac{AN}{SN} \;\Rightarrow\; \tan\frac{\angle ASB}{2} = \frac{AN}{\sqrt{(OS)^2 + (ON)^2}} \quad \text{(setting the value of SN from right ΔNOS)}$$

$$\Rightarrow\; tan\frac{\alpha}{2} = \frac{\frac{l}{2}}{\sqrt{h^2 + \left(\frac{b}{2}\right)^2}} \;\Rightarrow\; \boldsymbol{\sqrt{b^2 + 4h^2} = l\cot\frac{\alpha}{2}} \qquad \dots\dots\dots(9)$$

Similarly, in right ΔDPS (see fig-2 above)

$$\tan\angle DSP = \frac{DP}{SP} \;\Rightarrow\; \tan\frac{\angle ASD}{2} = \frac{DP}{\sqrt{(OS)^2 + (OP)^2}} \quad \text{(setting the value of SP from right ΔPOS)}$$

$$\Rightarrow\; \tan\frac{\beta}{2} = \frac{\frac{b}{2}}{\sqrt{h^2 + \left(\frac{l}{2}\right)^2}} \;\Rightarrow\; \boldsymbol{\sqrt{l^2 + 4h^2} = b\cot\frac{\beta}{2}} \qquad \dots\dots\dots(10)$$

Now, the solid angle (ω) subtended by the rectangular plane ABCD, having length l & width b lying at a normal distance $OS = h$, at the point-source S is given by the standard formula of rectangular plane. Eq.(3) as follows

$$\omega = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2 + 4h^2)(b^2 + 4h^2)}}\right)$$

Now, substituting the values of $\sqrt{l^2 + 4h^2}$ & $\sqrt{b^2 + 4h^2}$ from (9) and (10) in the above expression, we get

$$\omega = 4\sin^{-1}\left(\frac{lb}{\left(l\cot\frac{\alpha}{2}\right)\left(b\cot\frac{\beta}{2}\right)}\right) = 4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

Hence, the solid angle (ω) subtended by a beam with rectangular profile at the uniform point-source given the lateral beam angles α & β, is given as

$$\boldsymbol{\omega = 4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)} \qquad (\forall \quad \alpha, \beta \in [0, \pi)) \qquad \dots\dots\dots(11)$$

2.1. Surface area intercepted by the beam with a spherical surface: The surface area intercepted by the beam, having rectangular profile with lateral beam angles α & β, with the spherical surface having a radius R & centre at the point-source is given as

$$A_S = (\text{solid angle}) \times (\text{radius of spherical surface})^2 = \omega R^2 = 4R^2\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

$$\therefore \text{ Spherical surface area intercepted by the beam, } \boldsymbol{A_S = 4R^2 \sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)} \quad \dots\dots\dots\dots (12)$$

2.2. Beam with a circular profile equivalent to the beam with a rectangular profile emitted from the same uniform point-source: Let's consider a beam with circular profile subtending a cone angle θ_C & a solid angle ω at the uniform point-source S (as shown in the figure 3 above).

Now, equating the values of solid angles subtended by the beams with circular & rectangular profiles at the given uniform point source S as follows

$$2\pi\left(1-\cos\frac{\theta_C}{2}\right) = 4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

$$\Rightarrow \cos\frac{\theta_C}{2} = 1 - \frac{2}{\pi}\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right) \Rightarrow \theta_C = 2\cos^{-1}\left\{\frac{\pi - 2\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}{\pi}\right\}$$

Thus, for a given uniform point-source, a beam having rectangular profile with the lateral beam angles α & β, is equivalent to a beam having circular profile with cone angle θ_C given as

$$\boldsymbol{\theta_C = 2\cos^{-1}\left\{\frac{\pi - 2\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}{\pi}\right\}} \quad \dots\dots\dots\dots (13)$$

Conclusion: For the given values of horizontal and vertical beam angles θ_H & θ_V or the lateral beam angles α & β, we can easily calculate important parameters such as solid angle subtended by the beam at the point-source, total area intercepted by the beam with a spherical surface and cone angle of equivalent beam with circular section. These formulae are very useful for replacing the rectangular profile by circular profile of a beam emitted by a uniform point-source and vice versa without any change in the total radiation energy/luminous flux associated with the original beam (having either rectangular or circular profile). Thus the articles discussed & analysed here are very useful for the analysis of radiation energy and determining the intensity of (uniform) point-source in a given direction in Radiometry and for the analysis of luminous flux and luminous intensity of (uniform) point-source in a certain direction in Photometry. These are also very useful in replacing the rectangular aperture by a circular aperture and vice versa without any change in the total radiation energy/luminous flux passing through an original aperture and thus useful for the analysis and the designing of optical\beam emitting (like laser) devices based on (uniform) point-sources.

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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