Mathematical Analysis of Spherical Rectangle

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1. Introduction

A spherical rectangle is a 3-D figure, on a spherical surface, enclosed by four sides each as a great circle arc such that each pair of the non-parallel and opposite sides is equal in arc length. All of its four interior angles are equal in magnitude and each angle is greater than 90^o by the property of a spherical rectangle (as shown in figure 1 below). In this work, a set of analytical formulae is developed using elementary geometric and trigonometric methods for the systematic analysis of spherical rectangles [1-4]. The derived expressions enable straightforward computation of key geometric parameters, 3D surface area, interior angles, solid angle subtended at the centre of the sphere, and the corresponding values of plane rectangle. The results are shown to be equally applicable to the corresponding plane rectangles formed by joining the vertices of spherical rectangles with straight line segments, allowing for the evaluation of their fundamental geometric properties. In addition, the formulae are extended to the study of right pyramids obtained by joining the vertices of spherical rectangles to the centre of the sphere, providing closed-form expressions for quantities such as the normal height, angles between consecutive lateral edges, and the area of the rectangular base. Owing to their simplicity and broad applicability, the presented results offer a unified analytical framework for the geometric analysis of spherical rectangles and their associated planar and pyramidal constructions.

2. Analysis of spherical rectangle

Consider any spherical rectangle ABCD having its length & width (each as a great circle arc) as l & b $(\forall\, l \geq b)$ respectively on a spherical surface with a radius R such that each interior angle is θ $(\forall\, \theta > 90^o)$ (as shown in the figure 1).

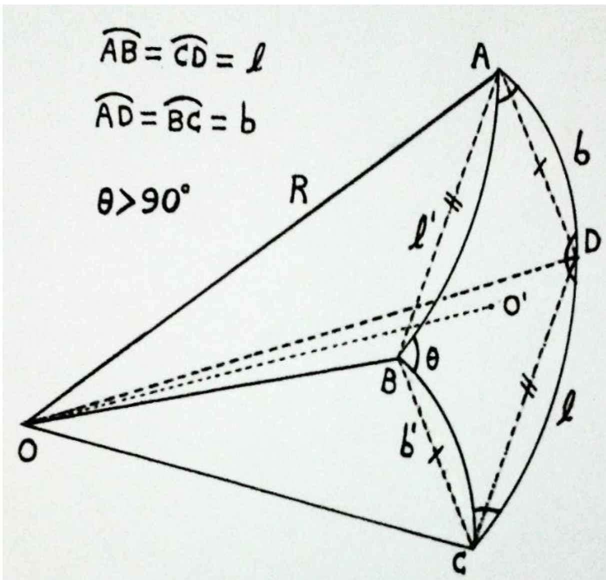

Figure 1: A spherical rectangle ABCD having its length and width, each in form of a great circle arc, as l & b respectively & each interior angle $\theta\ (\forall\ \theta > 90^o)$. A plane rectangle ABCD corresponding to the spherical rectangle ABCD is obtained by joining the vertices A, B, C and D by the dotted straight lines.

2.1. Interior angle (θ) of spherical rectangle

It is worth noticing that each interior angle of a spherical rectangle is the angle between the planes of great circle arcs representing any two of its consecutive sides. Now, join all the vertices A, B, C & D by the straight lines to obtain a corresponding plane rectangle ABCD (as shown by the dotted lines AB, BC, CD & DA) having centre at the point O'. Now the length l' & the width b' of the plane rectangle ABCD are calculated as follows

$$\angle AOB = \frac{\text{Arc length}}{\text{Radius}} = \frac{l}{R} \ \ \& \ \ \angle BOC = \frac{b}{R}$$

In (plane) isosceles ΔAOB (Fig. 1),

$$\sin\left(\frac{\angle AOB}{2}\right) = \frac{\left(\frac{AB}{2}\right)}{OA}$$

$$\sin\left(\frac{l}{2R}\right) = \frac{\left(\frac{l'}{2}\right)}{R} = \frac{l'}{2R}$$

$$l' = 2R\sin\left(\frac{l}{2R}\right) \ \ \& \text{ similarly, } \ b' = 2R\sin\left(\frac{b}{2R}\right)$$

$\therefore$ **Diagonal of the plane rectangle ABCD**, $AC = \sqrt{l'^2 + b'^2}$

$$\Rightarrow \boldsymbol{AC} = \sqrt{\left(2R\sin\left(\frac{l}{2R}\right)\right)^2 + \left(2R\sin\left(\frac{b}{2R}\right)\right)^2} = 2R\sqrt{\sin^2\left(\frac{l}{2R}\right) + \sin^2\left(\frac{b}{2R}\right)} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

