

On Continuous Dimensions and Infrared Mass Generation

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Abstract

We show that dimensional regularization of a generic self-interacting field theory induces an infrared mass scale, when the running spacetime dimension approaches the critical value $D = 4$. The mechanism is universal and does not rely on spontaneous symmetry breaking alone. Instead, it arises from logarithmic corrections generated by renormalization group (RG) flow near the bifurcation point at $\epsilon \equiv 4 - D \rightarrow 0$. We explicitly indicate how dimensional transmutation converts marginal couplings into dynamically generated mass scales. This framework unifies the origin of the Higgs vacuum expectation value and electroweak boson masses with the Yang–Mills mass gap and the QCD scale Λ_{QCD} . Step-by-step derivations are provided, without appealing to nonperturbative assumptions beyond RG consistency. The paper focuses on the regime bordering relativistic quantum field theory and complex dynamics.

Key words: dimensional regularization, electroweak model, Yang-Mills theory, Quantum Chromodynamics, Renormalization Group flow, bifurcations, transition to chaos, entropy production.

1. Introduction

A recurring theme across modern field theory is the emergence of characteristic mass scales from scale-invariant or nearly scale-invariant dynamics. This phenomenon appears in scalar field theories near criticality, in the electroweak sector of the Standard Model, and in non-Abelian gauge theories such as QCD.

The key observation of our paper is that the combined contribution of a) self-interaction and b) dimensional regularization near the upper critical dimension $D = 4$, lead to a finite infrared mass due to:

1. Marginality of couplings,
2. Logarithmic RG flow, and
3. Bifurcation of RG trajectories

We find that this is a generic result applicable to the three major paradigms of quantum field theory, namely,

- Scalar Landau–Ginzburg theory,
- The electroweak Higgs sector,
- Yang–Mills theory and QCD.

The regime described here lies at the border between the low-energy formulation of quantum field theory and the onset of complex dynamics somewhere above the Standard Model scale. Appendix B bridges the gap between the bifurcation mechanism of infrared mass generation and entropy production in the theory of dynamical systems.

2. Generic Self-Interacting Field Theory Near $D = 4$

A. Classical action

Consider a generic bosonic field Φ (scalar or gauge) with a nearly vanishing mass term:

$$S = \int d^D x \left[\frac{1}{2} (\partial\Phi)^2 + \frac{1}{2} m_0^2 \Phi^2 + \frac{g_0}{n!} \Phi^n \right], \quad (1)$$

with $m_0^2 \approx 0$ in appropriately normalized units.

B. Canonical dimensions

In D dimensions one has:

$$[\Phi] = \frac{D-2}{2}, \quad [g_0] = D - n \frac{D-2}{2}. \quad (2)$$

For $n = 4$,

$$[g_0] = 4 - D \equiv \epsilon. \quad (3)$$

It follows from (3) that, at $D = 4$, the interaction is *marginal*.

3. Dimensional Regularization and Renormalization

Next, we introduce a renormalization scale μ and define:

$$g_0 = \mu^\epsilon Z_g g(\mu), \quad m_0^2 = Z_m m^2(\mu). \quad (4)$$

The renormalized mass obeys the following flow equation:

$$\mu \frac{dm^2}{d\mu} = \gamma_m(g) m^2 + \delta(g) \mu^2. \quad (5)$$

The second term embodies the contribution of radiative corrections which contribute to the mass parameter even if $m^2 = 0$.

4. Beta Functions and RG Bifurcation

A. Generic beta function near $D = 4$

For a self-interacting theory:

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = -\epsilon g + b g^2 + O(g^3), \quad (6)$$

with $b > 0$ for asymptotically free (QCD-like) or critical theories.

B. Fixed points and bifurcation

The fixed points of (6) are given by solving $\beta(g) = 0$, which yields

$$g = 0, \quad g_* = \frac{\epsilon}{b}. \quad (7)$$

As $\epsilon \rightarrow 0$, the two fixed points coalesce, producing a *non-hyperbolic RG bifurcation*. Per Appendix B, a non-hyperbolic fixed point is likely to become unstable under mild or sustained perturbations. This is the root cause of *scale generation*.

