# Geometric Analysis of a Spherical Triangle

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## 1. Introduction

A spherical triangle is defined by three vertices lying on the surface of a sphere, with each side formed by an arc of a great circle [1,2]. Unlike a plane triangle, the sum of the interior angles of a spherical triangle exceeds $180^o$ (a fundamental property illustrated in Fig. 1. below). In this paper, the principal geometric parameters of a spherical triangle are derived using elementary geometry and trigonometry. The resulting formulae are simple, practical, and straightforward to apply for computing quantities such as the solid angle subtended at the centre [3,4], the spherical surface area enclosed [5], and the interior angles [6,7]. The analysis is further extended to the corresponding plane triangle obtained by joining the vertices of the spherical triangle with straight line segments, allowing for the evaluation of its geometric properties. In addition, the derived relations are applied to the right pyramid formed by connecting the vertices of the spherical triangle to the centre of the sphere, enabling the analytical determination of parameters such as the normal height, the angles between consecutive lateral edges, and the area of the base.

## 2. Analysis of spherical triangle given all of its sides

Consider any spherical $\Delta ABC$ having all its sides (each as a great circle arc) of lengths $a, b$ & $c$ $(\forall\ a \leq b \leq c)$ on a spherical surface with a radius $R$ such that its interior angles are $A, B$ & $C$ $(\forall\ A + B + C > \pi)$ (as shown in the figure 1).

### 2.1. Interior angles $A, B$ & $C$ of spherical triangle

We know that each interior angle of a spherical triangle is the angle between the planes of great circle arcs representing any two of its consecutive sides. Now, join the vertices A, B & C by straight lines to obtain a corresponding plane $\Delta ABC$ (as shown by the dotted lines AB, BC & CA). Similarly, we can extend the straight lines OA, OB & OC to obtain a plane $\Delta A'B'C'$ which is the base of tetrahedron OA'B'C'.

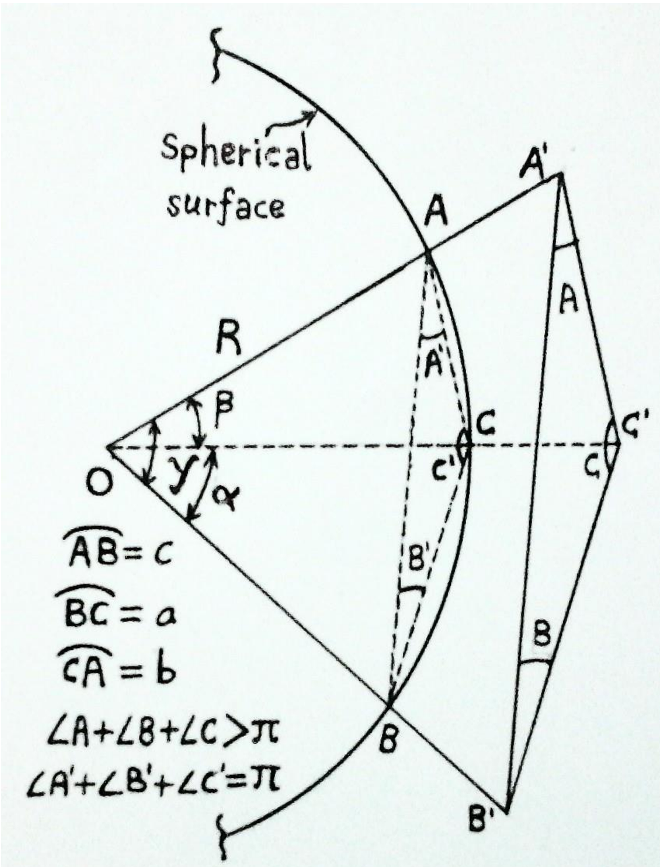


**Figure 1: A spherical triangle ABC having its sides (each as a great circle arc) of lengths $a, b$ & $c$ & its interior angles $A$, $B$ & $C$ respectively. A plane $\Delta ABC$ corresponding to the spherical triangle ABC is obtained by joining the vertices A, B & C by the straight lines.**

Now, consider the tetrahedron OA'B'C' having angles $\alpha, \beta$ & $\gamma$ between its consecutive lateral edges OB' & OC', OA' & OC' and OA' & OB' respectively. Now the angles $\alpha, \beta$ & $\gamma$ are the angles subtended by the sides (each as a great circle arc) of spherical triangle at the centre of sphere which are determined as follows

$$\boldsymbol{\alpha} = \frac{\text{Arc length}}{\text{Radius}} = \frac{\boldsymbol{a}}{\boldsymbol{R}}, \quad \boldsymbol{\beta} = \frac{\boldsymbol{b}}{\boldsymbol{R}} \ \& \ \boldsymbol{\gamma} = \frac{\boldsymbol{c}}{\boldsymbol{R}}$$

