Solid angles subtended by the platonic solids (regular polyhedra) at their vertices

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Introduction: We know that all five platonic solids i.e. regular tetrahedron, regular hexahedron (cube), regular octahedron, regular dodecahedron & regular icosahedron have all their vertices identical, hence the solid angle subtended by any platonic solid at any of its identical vertices will be equal in magnitude. If we treat all the edges meeting at any of the identical vertices of a platonic solid as the lateral edges of a right pyramid with a regular n-gonal base then the solid angle subtended by any of five platonic solids is calculated by using HCR's standard formula of solid angle. According to which, **solid angle (ω), subtended at the vertex (apex point) by a right pyramid with a regular n-gonal base & an angle α between any two consecutive lateral edges meeting at the same vertex,** is mathematically given by the standard (generalized) formula [1,2] as follows

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right)} \qquad \forall\ n \in N\ \&\ n \geq 3$$

Thus, by setting the value of the number of edges n meeting at any of the identical vertices of a platonic solid and the angle α between any two consecutive edges meeting at that vertex in the above formula, we can easily calculate the solid angle subtended by the given platonic solid at its vertex [3]. Let's assume that the eye of observer is located at any of the identical vertices of a given platonic solid & directed (focused) straight to the centre (of the platonic solid) (as shown in the Figures 1-5 below) then by setting the corresponding values of n & α in the above generalized formula we can analyse all five platonic solids as follows

1. Solid angle subtended by a regular tetrahedron at any of its four identical vertices: we know that a regular tetrahedron has 4 congruent equilateral triangular faces, 6 edges & 4 identical vertices. Three equilateral triangular faces meet at each vertex & hence 3 edges meet at each vertex & the angle between any two consecutive edges is 60^o (see Figure 1), thus in this case we have

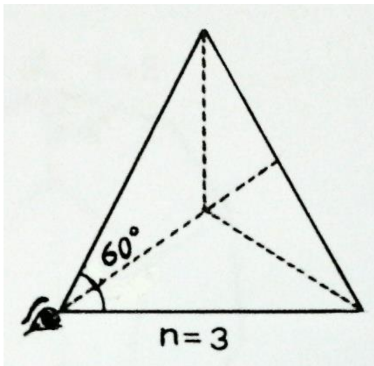

Figure 1: Eye of the observer is located at any of four identical vertices of a regular tetrahedron (in this case $\boldsymbol{n = 3}$ & $\boldsymbol{\alpha = 60^o}$).

$$\boldsymbol{n = 3}\ \ \&\ \ \boldsymbol{\alpha = 60^o}\ (\textbf{interior angle of equilateral triangular face})$$

$$\Rightarrow \omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2(3)\sin^{-1}\left(cos\frac{\pi}{3}\sqrt{tan^2\frac{\pi}{3} - tan^2\frac{60^o}{2}}\right) = 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\left(\sqrt{3}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{8}{3}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

Hence, the **solid angle (ω_T) subtended by a regular tetrahedron at its vertex** is given as

$$(\boldsymbol{\omega_T}) = \boldsymbol{2\pi - 6\sin^{-1}\left(\sqrt{\frac{2}{3}}\right) \approx 0.551285598\, sr} \qquad \dots\dots\dots\dots(1)$$

2. Solid angle subtended by a regular hexahedron (cube) at any of its eight identical vertices: we know that a regular hexahedron (cube) has 6 congruent square faces, 12 edges & 8 identical vertices. Three square faces meet at each vertex & hence 3 edges meet at each vertex & the angle between any two consecutive edges is 90^o (see Figure 2), thus in this case we have

$$\boldsymbol{n = 3} \;\; \& \;\; \boldsymbol{\alpha = 90^o} \quad (\textbf{interior angle of square face})$$

$$\Rightarrow \omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2(3)\sin^{-1}\left(cos\frac{\pi}{3}\sqrt{tan^2\frac{\pi}{3} - tan^2\frac{90^o}{2}}\right) = 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\left(\sqrt{3}\right)^2 - (1)^2}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{2}\right) = 2\pi - 6\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 2\pi - 6\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

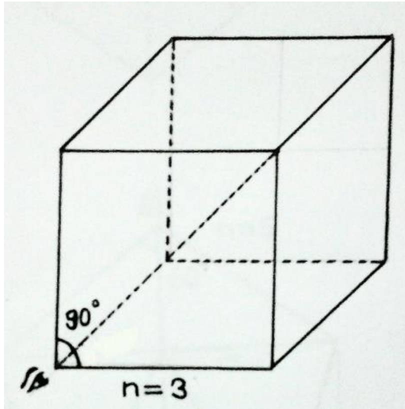

Figure 2: Eye of the observer is located at any of eight identical vertices of a regular hexahedron (cube) (in this case $\boldsymbol{n} = \boldsymbol{3}$ & $\boldsymbol{\alpha = 90^o}$).