5. Logarithmic Running and Dimensional Transmutation

Integrating Eq. (6) leads to

$$\int^{g(\mu)} \frac{dg}{-\epsilon g + bg^2} = \ln \frac{\mu}{\mu_0} \quad (8)$$

For $\epsilon \rightarrow 0$ we obtain

$$g(\mu) = \frac{g(\mu_0)}{1 - bg(\mu_0) \ln(\mu/\mu_0)} \quad (9)$$

The denominator vanishes at a finite scale:

$$\mu = \Lambda = \mu_0 \exp \left[-\frac{1}{bg(\mu_0)} \right] \quad (10)$$

(10) shows that a mass scale emerges from logarithmic corrections alone.

6. Emergent Infrared Mass

Per Appendix A, the renormalized mass satisfies:

$$m^2(\mu) = m^2(\mu_0) + cg(\mu)\mu^2 \ln \frac{\mu}{\mu_0} \quad (11)$$

Setting $m^2(\mu_0) \approx 0$ and $\mu \sim \Lambda$ leads to the *universal* result

$$m_{\text{IR}}^2 \sim c \Lambda^2 \quad (12)$$

7. Applications

A. Scalar Landau–Ginzburg theory

The inverse of the correlation length is given by:

$$\xi^{-1} \sim \Lambda = \mu_0 e^{-1/(bg_0)}, \quad (13)$$

explaining mass generation near criticality from first principles.

B. Electroweak sector

The Higgs quartic coupling obeys Eq. (6). The induced scale:

$$v^2 \sim \Lambda^2, \quad (14)$$

yields:

$$m_W = \frac{1}{2} g v, \quad m_Z = \frac{1}{2} \sqrt{g^2 + g'^2} v. \quad (15)$$

In this picture, the Higgs vacuum expectation value (vev) originates from RG bifurcation near $D = 4$, not from a fundamental mass input.

C. Yang–Mills theory and QCD

For non-Abelian gauge theory:

$$\beta(g) = -b_0 g^3 + O(g^5), b_0 > 0. \quad (16)$$

Integration yields:

$$\Lambda_{\text{QCD}} = \mu \exp \left[-\frac{1}{2b_0 g^2(\mu)} \right]. \quad (17)$$

This scale controls:

- Glueball masses,

- Infrared gluon screening,
- The Yang–Mills mass gap.

8. Universality of the Mechanism

The essential ingredients of this mechanism are:

1. Marginal interactions at $D = 4$,
2. Dimensional regularization,
3. Logarithmic RG flow,
4. Fixed-point bifurcation.

No assumption of confinement, condensation, or strong coupling is required here.

9. Conclusion

We have shown that dimensional regularization near $D = 4$ enforces infrared mass generation in any self-interacting field theory. Symmetry breaking producing the mass scale arises from RG bifurcations and

logarithmic corrections. This mechanism unifies the Higgs vev, electroweak boson masses, and the QCD mass gap under a single RG principle: *dimensional transmutation at a critical dimension*. Interested readers are invited to consult [1-8] for in-depth technical details and clarifications.

Appendix A: Detailed Derivation of the Emergent Infrared Mass

Here we explain how Eq. (12)

$$m_{\text{IR}}^2 \sim c \Lambda^2 \quad (12)$$

follows from Eq. (11)

$$m^2(\mu) = m^2(\mu_0) + c g(\mu) \mu^2 \ln \frac{\mu}{\mu_0}. \quad (11)$$

A. Origin of Eq. (11)

Equation (11) arises from the renormalization group equation for the mass operator:

$$\mu \frac{dm^2}{d\mu} = \gamma_m(g) m^2 + \delta(g) \mu^2, \quad (18)$$

where:

- $\gamma_m(g)$ is the anomalous dimension of the mass operator,
- $\delta(g)$ is generated by tadpole and self-energy diagrams.

