Analytical Solutions for a Non-Uniform Tetradecahedron with Two Regular Hexagonal Faces and 12 Trapezoidal Faces

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1. Introduction

This paper presents a complete geometric analysis of a non-uniform tetradecahedron consisting of two congruent regular hexagonal faces, twelve congruent trapezoidal faces, and eighteen vertices lying on a common circumscribed sphere of radius R. Each trapezoidal face possesses three equal edges, two equal acute angles each α ($\approx 79.45^o$), and two equal obtuse angles each β ($\approx 100.55^o$) (see the figure 1). The condition that all vertices are concyclic in three dimensions imposes strong geometric constraints that uniquely determine and correlate the fundamental parameters of the polyhedron, including the solid angle subtended by each face at the center, the normal distances of faces from the center, the circumscribed radius, the inscribed radius, the mean radius, as well as the total surface area and enclosed volume. It is shown that knowledge of a single independent edge length suffices to determine all remaining geometric quantities. In particular, when the edge length a of the regular hexagonal faces is specified, the analysis simplifies considerably. An explicit analytical relation between the hexagonal edge length a and the circumscribed radius R is derived using the generalized formula and axiom of solid angle [1,2,3], enabling all metric properties of the non-uniform tetradecahedron, including plane and solid angles associated with each face, to be expressed solely in terms of a.

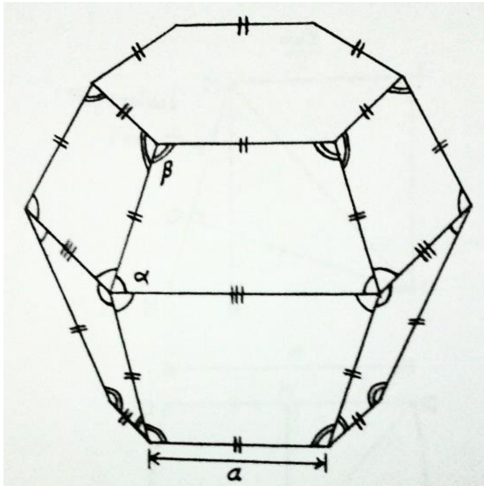

Figure 1: A non-uniform tetradecahedron has 2 congruent regular hexagonal faces each of edge length a & 12 congruent trapezoidal faces. All its 18 vertices eventually & exactly lie on a spherical surface with a certain radius.

2. Analysis of Non-Uniform Tetradecahedron

For ease of calculations & understanding, let there be a non-uniform tetradecahedron, with the centre O, having 2 congruent regular hexagonal faces each with edge length a & 12 congruent trapezoidal faces and all its 18 vertices lying on a spherical surface with a radius R_o. Now consider any of 12 congruent trapezoidal faces say ABCD ($AD = BC = CD = a$) & join the vertices A & D to the centre O (see the figure 2). Join the centre E of the top hexagonal face to the centre O & to the vertex D. Draw a perpendicular DF from the vertex D to the line AO, perpendicular EM from the centre E to the side CD, perpendicular ON from the centre O to the side AB & then join the mid-points M & N of the sides CD & AB respectively in order to obtain trapeziums ADEO & OEMN (see the figures 3 and 4 below). Now we have,

$$OA = OD = R_o\ ,\ \ BC = CD = AD = a,\ \ \angle CED = \angle AOB = \frac{360^o}{6} = 60^o$$

Hence, in equilateral triangles ΔCED & ΔAOB , we have

$$\boldsymbol{EC = ED = CD = a}\ \ \&\ \ \boldsymbol{OA = OB = AB = R_o}$$

In right ΔEMD (Fig. 2),

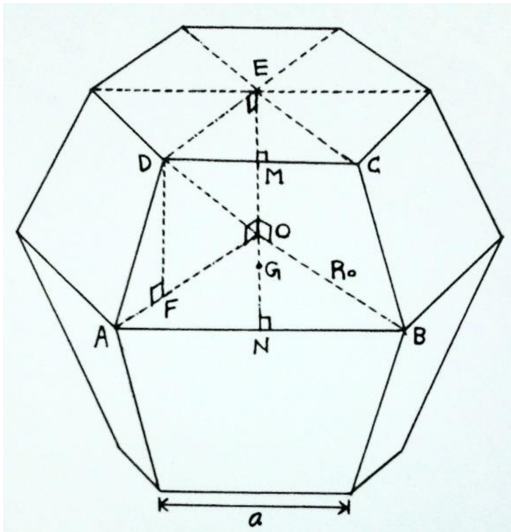

Figure 2: ABCD is one of 12 congruent trapezoidal faces with $AD = BC = CD$. ΔCED & ΔAOB are equilateral triangles. ADEO & OEMN are trapeziums.