Now the interior angles $A$, $B$ & $C$ of spherical triangle that are also the angles between consecutive lateral triangular faces of the tetrahedron OA'B'C' meeting at the vertex O (i.e. the centre of sphere), are determined by using Inverse Cosine Formula [6] according to which if $\alpha, \beta$ & $\gamma$ are the angles between consecutive lateral edges meeting at any of four vertices of a tetrahedron then the angle (opposite to $\alpha$) between two lateral faces is given as follows

$$\theta = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)$$

$$\therefore\ \boldsymbol{A} = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos\frac{a}{R} - \cos\frac{b}{R}\cos\frac{c}{R}}{\sin\frac{b}{R}\sin\frac{c}{R}}\right) \quad \dots\dots\dots\dots..(1)$$

$$\text{Similarly,}\quad \boldsymbol{B} = \cos^{-1}\left(\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos\frac{b}{R} - \cos\frac{a}{R}\cos\frac{c}{R}}{\sin\frac{a}{R}\sin\frac{c}{R}}\right) \quad \dots\dots\dots\dots..(2)$$

$$\boldsymbol{C} = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) = \cos^{-1}\left(\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}\right) \quad \dots\dots\dots\dots..(3)$$

**2.2. Area of spherical triangle**

In order to calculate area covered by spherical triangle ABC, let's first calculate the solid angle subtended by it at the centre of sphere. But if we join the vertices A, B & C of the spherical triangle by straight lines then we obtain a corresponding plane $\Delta ABC$ which exerts a solid angle equal to that subtended by the spherical triangle at the centre of sphere. Thus, we would calculate the solid angle subtended by the corresponding plane $\Delta ABC$ at the centre of sphere by two methods (1) Analytic, and (2) Graphical as given below.

**2.2.1. Analytic method for calculation of solid angle**

**1. Sides of corresponding plane $\boldsymbol{\Delta ABC}$:** Let the sides of corresponding plane $\Delta ABC$ be $a', b'$ & $c'$ opposite to its angles $A'$, $B'$ & $C'$ respectively ($\forall\ A' + B' + C' = \pi$).

In isosceles $\Delta OBC$ (Fig. 1 above),

$$\Rightarrow \sin\frac{\angle BOC}{2} = \frac{\left(\frac{BC}{2}\right)}{OB} \ \Rightarrow\ \sin\frac{\alpha}{2} = \frac{\left(\frac{a'}{2}\right)}{R} \ \Rightarrow\ a' = 2R\sin\frac{\alpha}{2} = 2R\sin\frac{a}{2R} \quad \left(\because \alpha = \frac{a}{R}\right)$$

$$\boldsymbol{a' = 2R\sin\frac{a}{2R}}$$

$$\text{Similarly,}\quad \boldsymbol{b' = 2R\sin\frac{b}{2R}} \ \&\ \boldsymbol{c' = 2R\sin\frac{c}{2R}}$$

Now from HCR's Axiom-2**,** we know that the perpendicular drawn from the centre of the sphere always passes through circumscribed centre of the plane triangle [7] (in this case plane $\Delta ABC$) obtained by joining the vertices of a spherical triangle to the centre of sphere (as shown in Figure 2 below).

Hence, the circumscribed radius ($R'$) of plane $\Delta ABC$ having its sides $a', b'$ & $c'$ (all known) is given as follows

$$R' = \frac{a'b'c'}{4\Delta}$$

Where,

Area of plane $\Delta ABC$, $\Delta = \sqrt{s(s-a')(s-b')(s-c')}$

$$s = \frac{a' + b' + c'}{2}$$

Hence, the normal height ($h$) of plane $\Delta ABC$ from the centre O of the sphere is given as follows

In right $\Delta OO'A$ (Fig. 2),

$$OO' = \sqrt{(OA)^2 - (AO')^2}$$

$$\therefore\ \boldsymbol{h = \sqrt{R^2 - R'^2}}$$

Now, in right $\Delta O'MB$ (Fig. 2),

$$\boldsymbol{O'M} = \sqrt{(BO')^2 - (MB)^2} = \sqrt{R'^2 - \left(\frac{a'}{2}\right)^2}$$

$$= \frac{\boldsymbol{\sqrt{4R'^2 - a'^2}}}{\boldsymbol{2}} \qquad \left(\because\ CM = MB = \frac{a'}{2}\right)$$

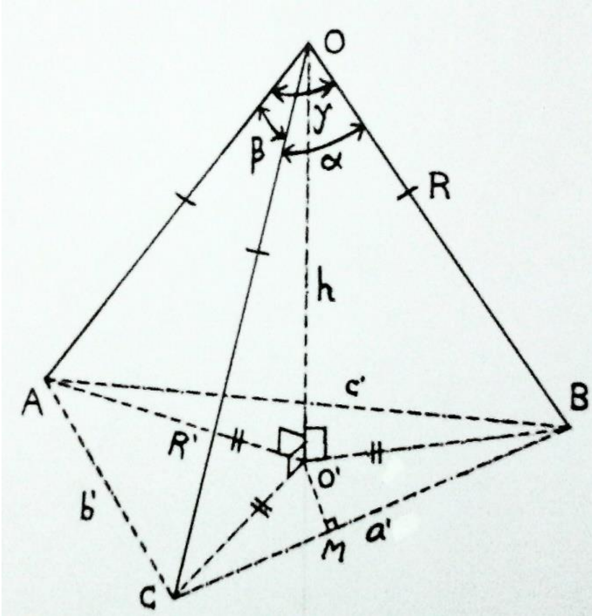


**Figure 2: The perpendicular OO' drawn from the centre O of the sphere to the plane $\Delta ABC$ always passes through its circumscribed centre O' according to HCR's Axiom-2.**