At one loop:

$$\delta(g) = cg + O(g^2), \quad (19)$$

with $c > 0$ for scalar, electroweak, and Yang–Mills theories. Integrating Eq. (18) perturbatively gives:

$$m^2(\mu) = Z_m(\mu, \mu_0) m^2(\mu_0) + c \int_{\mu_0}^{\mu} d\ln \mu' g(\mu') \mu'^2. \quad (20)$$

B. Nearly Vanishing Mass Term

We now impose the physical assumption:

$$m^2(\mu_0) \simeq 0, \quad (21)$$

corresponding to a nearly scale-invariant or critical theory. Thus:

$$m^2(\mu) = c \int_{\mu_0}^{\mu} d\ln \mu' g(\mu') \mu'^2. \quad (22)$$

C. Inserting the Running Coupling

Near $D = 4$, the beta function is:

$$\beta(g) = -\epsilon g + b g^2, \quad \epsilon = 4 - D. \quad (23)$$

For $\epsilon \rightarrow 0$, the solution is:

$$g(\mu) = \frac{1}{b \ln(\mu/\Lambda)}. \quad (24)$$

Substitute Eq. (24) into Eq. (22) to obtain:

$$m^2(\mu) = c \int_{\mu_0}^{\mu} \frac{\mu'^2}{b \ln(\mu'/\Lambda)} d\ln \mu'. \quad (25)$$

D. Dominance of the Infrared Scale

As $\mu \rightarrow \Lambda$, the logarithm in the denominator becomes small:

$$\ln(\mu/\Lambda) \ll 1. \quad (26)$$

Thus, the integral is *infrared dominated*, yielding:

$$m^2(\mu) \sim c \Lambda^2 \quad \text{for } \mu \sim \Lambda. \quad (27)$$

This proves:

$$\boxed{m_{\text{IR}}^2 \equiv m^2(\mu \sim \Lambda) \sim c \Lambda^2}$$

which is Eq. (12).

The key point here is that the mass scale arises even when $m^2(\mu_0) = 0$, from effects induced by the logarithmic RG flow.

Appendix A1: Dimensional Regularization and Fractal Spacetime

Dimensional regularization analytically continues spacetime dimension:

$$D = 4 - \epsilon, \quad \epsilon \in \mathbb{R}. \quad (\text{A1})$$

This implies:

- The integration measure $d^D x$ is extended to non-integer dimensions,
- Momentum space has effective dimension D ,
- Correlation functions scale with anomalous powers.

The theory effectively lives on a fractal spacetime with Hausdorff dimension:

$$d_H = 4 - \epsilon. \quad (\text{A2})$$

This has direct consequences, for instance, the momentum space propagator becomes

$$\int d^D k (k^2)^{-1} \sim \mu^{-\epsilon}, \quad (\text{A3})$$

producing logarithmic corrections as $\epsilon \rightarrow 0$. Note that (A3) follows from evaluation of scale-free loop integrals in $D - \epsilon$ dimensions.

Thus:

- Dimensional regularization is not merely a computational trick,
- It encodes continuous geometric scaling,

- The RG bifurcation at $D = 4$ acts a critical behavior in continuous dimensions.

Appendix A2: Lattice QCD and the Infrared Mass Gap

Nonperturbative lattice simulations of Yang–Mills theory show that:

$$m_{\text{glueball}} \sim \Lambda_{\text{QCD}}, \quad (\text{B1})$$

with:

$$\Lambda_{\text{QCD}} \approx 200\text{--}300 \text{ MeV}. \quad (\text{B2})$$

Key lattice results:

- Finite correlation length in the pure QCD theory,
- Infrared suppression of gluon propagator,
- Dynamical mass scale without explicit mass terms.

These observations confirm:

- The RG-generated scale Λ is physical,

- The QCD mass gap emerges from dimensional transmutation,
- No conventional symmetry breaking is required.

In this context, lattice QCD provides direct numerical confirmation of Eq. (12) in the strongly coupled regime.