$$cos\angle DEM = \frac{EM}{ED} \Rightarrow cos30^o = \frac{EM}{a} \Rightarrow \boldsymbol{EM = \frac{a\sqrt{3}}{2} = OH}$$

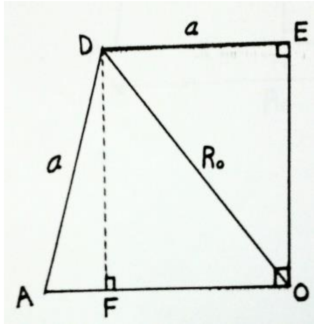

Figure 3: Trapezium ADEO with $\boldsymbol{AD = DE = OF = a}$, $\boldsymbol{OA = OD = R_o}$ & $\boldsymbol{DF = EO}$. The lines DE & AO are parallel.

Similarly, in right ΔANO (Fig. 2),

$$\Rightarrow \boldsymbol{ON = \frac{R_o\sqrt{3}}{2}}$$

In right ΔOED (Fig. 3),

$$EO = \sqrt{(OD)^2 - (DE)^2} = \sqrt{{R_o}^2 - a^2} \Rightarrow \boldsymbol{EO = DF = \sqrt{{R_o}^2 - a^2}}$$

In right ΔAFD (Fig. 3),

$$\Rightarrow (AD)^2 = (AF)^2 + (DF)^2 = (OA - OF)^2 + (DF)^2 = (R_o - a)^2 + \left(\sqrt{{R_o}^2 - a^2}\right)^2$$

$$\Rightarrow a^2 = {R_o}^2 + a^2 - 2aR_o + {R_o}^2 + a^2 = 2{R_o}^2 - 2aR_o$$

$2{R_o}^2 - 2aR_o - a^2 = 0$

$$\Rightarrow R_o = \frac{2a \pm \sqrt{(-2a)^2 + 8a^2}}{4} = \frac{2a \pm 2a\sqrt{3}}{4} = \frac{a(1 \pm \sqrt{3})}{2}$$

But, $R_o > a > 0$ by taking positive sign, we get

$$\therefore \quad \boldsymbol{R_o = \frac{(1+\sqrt{3})a}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (I)$$

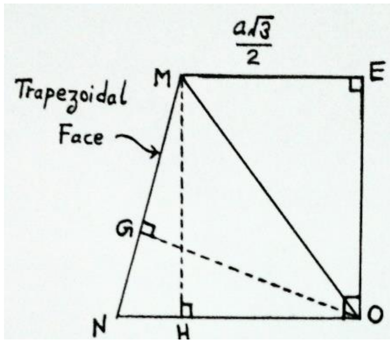

Figure 4: Trapezium OEMN with $\boldsymbol{EM = OH}$ & $\boldsymbol{EO = MH}$. The lines ME & NO are parallel.

Now, draw a perpendicular OG from the centre O to the trapezoidal face ABCD, perpendicular MH from the mid-point M of the side CD to the line ON. Thus in trapezium OEMN (see the figure 4), we have

$$MH = EO = \sqrt{{R_o}^2 - a^2} = \sqrt{\left(\frac{(1+\sqrt{3})a}{2}\right)^2 - a^2} = a\sqrt{\frac{(1+3+2\sqrt{3}) - 4}{4}} = a\sqrt{\frac{2\sqrt{3}}{4}} = a\sqrt{\frac{\sqrt{3}}{2}} = a\sqrt{\sqrt{\frac{3}{4}}}$$