Hence, the **solid angle $(\boldsymbol{\omega_s})$ subtended by a regular hexahedron (cube) at its vertex** is given as

$$(\boldsymbol{\omega_s}) = \frac{\boldsymbol{\pi}}{\boldsymbol{2}} \approx \boldsymbol{1.570796327\, sr} \qquad \dots\dots\dots\dots(2)$$

3. Solid angle subtended by a regular octahedron at any of its six identical vertices: we know that a regular octahedron has 8 congruent equilateral triangular faces, 12 edges & 6 identical vertices. Four equilateral triangular faces meet at each vertex & hence 4 edges meet at each vertex & the angle between any two consecutive edges is 60^o (see Figure 3), thus in this case we have

$$\boldsymbol{n = 4} \;\; \& \;\; \boldsymbol{\alpha = 60^o} \quad (\textbf{interior angle of equilateral triangular face})$$

$$\Rightarrow \omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2(4)\sin^{-1}\left(cos\frac{\pi}{4}\sqrt{tan^2\frac{\pi}{4} - tan^2\frac{60^o}{2}}\right) = 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{(1)^2 - \left(\frac{1}{\sqrt{3}}\right)^2}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}\right) = 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

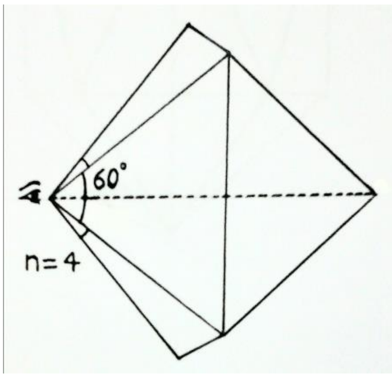

Figure 3: Eye of the observer is located at any of six identical vertices of a regular octahedron (in this case $\boldsymbol{n} = \boldsymbol{4}$ & $\boldsymbol{\alpha = 60^o}$).

Hence, the **solid angle $(\boldsymbol{\omega_o})$ subtended by a regular octahedron at its vertex** is given as

$$(\boldsymbol{\omega_o}) = \boldsymbol{2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)} \approx \boldsymbol{1.359347638\, sr} \qquad \dots\dots\dots\dots(3)$$

4. Solid angle subtended by a regular dodecahedron at any of its twenty identical vertices: we know that a regular dodecahedron has 12 congruent regular pentagonal faces, 30 edges & 20 identical vertices. Three regular pentagonal faces meet at each vertex & hence 3 edges meet at each vertex & the angle between any two consecutive edges is 108^o (see Figure 4), thus, in this case we have

$$\boldsymbol{n = 3} \;\; \& \;\; \boldsymbol{\alpha = 108^o} \quad (\textbf{interior angle of regular pentagonal face})$$

$$\Rightarrow \omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2(3)\sin^{-1}\left(cos\frac{\pi}{3}\sqrt{tan^2\frac{\pi}{3} - tan^2\frac{108^o}{2}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\left(\sqrt{3}\right)^2 - \left(\sqrt{\frac{5+2\sqrt{5}}{5}}\right)^2}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{10-2\sqrt{5}}{5}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)$$

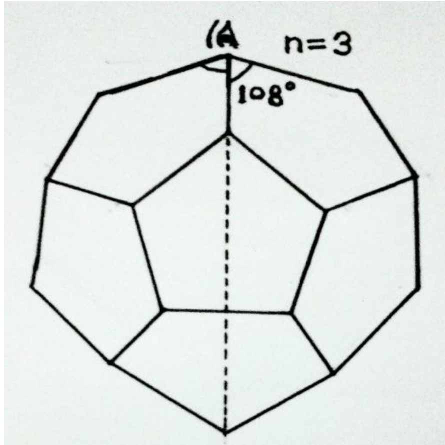

Figure 4: Eye of the observer is located at any of twenty identical vertices of a regular dodecahedron (in this case $n = 3$ & $\alpha = 108^o$).