Now, from HCR's Theory of Polygon [3,4], the solid angle subtended by the right triangle having its orthogonal sides $a$ & $b$ at any point lying at a height $h$ on the vertical axis passing through the vertex common to the side $a$ & the hypotenuse is given from standard formula as

$$\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2 + a^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2 + a^2}}\right)\left(\frac{h}{\sqrt{h^2 + a^2}}\right)\right\}$$

Hence, the solid angle ($\omega_{\Delta O'BC}$) subtended by the isosceles $\Delta O'BC$ at the centre O of the sphere

$$= \omega_{\Delta O'MB} + \omega_{\Delta O'MC} = 2(\omega_{\Delta O'MB}) = 2(\text{solid angle subtended by the right } \Delta O'MB)$$

Now, setting the corresponding values in the above formula, we obtain

$$\omega_{\Delta O'BC} = 2\left[\sin^{-1}\left(\frac{\frac{a'}{2}}{\sqrt{\left(\frac{a'}{2}\right)^2 + \left(\frac{\sqrt{4R'^2 - a'^2}}{2}\right)^2}}\right) - \sin^{-1}\left\{\left(\frac{\frac{a'}{2}}{\sqrt{\left(\frac{a'}{2}\right)^2 + \left(\frac{\sqrt{4R'^2 - a'^2}}{2}\right)^2}}\right)\left(\frac{\sqrt{R^2 - R'^2}}{\sqrt{\left(\sqrt{R^2 - R'^2}\right)^2 + \left(\frac{\sqrt{4R'^2 - a'^2}}{2}\right)^2}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{a'}{2\sqrt{\frac{a'^2}{4} + R'^2 - \frac{a'^2}{4}}}\right) - \sin^{-1}\left\{\left(\frac{a'}{2\sqrt{\frac{a'^2}{4} + R'^2 - \frac{a'^2}{4}}}\right)\left(\frac{\sqrt{R^2 - R'^2}}{\sqrt{R^2 - \frac{1}{4}a'^2}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left\{\left(\frac{a'}{2R'}\right)\left(\frac{\sqrt{R^2 - R'^2}}{\sqrt{R^2 - \frac{1}{4}\left(2R\sin\left(\frac{a}{2R}\right)\right)^2}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left\{\left(\frac{a'}{2R'}\right)\left(\frac{\sqrt{R^2 - R'^2}}{R\cos\left(\frac{a}{2R}\right)}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left(\left(\frac{a'}{2R'}\right)\sec\left(\frac{a}{2R}\right)\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\omega_{\Delta O'BC} = 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left(\left(\frac{a'}{2R'}\right)\sec\left(\frac{a}{2R}\right)\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right] = \omega_1 \text{ (let)} \qquad \ldots\ldots\ldots\ldots..(4)$$

Similarly, we have

$$\omega_{\Delta O'AC} = 2\left[\sin^{-1}\left(\frac{b'}{2R'}\right) - \sin^{-1}\left(\left(\frac{b'}{2R'}\right)\sec\left(\frac{a}{2R}\right)\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right] = \omega_2 \text{ (let)} \qquad \ldots\ldots\ldots\ldots..(5)$$

$$\omega_{\Delta O'AB} = 2\left[\sin^{-1}\left(\frac{c'}{2R'}\right) - \sin^{-1}\left(\left(\frac{c'}{2R'}\right)\sec\left(\frac{a}{2R}\right)\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right] = \omega_3 \text{ (let)} \qquad \ldots\ldots\ldots\ldots..(6)$$

Now, we must check out the nature of plane $\Delta ABC$ whether it is an acute, a right or an obtuse triangle. Since the largest side is $c'$ among $a'$ & $b'$ hence we can determine the largest angle $C'$ of plane $\Delta ABC$ using cosine formula as follows

$$\cos C' = \frac{a'^2 + b'^2 - c'^2}{2a'b'}$$

Thus, there arise two cases to calculate the solid angle subtended by the plane $\Delta ABC$ at the centre of sphere and so by the spherical triangle ABC as follows.