$$\therefore \boldsymbol{MH = EO = a\left(\frac{3}{4}\right)^{\frac{1}{4}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (II)$$

$$\boldsymbol{NH} = ON - OH = ON - EM = \frac{R_o\sqrt{3}}{2} - \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}(R_o - a)}{2} = \frac{\sqrt{3}\left(\frac{(1+\sqrt{3})a}{2} - a\right)}{2} \quad \text{(from eq(I))}$$

$$= \frac{a\sqrt{3}(1+\sqrt{3}-2)}{4} = \frac{a\sqrt{3}(\sqrt{3}-1)}{4} = \frac{(3-\sqrt{3})a}{4}$$

In right ΔMHN (Fig. 4),

$$MN = \sqrt{(MH)^2 + (NH)^2} = \sqrt{(EO)^2 + (NH)^2} = \sqrt{\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)^2 + \left(\frac{(3-\sqrt{3})a}{4}\right)^2}$$

$$= a\sqrt{\frac{\sqrt{3}}{2} + \frac{9+3-6\sqrt{3}}{16}} = a\sqrt{\frac{8\sqrt{3}+12-6\sqrt{3}}{16}} = a\sqrt{\frac{12+2\sqrt{3}}{16}} = a\sqrt{\frac{6+\sqrt{3}}{8}}$$

$$\therefore \quad \boldsymbol{MN = \frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}} \qquad \dots\dots\dots\dots\dots (III)$$

Now, area of ΔOMN can be calculated by two methods as follows (from figure 4 above),

$$\text{Area of } \Delta OMN = \frac{1}{2}[(MN)\times(OG)] = \frac{1}{2}[(ON)\times(MH)] \;\Rightarrow\; (MN)\times(OG) = (ON)\times(MH)$$

$$\Rightarrow \; OG = \frac{(ON)\times(MH)}{MN} = \frac{\left(\frac{R_o\sqrt{3}}{2}\right)\times\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)}{\left(\frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right)} = \left(\sqrt{3}\frac{(1+\sqrt{3})a}{2}\right)\sqrt{\frac{2\left(\frac{\sqrt{3}}{2}\right)}{6+\sqrt{3}}} = \frac{(3+\sqrt{3})a}{2}\sqrt{\frac{\sqrt{3}}{6+\sqrt{3}}}$$

$$= \frac{(3+\sqrt{3})a}{2}\sqrt{\frac{1}{2\sqrt{3}+1}} = \frac{(3+\sqrt{3})a}{2}\sqrt{\frac{(2\sqrt{3}-1)}{(2\sqrt{3}+1)(2\sqrt{3}-1)}} = \frac{a}{2}\sqrt{\frac{(2\sqrt{3}-1)(3+\sqrt{3})^2}{(12-1)}}$$

$$= \frac{a}{2}\sqrt{\frac{(2\sqrt{3}-1)(12+6\sqrt{3})}{11}} = \frac{a}{2}\sqrt{\frac{24+18\sqrt{3}}{11}} = a\sqrt{\frac{3(4+3\sqrt{3})}{22}}$$

$$\therefore \; \boldsymbol{OG = a\sqrt{\frac{3(4+3\sqrt{3})}{22}}} \qquad \dots\dots\dots\dots\dots (IV)$$

2.1. Normal distance (H_h) of regular hexagonal faces from the centre of non-uniform tetradecahedron

The normal distance (H_h) of each of 2 congruent regular hexagonal faces from the centre O of a non-uniform tetradecahedron is given as

$$H_h = EO = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \qquad \text{(from the eq(II) above)}$$

$$\therefore \; \boldsymbol{H_h = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a}$$

It's clear that both the congruent regular hexagonal faces are at an equal normal distance H_h from the centre of a non-uniform tetradecahedron.