Hence, the **solid angle (ω_D) subtended by a regular dodecahedron at its vertex** is given as

$$(\omega_D) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right) \approx 2.961739154\ sr \qquad \dots\dots\dots\dots(4)$$

5. Solid angle subtended by a regular icosahedron at any of its twelve identical vertices: we know that a regular icosahedron has 20 congruent equilateral triangular faces, 30 edges & 12 identical vertices. Five equilateral triangular faces meet at each vertex & hence 5 edges meet at each vertex & the angle between any two consecutive edges is 60^o (see Figure 5), thus in this case we have

$$n = 5 \ \ \& \ \ \alpha = 60^o \quad (\textbf{interior angle of equilateral triangular face})$$

$$\Rightarrow \omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2(5)\sin^{-1}\left(cos\frac{\pi}{5}\sqrt{tan^2\frac{\pi}{5} - tan^2\frac{60^o}{2}}\right)$$

$$= 2\pi - 10\sin^{-1}\left(\frac{\sqrt{5}+1}{4}\sqrt{\left(\sqrt{5-2\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2}\right)$$

$$= 2\pi - 10\sin^{-1}\left(\frac{\sqrt{5}+1}{4}\sqrt{\frac{\left(3-\sqrt{5}\right)^2}{3}}\right) = 2\pi - 10\sin^{-1}\left(\frac{\sqrt{5}-1}{2\sqrt{3}}\right)$$

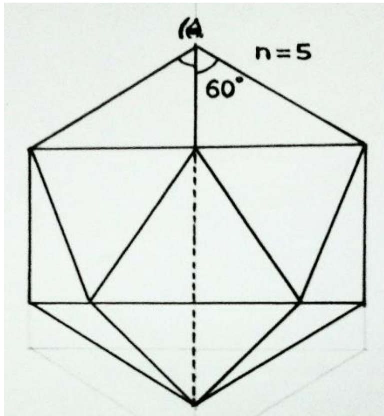

Figure 5: Eye of the observer is located at any of twelve identical vertices of a regular icosahedron (in this case $n = 5$ & $\alpha = 60^o$).

Hence, the **solid angle (ω_{Icos}) subtended by a regular icosahedron at its vertex** is given as

$$(\omega_{Icos}) = 2\pi - 10\sin^{-1}\left(\frac{\sqrt{5}-1}{2\sqrt{3}}\right) \approx 2.634547026\ sr \qquad \dots\dots\dots\dots(5)$$

Conclusions: The solid angles subtended at the vertices by all five platonic solids (regular polyhedrons) have been calculated by the author using the standard formula of solid angle. These are the standard values of solid angles for all five platonic solids i.e. regular tetrahedron, regular hexahedron (cube), regular octahedron, regular dodecahedron & regular icosahedron useful for the analysis of platonic solids.

Summary: All the above results of the solid angles subtended by all five Platonic solids or regular polyhedrons at their vertices can be tabulated as follows

Platonic solid (regular polyhedron)	**Regular polygonal face**	**No. of congruent faces**	**No. of equal edges**	**No. of identical vertices**	**Solid angle subtended by the platonic solid at its each vertex (in Ste-radian (sr))**
Regular tetrahedron	Equilateral triangle	4	6	4	$2\pi - 6\sin^{-1}\left(\sqrt{\frac{2}{3}}\right) \approx 0.551285598\ sr$
Regular hexahedron (cube)	Square	6	12	8	$\frac{\pi}{2} \approx 1.570796327\ sr$
Regular octahedron	Equilateral triangle	8	12	6	$2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 1.359347638\ sr$
Regular dodecahedron	Regular pentagon	12	30	20	$2\pi - 6\sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right) \approx 2.961739154\ sr$
Regular icosahedron	Equilateral triangle	20	30	12	$2\pi - 10\sin^{-1}\left(\frac{\sqrt{5}-1}{2\sqrt{3}}\right) \approx 2.634547026\ sr$

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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