**Case 1: Corresponding plane $\Delta ABC$ is an acute or a right triangle** ($\forall\ c' \geq b' \geq a'\ \&\ C' \leq 90^o$)

In this case, the foot point O' of the perpendicular drawn from the centre of sphere to the acute plane $\Delta ABC$ lies within or on the boundary of this triangle. All the values of solid angles $\omega_1$ , $\omega_2$ & $\omega_3$ corresponding to all the sides $a', b'$ & $c'$ respectively of acute plane $\Delta ABC$ are taken as positive. Hence, the solid angle ($\omega_{\Delta ABC}$) subtended by the acute plane $\Delta ABC$ at the centre of sphere is given as the sum of magnitudes of solid angles from the above equations (4), (5), and (6) as follows

$$\omega = \omega_{\Delta ABC} = \omega_{\Delta O'BC} + \omega_{\Delta O'AC} + \omega_{\Delta O'AB} = \omega_1 + \omega_2 + \omega_3$$

$$\therefore \text{Area covered by the spherical triangle } \boldsymbol{A}BC = \omega R^2 = R^2(\omega_1 + \omega_2 + \omega_3) \qquad \ldots\ldots\ldots\ldots..(7)$$

**Case 2: Corresponding plane $\Delta ABC$ is an obtuse triangle** ($\forall\ c' > b' \geq a'\ \&\ C' > 90^o$)

In this case, the foot point O' of the perpendicular drawn from the centre of sphere to the obtuse plane $\Delta ABC$ lies outside the boundary of this triangle (as shown in the Figure 3 below). In this case, solid angles $\omega_1$ & $\omega_2$ corresponding to the sides $a'$ & $b'$ respectively are taken as positive while solid angle $\omega_3$ corresponding to the largest side $c'$ of obtuse plane $\Delta ABC$ is taken as negative. Hence, the solid angle ($\omega_{\Delta ABC}$) subtended by the obtuse plane $\Delta ABC$ at the centre of sphere is given as the algebraic sum of solid angles [3,4] as follows

$$\omega = \omega_{\Delta ABC} = \omega_{\Delta O'BC} + \omega_{\Delta O'AC} - \omega_{\Delta O'AB} = \omega_1 + \omega_2 - \omega_3$$

 

$$\therefore \text{Area covered by the spherical triangle } ABC = \omega R^2 = R^2(\omega_1 + \omega_2 - \omega_3) \quad \dots\dots\dots\dots..(8)$$

### 2.2.2. Graphical method for calculation of solid angle

In this method, we first plot the diagram of corresponding plane $\Delta ABC$ having known sides $a', b'$ & $c'$ & then specify the location of foot of perpendicular (F.O.P.) i.e. the circumscribed centre of plane $\Delta ABC$ then draw the perpendiculars from circumscribed centre to all the opposite sides to divide it (i.e. plane $\Delta ABC$) into elementary right triangles. Now, using standard formula-1 of right triangle for calculating the solid angle subtended by each of the elementary right triangles at the centre of sphere which is given by following formula [3,4],

$$\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2+a^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+a^2}}\right)\left(\frac{h}{\sqrt{h^2+a^2}}\right)\right\}$$

Then find out the algebraic sum $(\omega)$ of the solid angles subtended by the elementary right triangles at the centre of the sphere & hence the area covered by the spherical triangle ABC is given as follows

$$\text{Area covered by the spherical triangle } ABC = \omega R^2 \quad \dots\dots\dots\dots..(9)$$

## 3. Analysis of spherical triangle (when two of its sides & an interior angle between them are known)

Consider any spherical triangle $\Delta ABC$, having its two sides (each as a great circle arc) of lengths $a$ & $b$ and an interior angle $C$ between them, on a spherical surface with a radius $R$. Now we can easily determine all its unknown parameters i.e. unknown side $(c)$, two interior angles $A$ & $B$ and area covered by it.

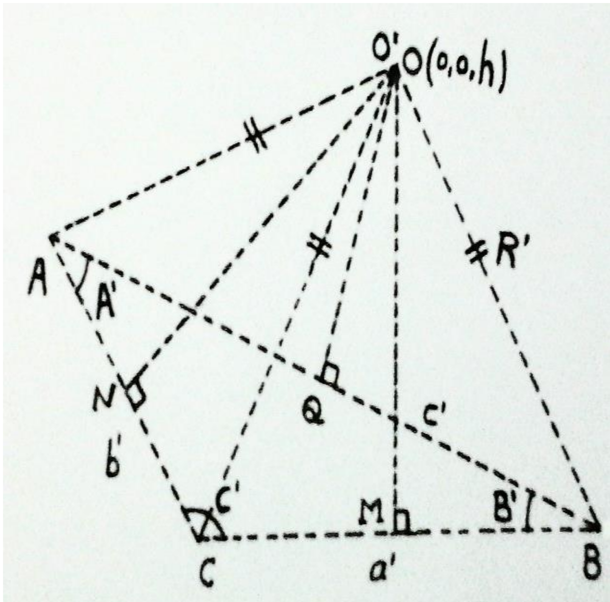

**Figure 3: Corresponding plane $\Delta ABC$ is an obtuse triangle $\forall\ c' > b' \geq a'$ & $C' > 90^o$. The centre O $(0, 0, h)$ of the sphere is lying at a height *h* perpendicular outwards to the plane of paper.**