In (plane) isosceles ΔAOC (Fig. 1),

$$\sin\left(\frac{\angle AOC}{2}\right) = \frac{\left(\frac{AC}{2}\right)}{OA} \ \Rightarrow \ \sin\left(\frac{\text{acr } AC}{2R}\right) = \frac{AC}{2R} \qquad \left(\because \angle AOC = \frac{\text{arc } AC}{\text{radius}} = \frac{\text{arc } AC}{R}\right)$$

$$\Rightarrow \frac{\text{arc } AC}{2R} = \sin^{-1}\left(\frac{AC}{2R}\right) \ \Rightarrow \ \text{arc } AC = 2R\sin^{-1}\left(\frac{2R\sqrt{\sin^2\left(\frac{l}{2R}\right) + \sin^2\left(\frac{b}{2R}\right)}}{2R}\right) \quad \text{(from Eq(1))}$$

$$\therefore \ \text{arc } AC = 2R\sin^{-1}\left(\sqrt{\sin^2\left(\frac{l}{2R}\right) + \sin^2\left(\frac{b}{2R}\right)}\right) \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Now, consider the tetrahedron OABC having angles $\angle AOB, \angle BOC \ \& \ \angle AOC$ between its consecutive lateral edges OA & OB, OB & OC and OA & OC respectively meeting at the vertex O (i.e. centre of the sphere). Now these are the angles subtended by the sides (each as a great circle arc) AB, BC & CA respectively of the spherical triangle ABC at the centre of sphere which are determined as follows

$$\angle AOB = \frac{\text{arc length}}{\text{radius}} = \frac{l}{R}, \ \ \angle BOC = \frac{b}{R} \ \ \& \ \ \angle AOC = \frac{\text{arc } AC}{R}$$

Now each of the interior angles θ of the spherical rectangle ABCD is equal to the angle between consecutive lateral triangular faces $\Delta AOB \ \& \ \Delta BOC$ of the tetrahedron OABC meeting at the vertex O (i.e. the centre of sphere), are determined by using HCR's Inverse Cosine Formula [1] according to which if $\alpha, \beta \ \& \ \gamma$ are the angles between consecutive lateral edges meeting at any of the four vertices of a tetrahedron then the angle (opposite to α) between two lateral faces is given as follows

$$\theta = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\cos\left(\frac{\text{arc } AC}{R}\right) - \cos\left(\frac{l}{R}\right)\cos\left(\frac{b}{R}\right)}{\sin\left(\frac{l}{R}\right)\sin\left(\frac{b}{R}\right)}\right)$$

$$= \cos^{-1}\left(\frac{\cos\left(\frac{2R\sin^{-1}\left(\sqrt{\sin^2\left(\frac{l}{2R}\right) + \sin^2\left(\frac{b}{2R}\right)}\right)}{R}\right) - \cos\left(\frac{l}{R}\right)\cos\left(\frac{b}{R}\right)}{\sin\left(\frac{l}{R}\right)\sin\left(\frac{b}{R}\right)}\right)$$

$$\theta = \cos^{-1}\left(\frac{\cos\left(2\sin^{-1}\left(\sqrt{\sin^2\left(\frac{l}{2R}\right)+\sin^2\left(\frac{b}{2R}\right)}\right)\right)-\cos\left(\frac{l}{R}\right)\cos\left(\frac{b}{R}\right)}{\sin\left(\frac{l}{R}\right)\sin\left(\frac{b}{R}\right)}\right) \quad \dots\dots\dots\dots\dots(3)$$

$$\forall\ l, b, R > 0\ \&\ \ l + b < \pi R \ \Rightarrow\ 90^o < \theta < 180^o$$

Above Eq.(3) is the required expression to determine each of the interior angles θ of any spherical rectangle having length l & width b (each as a great circle arc) on a spherical surface with a radius R.

2.2. Area covered by the spherical rectangle

In order to calculate area covered by the spherical rectangle ABCD, let's first calculate the solid angle subtended by it at the centre O of the sphere. But if we join the all the vertices A, B, C & D of the spherical rectangle ABCD by the straight lines then we obtain a corresponding plane rectangle ABCD which exerts a solid angle equal to that subtended by the spherical rectangle ABCD at the centre of sphere. Now, the normal height $OO' = h$ of the plane rectangle ABCD with centre O' from the centre O of the sphere is found out as follows.