2.2. Solid angle (ω_h) subtended by each of 2 congruent regular hexagonal faces at the centre of non-uniform tetradecahedron

We know that the solid angle (ω) subtended by any regular polygon with each side of length a at any point lying at a distance H on the vertical axis passing through the centre of plane is given by generalized formula [1,2] as follows

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)}$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle subtended by each regular hexagonal face at the centre of the non-uniform tetradecahedron as follows

$$\omega_h = 2\pi - 2\times 6\sin^{-1}\left(\frac{2(EO)\sin\frac{\pi}{6}}{\sqrt{4(EO)^2 + a^2\cot^2\frac{\pi}{6}}}\right) = 2\pi - 12\sin^{-1}\left(\frac{2\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)\left(\frac{1}{2}\right)}{\sqrt{4\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)^2 + a^2\left(\sqrt{3}\right)^2}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\frac{\left(\frac{3}{4}\right)^{\frac{1}{4}}}{\sqrt{4\left(\frac{\sqrt{3}}{2}\right) + 3}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(\frac{\sqrt{3}}{2}\right)}{2\sqrt{3} + 3}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{1}{2\left(2+\sqrt{3}\right)}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(2-\sqrt{3}\right)}{2\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(2-\sqrt{3}\right)}{2(4-3)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right)$$

$$\boldsymbol{\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right) \approx 1.786372116\ sr}$$

2.3. Normal distance (H_t) of trapezoidal faces from the centre of non-uniform tetradecahedron

The normal distance (H_t) of each of 12 congruent trapezoidal faces from the centre of non-uniform tetradehedron is given as

$$H_t = OG = a\sqrt{\frac{3\left(4 + 3\sqrt{3}\right)}{22}} \qquad \text{(from the eq(IV) above)}$$

$$\Rightarrow \boldsymbol{H_t = a\sqrt{\frac{3\left(4 + 3\sqrt{3}\right)}{22}} \approx 1.119830695a}$$

It's clear that all 12 congruent trapezoidal faces are at an equal normal distance H_t from the centre of a non-uniform tetradecahedron.

2.4. Solid angle (ω_t) subtended by each of 12 congruent trapezoidal faces at the centre of non-uniform tetradecahedron

Since a non-uniform tetradecahedron is a closed surface & we know that the total solid angle, subtended by any closed surface at any point lying inside it, is $4\pi\ sr$ (Ste-radian) [3] hence the sum of solid angles subtended by 2 congruent regular hexagonal & 12 congruent trapezoidal faces at the centre of the non-uniform tetradecahedron must be $4\pi\ sr$. Thus we have

$$2[\omega_{hexagon}] + 12[\omega_{trapezium}] = 4\pi \ \Rightarrow\ 12[\omega_{trapezium}] = 4\pi - 2[\omega_{hexagon}]$$

$$\omega_t = \frac{2\pi - \omega_{hexagon}}{6} = \frac{2\pi - \left[2\pi - 12\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right)\right]}{6} = 2\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right)$$

$$\therefore\ \boldsymbol{\omega_t = 2\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right) \approx 0.749468865\ sr}$$

It's clear from the above results that the solid angle subtended by each of 2 regular hexagonal faces is greater than the solid angle subtended by each of 12 trapezoidal faces at the centre of a non-uniform tetradecahedron.

It's also clear from the above results that $H_t > H_h$ i.e. the normal distance (H_t) of trapezoidal faces is greater than the normal distance H_h of the regular hexagonal faces from the centre of a non-uniform tetradecahedron i.e. regular hexagonal faces are closer to the centre as compared to the trapezoidal faces in a non-uniform tetradecahedron.

2.5. Interior angles ($\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$) of the trapezoidal faces of non-uniform tetradecahedron

From above figures 1 & 2, let α be acute angle & β be obtuse angle. Acute angle α is determined as follows

$$\sin\angle BAD = \frac{MN}{AD} \Rightarrow sin\alpha = \frac{\left(\frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right)}{a} \qquad \text{(from eq(III) above)}$$

$$= \frac{1}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\ \ or\ \alpha = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right)$$

$$\Rightarrow \text{Acute angle,}\ \ \boldsymbol{\alpha = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right) \approx 79.45470941^o \approx 79^o27'16.95''}$$

In trapezoidal face ABCD, we know that the sum of all interior angles (of a quadrilateral) is 360^o

$$\therefore 2\alpha + 2\beta = 360^o\ \ or\ \ \beta = 180^o - \alpha$$

$$\therefore \text{Obtuse angle,}\ \ \boldsymbol{\beta = 180^o - \alpha \approx 100.5452906^o \approx 100^o32'43.05''}$$