Now the angles $\alpha, \beta$ & $\gamma$ are the angles subtended by the sides (each as a great circle arc) of spherical triangle at the centre of sphere which are determined as follows (as shown in Figure 2 above),

$$\boldsymbol{\alpha} = \frac{\textbf{Arc length}}{\textbf{Radius}} = \frac{\boldsymbol{a}}{\boldsymbol{R}}, \quad \boldsymbol{\beta} = \frac{\boldsymbol{b}}{\boldsymbol{R}} \quad \& \quad \boldsymbol{\gamma} = \frac{\boldsymbol{c}}{\boldsymbol{R}} = ?\ (\because \boldsymbol{c} = ?)$$

Now, apply HCR's Inverse cosine formula [6] for known interior angle $C$ as follows

$$C = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) = \cos^{-1}\left(\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}\right)$$

$$\Rightarrow \quad \frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}} = \cos C$$

$$\Rightarrow \quad \cos\frac{c}{R} = \sin\frac{a}{R}\sin\frac{b}{R}\cos C + \cos\frac{a}{R}\cos\frac{b}{R}$$

$$\therefore \ c = R\cos^{-1}\left(\sin\frac{a}{R}\sin\frac{b}{R}\cos C + \cos\frac{a}{R}\cos\frac{b}{R}\right) \quad \& \quad \gamma = \frac{c}{R} = \cos^{-1}\left(\sin\frac{a}{R}\sin\frac{b}{R}\cos C + \cos\frac{a}{R}\cos\frac{b}{R}\right)$$

Similarly, applying HCR's Inverse cosine formula [6] for calculating the unknown interior angle $A$ & $B$ as follows

$$\boldsymbol{A} = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos\frac{a}{R} - \cos\frac{b}{R}\cos\frac{c}{R}}{\sin\frac{b}{R}\sin\frac{c}{R}}\right) \quad \dots\dots\dots\dots..(10)$$

$$\boldsymbol{B} = \cos^{-1}\left(\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos\frac{b}{R} - \cos\frac{a}{R}\cos\frac{c}{R}}{\sin\frac{a}{R}\sin\frac{c}{R}}\right) \quad \dots\dots\dots\dots..(11)$$

### 3.1. Area of spherical triangle

In order to calculate area covered by the spherical triangle ABC, let's first calculate the solid angle subtended by it at the centre of sphere. But if we join the vertices A, B & C of spherical triangle by the straight lines then we obtain a corresponding plane $\Delta ABC$ which exerts a solid angle equal to that subtended by the spherical triangle ABC at the centre of sphere. Now all the sides $a'$, $b'$ & $c'$ of the plane $\Delta ABC$ can be calculated by following the previous method (as mentioned above) as follows

$$\boldsymbol{a'} = \boldsymbol{2R}\sin\frac{\boldsymbol{a}}{\boldsymbol{2R}}, \quad \boldsymbol{b'} = \boldsymbol{2R}\sin\frac{\boldsymbol{b}}{\boldsymbol{2R}} \ \ \& \ \boldsymbol{c'} = \boldsymbol{2R}\sin\frac{\boldsymbol{c}}{\boldsymbol{2R}}$$

Thus, we can calculate the solid angle subtended by the corresponding plane $\Delta ABC$ & so by the spherical triangle ABC at the centre of sphere by following the previous two methods (1) Analytic, and (2) Graphical. The same procedure as mentioned above the). Hence, we can calculate the area covered by the given spherical triangle.

### 3.2. Illustrative Examples

These examples are based on all above articles which are very practical and directly and simply applicable to calculate the different parameters of a spherical triangle. For ease of understanding and the calculations, the value of side $c$ of the spherical triangle ABC is taken as the largest one.

**Example 1**: Calculate the area & each of the interior angles of a spherical triangle, having its sides (each as a great circle arc) of lengths 12, 18 & 20 units, on the spherical surface with a radius 50 units.

**Sol.** Here, we have

$$R = 50\,units, \ \ a = 12\,units, \ \ b = 18\,units, \ \ c = 20\,units \ \Rightarrow \ A, B, C = ? \ \& \ \text{Area} = ?$$

Now, all the interior angles of spherical triangle can be easily calculated by using inverse cosine formula as follows

$$\Rightarrow \ \boldsymbol{A} = cos^{-1}\left(\frac{cos\frac{a}{R} - cos\frac{b}{R}cos\frac{c}{R}}{sin\frac{b}{R}sin\frac{c}{R}}\right) = cos^{-1}\left(\frac{cos\frac{12}{50} - cos\frac{18}{50}cos\frac{20}{50}}{sin\frac{18}{50}sin\frac{20}{50}}\right) \approx \boldsymbol{37.165231^o} \approx \boldsymbol{37^o9'54.83''}$$