In right $\Delta AO'O$ ($\perp$ to the plane of paper, Fig. 2),

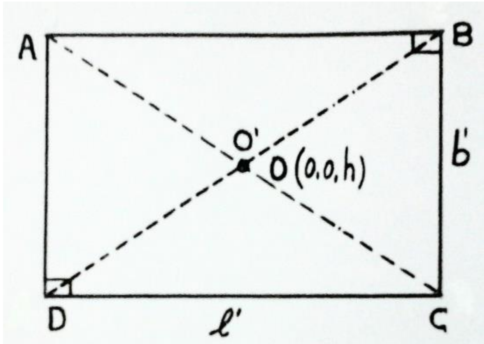

Figure 2: Plane rectangle ABCD is obtained by joining all the vertices A, B, C & D of spherical rectangle ABCD by the straight lines. The centre point O $(0, 0, h)$ is lying at a height h perpendicularly outwards to the plane of paper.

$$OO' = \sqrt{(OA)^2 - (AO')^2} = \sqrt{(OA)^2 - \left(\frac{AC}{2}\right)^2} \qquad \left(\because AO' = O'C = \frac{AC}{2}\right)$$

$$h = \sqrt{(R)^2 - \left(\frac{2R\sqrt{\sin^2\left(\frac{l}{2R}\right)+\sin^2\left(\frac{b}{2R}\right)}}{2}\right)^2} \qquad \text{(from Eq(1))}$$

$$= \sqrt{R^2 - R^2\left(\sin^2\left(\frac{l}{2R}\right)+\sin^2\left(\frac{b}{2R}\right)\right)} = R\sqrt{1-\sin^2\left(\frac{l}{2R}\right)-\sin^2\left(\frac{b}{2R}\right)}$$

$$h = R\sqrt{\cos^2\left(\frac{b}{2R}\right)-\sin^2\left(\frac{l}{2R}\right)} \quad \dots\dots\dots\dots\dots\dots(4)$$

It is worth noticing that the ***cone of vision*** (*i.e. an imaginary cone formed by straight lines starting from the observation point and passing through all the points of a given object in 3D space* [2]) of both the spherical and plane rectangles are the same. Therefore, the solid angle (ω) subtended by the spherical rectangle ABCD at the centre of sphere is equal to the solid angle subtended by corresponding plane rectangle ABCD at the centre O, which is obtained from HCR's Theory of Polygon according to which the solid angle (ω) subtended by any rectangular plane having length & width, l & b respectively at any point, lying on the perpendicular, at a normal height h from the centre is given from the standard formula [3,4],

$$\omega = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2+4h^2)(b^2+4h^2)}}\right)$$

Now by setting the corresponding values, l, b and h in the above expression, the solid angle subtended by the plane rectangle ABCD at the centre of sphere is obtained as follows

$$\omega = 4\sin^{-1}\left(\frac{l'b'}{\sqrt{\left(l'^2+4h^2\right)\left(b'^2+4h^2\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\left(2R\sin\left(\frac{l}{2R}\right)\right)\left(2R\sin\left(\frac{b}{2R}\right)\right)}{\sqrt{\left(\left(2R\sin\left(\frac{l}{2R}\right)\right)^2+4\left(R\sqrt{\cos^2\left(\frac{b}{2R}\right)-\sin^2\left(\frac{l}{2R}\right)}\right)^2\right)\left(\left(2R\sin\left(\frac{b}{2R}\right)\right)^2+4\left(R\sqrt{\cos^2\left(\frac{b}{2R}\right)-\sin^2\left(\frac{l}{2R}\right)}\right)^2\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{4R^2\sin\left(\frac{l}{2R}\right)\sin\left(\frac{b}{2R}\right)}{4R^2\sqrt{\left(\sin^2\left(\frac{l}{2R}\right)+\cos^2\left(\frac{b}{2R}\right)-\sin^2\left(\frac{l}{2R}\right)\right)\left(\sin^2\left(\frac{b}{2R}\right)+\cos^2\left(\frac{b}{2R}\right)-\sin^2\left(\frac{l}{2R}\right)\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sin\left(\frac{l}{2R}\right)\sin\left(\frac{b}{2R}\right)}{\sqrt{\left(\cos^2\left(\frac{b}{2R}\right)\right)\left(1-\sin^2\left(\frac{l}{2R}\right)\right)}}\right) = 4\sin^{-1}\left(\frac{\sin\left(\frac{l}{2R}\right)\sin\left(\frac{b}{2R}\right)}{\sqrt{\left(\cos^2\left(\frac{b}{2R}\right)\right)\left(\cos^2\left(\frac{l}{2R}\right)\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sin\left(\frac{l}{2R}\right)\sin\left(\frac{b}{2R}\right)}{\cos\left(\frac{l}{2R}\right)\cos\left(\frac{b}{2R}\right)}\right)$$

$$\Rightarrow\ \omega = 4\sin^{-1}\left(\tan\left(\frac{l}{2R}\right)\tan\left(\frac{b}{2R}\right)\right) \qquad \dots\dots\dots\dots..(5)$$

Above Eq.(5) is the required expression for calculating the solid angle subtended by any spherical rectangle having length l & width b (each as a great circle arc) at the centre of a spherical surface with a radius R.