2.6. Sides of each trapezoidal face of non-uniform tetradecahedron

All the sides of each trapezoidal face can be determined as follows (see figures (3) & (5) above),

$$\boldsymbol{CD = AD = BC = a}\ \ \&\ \ \boldsymbol{AB = R_o = \frac{(1+\sqrt{3})a}{2}} \quad \text{(from eq(I) above)}$$

Distance between parallel sides AB & CD of trapezoidal face ABCD

$$\therefore\ \boldsymbol{MN} = \frac{\boldsymbol{a}}{\boldsymbol{2}}\sqrt{\frac{\boldsymbol{6+\sqrt{3}}}{\boldsymbol{2}}} \qquad \text{(from eq(III)above)}$$

Hence, the area of each of 12 congruent trapezoidal faces of a non-uniform tetradecahedron is given as follows

$$\textbf{Area of trapezium ABCD} = \frac{1}{2}(\text{sum of parallel sides}) \times (\text{normal distance between parallel sides})$$

$$= \frac{1}{2}(AB+CD)(MN) = \frac{1}{2}(R_o+a)\left(\frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right) = \frac{a}{4}\left(\frac{(1+\sqrt{3})a}{2}+a\right)\sqrt{\frac{6+\sqrt{3}}{2}}$$

$$= \frac{a^2}{8}(1+\sqrt{3}+2)\sqrt{\frac{6+\sqrt{3}}{2}} = \frac{a^2}{8}(3+\sqrt{3})\sqrt{\frac{6+\sqrt{3}}{2}} = \frac{a^2}{8}\sqrt{\frac{(6+\sqrt{3})(3+\sqrt{3})^2}{2}} = \frac{a^2}{8}\sqrt{\frac{(6+\sqrt{3})(12+6\sqrt{3})}{2}}$$

$$= \frac{a^2}{8}\sqrt{\frac{(6+\sqrt{3})(12+6\sqrt{3})}{2}} = \frac{a^2}{8}\sqrt{\frac{90+48\sqrt{3}}{2}} = \frac{a^2}{8}\sqrt{45+24\sqrt{3}}$$

$$= \frac{\boldsymbol{a^2}}{\boldsymbol{8}}\sqrt{\boldsymbol{45+24\sqrt{3}}} \approx \boldsymbol{1.163032266a^2}$$

3. Important parameters of a non-uniform tetradecahedron

1. **Inner (inscribed) radius ($\boldsymbol{R_i}$)**: It is the radius of the largest sphere inscribed (trapped inside) by a non-uniform tetradecahedron. The largest inscribed sphere always touches both the congruent regular hexagonal faces but does not touch any of 12 congruent trapezoidal faces at all since both the hexagonal faces are closer to the centre as compared to all 12 trapezoidal faces. Thus, inner radius is always equal to the normal distance (H_h) of the regular hexagonal faces from the centre of a non-uniform tetradecahedron & is given as follows

$$\boldsymbol{R_i = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a}$$

Hence, the volume of inscribed sphere is given as

$$\boldsymbol{V}_{\textbf{inscribed}} = \frac{4}{3}\pi(R_i)^3 = \frac{\boldsymbol{4}}{\boldsymbol{3}}\boldsymbol{\pi}\left(\boldsymbol{a}\left(\frac{\boldsymbol{3}}{\boldsymbol{4}}\right)^{\frac{1}{4}}\right)^3 \approx \boldsymbol{3.375861004a^3}$$

2. **Outer (circumscribed) radius ($\boldsymbol{R_o}$)**: It is the radius of the smallest sphere circumscribing a non-uniform tetradecahedron or it's the radius of a spherical surface passing through all 18 vertices of a non-uniform tetradecahedron. It is given as follows

$$\boldsymbol{R_o = \frac{(1+\sqrt{3})a}{2} \approx 1.366025404a}$$

Hence, the volume of circumscribed sphere is given as

$$\boldsymbol{V}_{\textbf{circumscribed}} = \frac{4}{3}\pi(R_o)^3 = \frac{\boldsymbol{4}}{\boldsymbol{3}}\boldsymbol{\pi}\left(\frac{\boldsymbol{(1+\sqrt{3})a}}{\boldsymbol{2}}\right)^3 = \boldsymbol{10.67738585a^3}$$