$$\boldsymbol{B} = cos^{-1}\left(\frac{cos\frac{b}{R} - cos\frac{a}{R}cos\frac{c}{R}}{sin\frac{a}{R}sin\frac{c}{R}}\right) = cos^{-1}\left(\frac{cos\frac{18}{50} - cos\frac{12}{50}cos\frac{20}{50}}{sin\frac{12}{50}sin\frac{20}{50}}\right) \approx \boldsymbol{63.54656423^o} \approx \boldsymbol{63^o32'47.63''}$$

$$\boldsymbol{C} = cos^{-1}\left(\frac{cos\frac{c}{R} - cos\frac{a}{R}cos\frac{b}{R}}{sin\frac{a}{R}sin\frac{b}{R}}\right) = cos^{-1}\left(\frac{cos\frac{20}{50} - cos\frac{12}{50}cos\frac{18}{50}}{sin\frac{12}{50}sin\frac{18}{50}}\right) \approx \boldsymbol{81.76846174^o} \approx \boldsymbol{81^o46'6.46''}$$

$$\Rightarrow \boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} > \boldsymbol{180^o} \quad \text{(Property of spherical triangle)}$$

Now, the sides of corresponding plane $\Delta ABC$ are calculated as follows

$$a' = 2Rsin\frac{a}{2R} = 2(50)sin\frac{12}{100} \approx 11.97122073$$

$$b' = 2Rsin\frac{b}{2R} = 2(50)sin\frac{18}{100} \approx 17.90295734$$

$$c' = 2Rsin\frac{c}{2R} = 2(50)sin\frac{20}{100} \approx 19.86693308$$

$$s = \text{semiperimeter} = \frac{a'+b'+c'}{2} \approx \frac{11.97122073+17.90295734+19.86693308}{2} \approx 24.87055558$$

Area of plane $\Delta ABC$ is given as

$$\Delta = \sqrt{s(s-a')(s-b')(s-c')}$$

$$\approx \sqrt{24.87055558(24.87055558-11.97122073)(24.87055558-17.90295734)(24.87055558-19.86693308)}$$
$$\approx \sqrt{24.87055558 \times 12.89933485 \times 6.96759824 \times 5.0036225} \approx 105.7572673$$

$$\therefore \text{Circumscribed radius}, \boldsymbol{R'} = \frac{a'b'c'}{4\Delta} \approx \frac{11.97122073 \times 17.90295734 \times 19.86693308}{4 \times 105.7572673} \approx 10.06523299$$

Since, the largest side of plane $\Delta ABC$ is $c' \approx 19.86693308$ hence the largest angle of the plane $\Delta ABC$ is $C'$ which is calculated by using cosine formula as follows

$$cosC' = \frac{a'^2+b'^2-c'^2}{2a'b'} \Rightarrow C' = \cos^{-1}\left(\frac{a'^2+b'^2-c'^2}{2a'b'}\right)$$

$$C' \approx \cos^{-1}\left(\frac{(11.97122073)^2+(17.90295734)^2-(19.86693308)^2}{2(11.97122073)(17.90295734)}\right) \approx 80.71882239^o < 90^o$$

Hence, the plane $\Delta ABC$ is an acute angled triangle.

**Note:** If all the interior angles $A, B$ & $C$ of any spherical triangle are acute then definitely the corresponding plane $\Delta ABC$ will also be an acute angled triangle. It is not required to check it out by calculating the largest angle $C'$ of plane $\Delta ABC$. (As in above example 1, we need not calculate the largest angle $C'$ to check out the nature of the plane $\Delta ABC$ we can directly say on the basis of values of interior angles A, B & C of the spherical surface that the plane $\Delta ABC$ is an acute if each of A, B & C is an acute angle).

Hence the foot of perpendicular (F.O.P.) drawn from the centre of sphere to the plane $\Delta ABC$ will lie within the boundary of plane $\Delta ABC$ (as shown in Figure 2 above) hence, the solid angle subtended by it at the centre of sphere is calculated as follows

$$\boldsymbol{\omega_1} = 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left(\left(\frac{a'}{2R'}\right)sec\frac{a}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{11.97122073}{2(10.06523299)}\right) - \sin^{-1}\left(\left(\frac{11.97122073}{2(10.06523299)}\right)sec\frac{12}{2(50)}\sqrt{1-\left(\frac{10.06523299}{50}\right)^2}\right)\right]$$
$$\approx \boldsymbol{0.019716827\ sr}$$

$$\boldsymbol{\omega_2} = 2\left[\sin^{-1}\left(\frac{b'}{2R'}\right) - \sin^{-1}\left(\left(\frac{b'}{2R'}\right)sec\frac{b}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{17.90295734}{2(10.06523299)}\right) - \sin^{-1}\left(\left(\frac{17.90295734}{2(10.06523299)}\right)sec\frac{18}{2(50)}\sqrt{1-\left(\frac{10.06523299}{50}\right)^2}\right)\right]$$

$$\approx \mathbf{0.016922497}\ \boldsymbol{sr}$$

$$\boldsymbol{\omega_3} = 2\left[\sin^{-1}\left(\frac{c'}{2R'}\right) - \sin^{-1}\left(\left(\frac{c'}{2R'}\right)sec\frac{c}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{19.86693308}{2(10.06523299)}\right) - \sin^{-1}\left(\left(\frac{19.86693308}{2(10.06523299)}\right)sec\frac{20}{2(50)}\sqrt{1-\left(\frac{10.06523299}{50}\right)^2}\right)\right]$$

$$\approx \mathbf{0.00664932472}\ \boldsymbol{sr}$$

Note: In this case, all the values of solid angles $\omega_1$ , $\omega_2$ & $\omega_3$ corresponding to all the sides $a'$, $b'$ & $c'$ respectively of the acute plane $\Delta ABC$ are taken as positive.