Hence, the area (A) covered by the spherical rectangle ABCD is given as

$$A = \omega \times (\text{radius})^2 = \omega R^2 = 4\sin^{-1}\left(\tan\left(\frac{l}{2R}\right)\tan\left(\frac{b}{2R}\right)\right)\times R^2$$

$$A = 4R^2\sin^{-1}\left(\tan\left(\frac{l}{2R}\right)\tan\left(\frac{b}{2R}\right)\right) \quad \forall\ l, b, R > 0\ \&\ l + b < \pi R \qquad \dots\dots\dots\dots..(6)$$

Above Eq.(6) is the required expression for calculating the area covered by any spherical rectangle having length l & width b (each as a great circle arc) on a spherical surface with a radius R.

3. Illustrative Numerical Example

This example is based on all above articles which are very practical and directly & simply applicable to calculate the different parameters of any spherical rectangle such as the interior angle & the area covered by it.

Example: Calculate the area & each of the interior angles of a spherical rectangle, having its length & width (each as a great circle arc) of 15 & 4 units respectively, on a spherical surface with a radius 40 units.

Sol. In this case, we have

$$R = 40 \text{ units}, \quad l = 15 \text{ units}, \quad b = 4 \text{ units} \Rightarrow \theta = ? \,\&\, Area\ (A) = ?$$

Now, each of the interior angles (θ) of the given spherical rectangle can be easily calculated from (3) as follows

$$\theta = \cos^{-1}\left(\frac{\cos\left(2\sin^{-1}\left(\sqrt{\sin^2\left(\frac{l}{2R}\right)+\sin^2\left(\frac{b}{2R}\right)}\right)\right)-\cos\left(\frac{l}{R}\right)\cos\left(\frac{b}{R}\right)}{\sin\left(\frac{l}{R}\right)\sin\left(\frac{b}{R}\right)}\right)$$

Now by setting the corresponding values of R, l & b, we get

$$\theta = \cos^{-1}\left(\frac{\cos\left(2\sin^{-1}\left(\sqrt{\sin^2\left(\frac{15}{2(40)}\right)+\sin^2\left(\frac{4}{2(40)}\right)}\right)\right)-\cos\left(\frac{15}{40}\right)\cos\left(\frac{4}{40}\right)}{\sin\left(\frac{15}{40}\right)\sin\left(\frac{4}{40}\right)}\right) \approx 90.543994^o$$

$$\boldsymbol{\theta \approx 90^o 32' 38.38''} \Rightarrow \boldsymbol{\theta > 90^o} \qquad \text{(Property of a spherical rectangle)}$$

Now, the area (A) covered by the spherical rectangle on the spherical surface is given from (6) as follows

$$\boldsymbol{A} = 4R^2\sin^{-1}\left(\tan\left(\frac{l}{2R}\right)\tan\left(\frac{b}{2R}\right)\right) = 4(40)^2\sin^{-1}\left(\tan\left(\frac{15}{2(40)}\right)\tan\left(\frac{4}{2(40)}\right)\right)$$

$$= 6400\sin^{-1}\left(\tan\left(\frac{3}{16}\right)\tan\left(\frac{1}{20}\right)\right) \approx \mathbf{60.76471331\ unit^2}$$

While the solid angle subtended by the spherical rectangle at the centre of sphere is given from (5) as follows

$$\boldsymbol{\omega} = 4\sin^{-1}\left(\tan\left(\frac{l}{2R}\right)\tan\left(\frac{b}{2R}\right)\right) = 4\sin^{-1}\left(\tan\left(\frac{3}{16}\right)\tan\left(\frac{1}{20}\right)\right) \approx \boldsymbol{0.037977945\ sr}$$

The above value of area implies that the given spherical rectangle covers ≈ 60.76471331 unit^2 of the total surface area $= 4\pi(40)^2 \approx 20106.19298$ unit^2& subtends a solid angle $\approx 0.037977945\ sr$ at the centre of the sphere with a radius 40 units**.**

Conclusion: All the results presented in this work have been derived using elementary geometric and trigonometric principles. The analytical formulae obtained are simple, practical, and straightforward to apply for determining the fundamental parameters of a spherical rectangle, including the solid angle, surface area covered on the sphere, and interior angles. These results are also directly applicable to the corresponding plane rectangle formed by joining the vertices of the spherical rectangle with straight line segments, enabling the evaluation of its geometric parameters. Furthermore, the derived expressions can be extended to the analysis of a right pyramid formed by joining the vertices of a spherical rectangle to the centre of the sphere. In this context, the formulae allow for the analytical computation of key quantities such as the normal height, the angles between consecutive lateral edges, and the area of the plane rectangular base. As a result, the presented framework provides a unified and efficient approach for analysing spherical rectangles, their planar counterparts, and the associated pyramidal structures.

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering) M.M.M. University of Technology, Gorakhpur-273010 (UP) India* *16 Feb 2015*

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