3. **Surface area (A_s):** We know that a non-uniform tetradecahedron has 2 congruent regular hexagonal faces & 12 congruent trapezoidal faces. Hence, its total surface area is given as follows

$$A_s = 2(\text{area of regular hexagon}) + 12(\text{area of trapezium } ABCD) \qquad \text{(see figure 2 above)}$$

We know that area of any regular n-polygon with each side of length a is given by generalized formula [4],

$$\boldsymbol{A = \frac{1}{4}na^2 cot\frac{\pi}{n}} \qquad \text{(for a regular hexagon, } n = 6)$$

Hence, by substituting all the corresponding values in the above expression, we obtain the total surface area,

$$A_s = 2 \times \left(\frac{1}{4} \times 6a^2 cot\frac{\pi}{6}\right) + 12 \times \left(\frac{1}{2}(AB + CD)(MN)\right) = 3a^2\sqrt{3} + 12 \times \left(\frac{a^2}{8}\sqrt{45 + 24\sqrt{3}}\right)$$

$$= 3a^2\sqrt{3} + \frac{3a^2}{2}\sqrt{45 + 24\sqrt{3}} = \frac{3a^2}{2}\left(2\sqrt{3} + \sqrt{45 + 24\sqrt{3}}\right)$$

$$\therefore \ \boldsymbol{A_s = \frac{3a^2}{2}\left(2\sqrt{3} + \sqrt{45 + 24\sqrt{3}}\right) \approx 19.15253962a^2}$$

4. **Volume (V):** We know that a non-uniform tetradecahedron has 2 congruent regular hexagonal & 12 congruent trapezoidal faces. Hence, the volume (V) of the non-uniform tetradecahedron is the sum of volumes of all its elementary right pyramids with regular hexagonal and trapezoidal bases [4] given as follows

$$\text{V} = 2(\text{volume of right pyramid with regular hexagonal base}) + 12(\text{volume of right pyramid with trapezoidal base ABCD})$$

$$= 2\left(\frac{1}{3}(\text{area of regular hexagon}) \times H_h\right) + 12\left(\frac{1}{3}(\text{area of trapezium ABCD}) \times H_t\right)$$

$$= 2\left(\frac{1}{3}\left(\frac{1}{4} \times 6a^2 cot\frac{\pi}{6}\right) \times a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right) + 12\left(\frac{1}{3}\left(\frac{a^2}{8}\sqrt{45 + 24\sqrt{3}}\right) \times a\sqrt{\frac{3(4 + 3\sqrt{3})}{22}}\right)$$

$$= a^3(\sqrt{3})\left(\frac{3}{4}\right)^{\frac{1}{4}} + \frac{a^3}{2}\sqrt{45 + 24\sqrt{3}}\sqrt{\frac{3(4 + 3\sqrt{3})}{22}} = a^3\sqrt{\frac{3\sqrt{3}}{2}} + \frac{a^3}{2}\sqrt{\frac{3(4 + 3\sqrt{3})(45 + 24\sqrt{3})}{22}}$$

$$= a^3\sqrt{\frac{3\sqrt{3}}{2}} + \frac{a^3}{2}\sqrt{\frac{3(396 + 231\sqrt{3})}{22}} = a^3\sqrt{\frac{3\sqrt{3}}{2}} + \frac{a^3}{2}\sqrt{\frac{3(396 + 231\sqrt{3})}{22}}$$

$$= a^3\sqrt{\frac{3\sqrt{3}}{2}} + \frac{a^3}{2}\sqrt{\frac{99(12 + 7\sqrt{3})}{22}} = a^3\left(\sqrt{\frac{3\sqrt{3}}{2}} + \frac{3}{2}\sqrt{\frac{12 + 7\sqrt{3}}{2}}\right) = a^3\left(\frac{2\sqrt{6\sqrt{3}} + 3\sqrt{24 + 14\sqrt{3}}}{4}\right)$$

$$\therefore \ \boldsymbol{V = a^3\left(\frac{2\sqrt{6\sqrt{3}} + 3\sqrt{24 + 14\sqrt{3}}}{4}\right) \approx 6.821451822a^3}$$