Hence, the solid angle $(\omega_{\Delta ABC})$ subtended by the acute plane $\Delta ABC$ or spherical triangle ABC at the centre of sphere is given as the sum of magnitudes of solid angles as follows

$$\boldsymbol{\omega} = \omega_1 + \omega_2 + \omega_3 \approx 0.019716827 + 0.016922497 + 0.00664932472 \approx \mathbf{0.043288648}\ \boldsymbol{sr}$$

$\therefore$ **Area covered by the spherical triangle** $\boldsymbol{ABC = \omega R^2} \approx 0.043288648 \times 50^2$

$\approx \mathbf{108.2216218\ unit^2}$ ***Ans.***

The above value of area implies that the given spherical triangle covers $\approx 108.2216218\,unit^2$ of the total surface area $= 4\pi(50)^2 \approx 31415.92654\,unit^2$ & subtends a solid angle $\approx 0.043288648\,sr$ at the centre of the sphere with a radius 50 units.

**Example 2**: A spherical triangle, having its two sides (each as a great circle arc) of lengths 25 & 38 units and an interior angle $160^o$ included by them, on the spherical surface with a radius 200 units. Calculate the unknown side, interior angles & the area covered by it.

**Sol.** Here, we have

$$R = 200\,units,\ \ a = 25\,units,\ \ b = 38\,units,\ \ \ C = 160^o = \frac{8\pi}{9}\ \Rightarrow c = ?,\ \ A, B\ = ?\ \&\ \text{Area} = ?$$

Now in order to calculate unknown side c, apply HCR's Inverse cosine formula for known interior angle $C$ as follows

$$C = cos^{-1}\left(\frac{cos\frac{c}{R} - cos\frac{a}{R}cos\frac{b}{R}}{sin\frac{a}{R}sin\frac{b}{R}}\right) \Rightarrow\ c = R\cos^{-1}\left(sin\frac{a}{R}sin\frac{b}{R}cosC + cos\frac{a}{R}cos\frac{b}{R}\right)$$

$$\boldsymbol{c} = 200\cos^{-1}\left(sin\frac{25}{200}sin\frac{38}{200}cos\frac{8\pi}{9} + cos\frac{25}{200}cos\frac{38}{200}\right) \approx \mathbf{62.07679003}$$

Similarly, applying HCR's Inverse cosine formula for calculating the unknown interior angle $A$ & $B$ as follows

$$\boldsymbol{A} = cos^{-1}\left(\frac{cos\frac{a}{R} - cos\frac{b}{R}cos\frac{c}{R}}{sin\frac{b}{R}sin\frac{c}{R}}\right) = cos^{-1}\left(\frac{cos\frac{25}{200} - cos\frac{38}{200}cos\frac{62.07679003}{200}}{sin\frac{38}{200}sin\frac{62.07679003}{200}}\right) \approx \boldsymbol{8^o1'31.68''}$$

$$\boldsymbol{B} = cos^{-1}\left(\frac{cos\frac{b}{R} - cos\frac{a}{R}cos\frac{c}{R}}{sin\frac{a}{R}sin\frac{c}{R}}\right) = cos^{-1}\left(\frac{cos\frac{38}{200} - cos\frac{25}{200}cos\frac{62.07679003}{200}}{sin\frac{25}{200}sin\frac{62.07679003}{200}}\right) \approx \boldsymbol{12^o12'34.43''}$$

$$\Rightarrow \boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} > \boldsymbol{180^o} \quad \text{(Property of spherical triangle)}$$

Now, the **sides of corresponding plane** $\Delta\boldsymbol{ABC}$ are calculated as follows

$$a' = 2Rsin\frac{a}{2R} = 2(200)sin\frac{25}{400} \approx 24.98372714$$

$$b' = 2Rsin\frac{b}{2R} = 2(200)sin\frac{38}{400} \approx 37.94286745$$

$$c' = 2Rsin\frac{c}{2R} = 2(200)sin\frac{62.07679003}{400} \approx 61.82790801$$

$$s = \text{semiperimeter} = \frac{a' + b' + c'}{2} \approx \frac{24.98372714 + 37.94286745 + 61.82790801}{2} \approx 62.3772513$$

**Area of plane** $\Delta\boldsymbol{ABC}$ is given as

$$\Delta = \sqrt{s(s-a')(s-b')(s-c')}$$

$$\approx \sqrt{62.3772513(62.3772513 - 24.98372714)(62.3772513 - 37.94286745)(62.3772513 - 61.82790801)}$$
$$\approx \sqrt{62.3772513 \times 37.39352416 \times 24.43438385 \times 0.54934329} \approx 176.9432188$$

$$\therefore \text{Circumscribed radius}, \boldsymbol{R'} = \frac{a'b'c'}{4\Delta} \approx \frac{24.98372714 \times 37.94286745 \times 61.82790801}{4 \times 176.9432188} \approx 82.80909039$$

Since, the largest side of plane $\Delta ABC$ is $c' \approx 61.82790801$ hence the largest angle of the plane $\Delta ABC$ is $C'$ which is calculated by using cosine formula as follows

$$cosC' = \frac{a'^2 + b'^2 - c'^2}{2a'b'} \Rightarrow C' = \cos^{-1}\left(\frac{a'^2 + b'^2 - c'^2}{2a'b'}\right)$$

$$C' \approx \cos^{-1}\left(\frac{(24.98372714)^2 + (37.94286745)^2 - (61.82790801)^2}{2(24.98372714)(37.94286745)}\right) \approx 158.0797337^o > 90^o$$

Hence, the **plane** $\Delta\boldsymbol{ABC}$ **is an obtuse angled triangle**.

Hence the foot of perpendicular (F.O.P.) drawn from the centre of sphere to the plane $\Delta ABC$ will lie outside the boundary of plane $\Delta ABC$ (See the figure 3 above) hence, the solid angle subtended by it at the centre of sphere is calculated by taking algebraic sum as follows

$$\boldsymbol{\omega_1} = 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left(\left(\frac{a'}{2R'}\right)sec\frac{a}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{24.98372714}{2(82.80909039)}\right)-\sin^{-1}\left(\left(\frac{24.98372714}{2(82.80909039)}\right)sec\frac{25}{2(200)}\sqrt{1-\left(\frac{82.80909039}{200}\right)^2}\right)\right]$$

$$\approx \mathbf{0.026819267}\ \boldsymbol{sr}$$

$$\boldsymbol{\omega_2}=2\left[\sin^{-1}\left(\frac{b'}{2R'}\right)-\sin^{-1}\left(\left(\frac{b'}{2R'}\right)sec\frac{b}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{37.94286745}{2(82.80909039)}\right)-\sin^{-1}\left(\left(\frac{37.94286745}{2(82.80909039)}\right)sec\frac{38}{2(200)}\sqrt{1-\left(\frac{82.80909039}{200}\right)^2}\right)\right]$$

$$\approx \mathbf{0.04021067}\ \boldsymbol{sr}$$

$$\boldsymbol{\omega_3}=2\left[\sin^{-1}\left(\frac{c'}{2R'}\right)-\sin^{-1}\left(\left(\frac{c'}{2R'}\right)sec\frac{c}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{61.82790801}{2(82.80909039)}\right)-\sin^{-1}\left(\left(\frac{61.82790801}{2(82.80909039)}\right)sec\frac{62.07679003}{2(200)}\sqrt{1-\left(\frac{82.80909039}{200}\right)^2}\right)\right]$$

$$\approx \mathbf{0.062927892}\ \boldsymbol{sr}$$

**Note:** In this case, solid angles $\omega_1$ & $\omega_2$ corresponding to the sides $a'$ & $b'$ respectively are taken as positive while solid angle $\omega_3$ corresponding to the largest side $c'$ of obtuse plane $\Delta ABC$ is taken as negative.

Hence, the solid angle $(\omega_{\Delta ABC})$ subtended by the obtuse plane $\Delta ABC$ or spherical triangle ABC at the centre of sphere is given as the algebraic sum of solid angles as follows

$$\omega=\omega_1+\omega_2-\omega_3\approx 0.026819267+0.04021067-0.062927892\approx 0.004102045\ sr$$

$$\therefore \text{Area covered by the spherical triangle } ABC=\omega R^2\approx 0.00646329\times 200^2$$

$$\approx \mathbf{164.0818\ unit^2} \qquad\qquad \boldsymbol{Ans.}$$

The above value of area implies that the given spherical triangle covers $\approx 164.0818\ unit^2$ of the total surface area $=4\pi(200)^2\approx 502654.8246\ unit^2$ & subtends a solid angle $\approx 0.004102045\ sr$ at the centre of the sphere with a radius 200 units.

**Conclusion:** All the results presented in this work are derived using elementary geometry and trigonometry. The obtained formulae are simple and practical for evaluating the key parameters of a spherical triangle, including solid angle, covered surface area, and interior angles. These relations are also applicable to the corresponding plane triangle formed by joining the vertices with straight line segments. In addition, the formulae enable the analytical computation of the main parameters of the right pyramid formed by joining the vertices of a spherical triangle to the centre of the sphere, such as the normal height, angles between consecutive lateral edges, and the area of the plane triangular base.

**Note:** Above articles had been derived *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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