5. **Mean radius (R_m):** It is the radius of the sphere having a volume equal to that of a non-uniform tetradecahedron. It is calculated as follows

$$\text{Volume of sphere with mean radius } R_m = \text{Volume of the non uniform tetradecahedron}$$

$$\frac{4}{3}\pi(R_m)^3 = a^3\left(\frac{2\sqrt{6\sqrt{3}} + 3\sqrt{24 + 14\sqrt{3}}}{4}\right) \Rightarrow (R_m)^3 = \left(\frac{6\sqrt{6\sqrt{3}} + 9\sqrt{24 + 14\sqrt{3}}}{16\pi}\right)a^3$$

$$\therefore \boldsymbol{R_m = a\left(\frac{6\sqrt{6\sqrt{3}} + 9\sqrt{24 + 14\sqrt{3}}}{16\pi}\right)^{\frac{1}{3}} \approx 1.176511208a}$$

It's clear from above results that $R_i < R_m < R_o$.

4. Construction of a solid non-uniform tetradecahedron

In order to construct a non-uniform tetradecahedron with 14 faces, there are two methods as given below.

4.1. Construction from elementary right pyramids: In this method, first we construct all elementary right pyramids as follows

Step 1. Construct 2 congruent right pyramids with regular hexagonal base of side length a & normal height (H_h)

$$H_h = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a$$

Step 2. Construct 12 congruent right pyramids with trapezoidal base ABCD of sides AB, $BC = AD = CD$ and normal height (H_t)

$$H_t = a\sqrt{\frac{3(4 + 3\sqrt{3})}{22}} \approx 1.119830695a$$

$$AD = BC = CD = a \;\&\; AB = \frac{(1 + \sqrt{3})a}{2} \approx 1.366025404a \qquad \text{(see figure 2 above)}$$

$$\text{Acute angle}, \alpha \approx 79.45470941^o$$

$$AB \;\&\; CD \text{ are parallel sides, and } AD \;\&\; AC \text{ are equal but non parallel sides}$$

Step 3. Now, bond by joining all these elementary right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that all 6 edges of each regular hexagonal base (face) coincide with the edges of 6 trapezoidal bases (faces). Thus a solid non-uniform tetradecahedron, with 2 congruent regular hexagonal faces, 12 congruent trapezoidal faces & 18 vertices lying on a spherical surface, is obtained.

4.2. Facing a solid sphere: It is a method of facing, first we select a blank as a solid sphere of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of the hexagonal face of a non-uniform tetradecahedron. Then, we perform the facing operations on the solid sphere to generate 2 congruent regular hexagonal faces each with edge length a & 12 congruent trapezoidal faces.

4.2.1. Edge length: Let there be a blank as a solid sphere with a diameter D. Then the edge length a, of each regular hexagonal face of a non-uniform tetradecahedron of the maximum volume to be produced, can be co-related with the diameter D by relation of outer radius (R_o) with edge length (a) of the hexagonal face as follows

$$R_o = \frac{(1 + \sqrt{3})a}{2}$$

Now, substituting $R_o = {}^{D}/_{2}$ in the above expression, we have

$$\frac{D}{2} = \frac{(1+\sqrt{3})a}{2} \quad or \quad a = \frac{D}{(\sqrt{3}+1)} = \frac{D(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{D(\sqrt{3}-1)}{2}$$

$$\boldsymbol{a = \frac{D(\sqrt{3}-1)}{2} \approx 0.366025403D}$$

The above relation is very useful for determining the edge length a of regular hexagonal face of a non-uniform tetradecahedron to be produced from a solid sphere with known diameter D for manufacturing purpose.

4.2.2. Maximum volume of non-uniform tetradecahedron produced from a solid sphere is given as follows

$$V_{max} = a^3\left(\frac{2\sqrt{6\sqrt{3}}+3\sqrt{24+14\sqrt{3}}}{4}\right) = \left(\frac{D(\sqrt{3}-1)}{2}\right)^3\left(\frac{2\sqrt{6\sqrt{3}}+3\sqrt{24+14\sqrt{3}}}{4}\right)$$

$$= D^3(6\sqrt{3}-10)\left(\frac{2\sqrt{6\sqrt{3}}+3\sqrt{24+14\sqrt{3}}}{32}\right) = D^3(3\sqrt{3}-5)\left(\frac{2\sqrt{6\sqrt{3}}+3\sqrt{24+14\sqrt{3}}}{16}\right)$$

$$= D^3\left(\frac{2\sqrt{6\sqrt{3}(3\sqrt{3}-5)^2}+3\sqrt{(24+14\sqrt{3})(3\sqrt{3}-5)^2}}{16}\right) = D^3\left(\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{(2\sqrt{3}-3)}}{8}\right)$$

$$\boldsymbol{V_{max} = D^3\left(\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right) \approx 0.334511075D^3}$$

4.2.3. Minimum volume of material removed is given as

$$(V_{removed})_{min} = (\text{Volume of parent sphere with diameter } D) - (\text{Volume of nonuniform tetradecahedron})$$

$$= \frac{\pi}{6}D^3 - D^3\left(\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right) = \left(\frac{\pi}{6}-\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right)D^3$$

$$\boldsymbol{(V_{removed})_{min} = \left(\frac{\pi}{6}-\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right)D^3 \approx 0.1890877D^3}$$

4.2.4. Percentage $(\%)$ **of minimum volume of material removed**

$$\% \text{ of } \boldsymbol{V_{removed}} = \frac{\text{Minimum volume removed}}{\text{Total volume of sphere}} \times 100$$

$$= \frac{\left(\frac{\pi}{6}-\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right)D^3}{\frac{\pi}{6}D^3} \times 100 = \boldsymbol{\left(1-\frac{6\sqrt{3(26\sqrt{3}-45)}+9\sqrt{2\sqrt{3}-3}}{4\pi}\right) \times 100 \approx 36.11\%}$$

It's obvious that when a solid non-uniform tetradecahedron of the maximum volume is produced from a solid sphere then about 36.11% of material is removed as scraps. Thus, we can select optimum diameter of blank as a solid sphere to produce a solid non-uniform tetradecahedron of the maximum volume (or with the maximum desired edge length a of regular hexagonal face).

Conclusions: Let there be a non-uniform tetradecahedron having 2 congruent regular hexagonal faces each with edge length a, 12 congruent trapezoidal faces & 18 vertices lying on a spherical surface with certain radius then all its important parameters are calculated/determined as tabulated below.

Congruent polygonal faces	No. of faces	Normal distance of each face from the centre of the non-uniform tetradecahedron	Solid angle subtended by each face at the centre of the non-uniform tetradecahedron
Regular hexagon	2	$a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a$	$2\pi - 12\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right) \approx 1.786372116\ sr$
Trapezium	12	$a\sqrt{\frac{3(4+3\sqrt{3})}{22}} \approx 1.119830695a$	$2\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right) \approx 0.749468865\ sr$

Inner (inscribed) radius $(\boldsymbol{R_i})$	$R_i = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a$
Outer (circumscribed) radius $(\boldsymbol{R_o})$	$R_o = \frac{(1+\sqrt{3})a}{2} \approx 1.366025404a$
Mean radius $(\boldsymbol{R_m})$	$R_m = a\left(\frac{6\sqrt{6\sqrt{3}} + 9\sqrt{24+14\sqrt{3}}}{16\pi}\right)^{\frac{1}{3}} \approx 1.176511208a$
Surface area $(\boldsymbol{A_s})$	$A_s = \frac{3a^2}{2}\left(2\sqrt{3} + \sqrt{45+24\sqrt{3}}\right) \approx 19.15253962a^2$
Volume $(\boldsymbol{V})$	$V = a^3\left(\frac{2\sqrt{6\sqrt{3}} + 3\sqrt{24+14\sqrt{3}}}{4}\right) \approx 6.821451822a^3$

Note: Above articles had been developed & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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