To summarize the content of Appendix A, A1 and A2,

1. Eq. (12) follows from Eq. (11) via IR-dominated logarithmic integrals.
2. Dimensional regularization enforces fractal spacetime scaling near $D = 4$.
3. RG flow undergoes a bifurcation at $\epsilon = 0$.
4. The emergent scale Λ universally generates IR masses.
5. Lattice QCD independently confirms this mechanism.

Appendix B: RG Bifurcation at $D = 4$ as a Source of Entropy Production

1. What “Entropy” Means in RG Language

In quantum and classical field theory, entropy production is not an effect rooted in Thermodynamics. It arises when:

1. Degrees of freedom become effectively unresolvable under coarse graining,
2. RG flow becomes irreversible, and
3. Information about UV initial conditions is encoded into IR collective variables.

Formally, entropy production is tied to *loss of phase-space resolution* under RG transformations.

Let $\rho[\Phi; \mu]$ be the probability functional of field configurations at RG scale μ .

Define the coarse-grained entropy:

$$S(\mu) = -\int \mathcal{D}\Phi \rho(\Phi; \mu) \ln \rho(\Phi; \mu). \quad (31)$$

Entropy production corresponds to:

$$\frac{dS}{d \ln \mu} > 0. \quad (32)$$

2. RG Flow as a Dynamical System

The RG equations define a flow in coupling space:

$$\frac{dg_i}{d \ln \mu} = \beta_i(\{g\}), \quad (33)$$

which generically represents a *nonlinear dynamical system*. Near a fixed point

g_* :

$$\frac{d \delta g_i}{d \ln \mu} = \sum_j M_{ij} \delta g_j + O(\delta g^2), \quad (34)$$

where:

$$M_{ij} = \left. \frac{\partial \beta_i}{\partial g_j} \right|_{g_*}. \quad (35)$$

3. Hyperbolic vs Non-Hyperbolic RG Flow

Hyperbolic fixed point

All eigenvalues of M_{ij} satisfy:

$$\Re(\lambda_i) \neq 0. \quad (36)$$

where λ_i are the Lyapunov exponents. Then:

- Stable/unstable manifolds exist,
- RG flow is reversible under scale inversion,
- Entropy production is minimal or zero.

Non-hyperbolic fixed point (Bifurcation)

If, at $D = 4$:

$$\beta(g) = bg^2, \quad (37)$$

then

$$\left. \frac{d\beta}{dg} \right|_{g=0} = 0. \quad (38)$$

This implies:

- Vanishing Lyapunov exponents,
- Breakdown of linear stability theory,
- Emergence of marginal directions.

These define the conditions enabling an RG bifurcation.

4. Why Bifurcation Implies Entropy Production

At a non-hyperbolic fixed point there is *irreversible information loss* because:

- RG trajectories separate logarithmically,
- Coarse graining integrates out infinitely many nearly marginal modes,
- UV information cannot be uniquely reconstructed from IR data.

Define the RG Jacobian:

$$J(\mu) = \det \left(\frac{\partial g_i(\mu)}{\partial g_j(\mu_0)} \right). \quad (39)$$

(39) measures the contraction of phase-space volume in coupling space under RG flow. The entropy production rate is:

$$\frac{dS}{d \ln \mu} = - \frac{d}{d \ln \mu} \ln |J(\mu)|. \quad (40)$$

At a hyperbolic fixed point:

$$J(\mu) \sim e^{\sum \lambda_i \ln(\mu/\mu_0)} \Rightarrow \frac{dS}{d \ln \mu} = 0. \quad (41)$$

At the bifurcation point:

$$J(\mu) \sim [\ln(\mu/\Lambda)]^{-N}, \quad (42)$$

where N is the number of near-marginal RG directions at the bifurcation. It follows that

$$\frac{dS}{d \ln \mu} \sim \frac{N}{\ln(\mu/\Lambda)} > 0. \quad (43)$$

It is seen that entropy production diverges logarithmically as $\mu \rightarrow \Lambda$.

5. Entropy Production and Mass Generation Are the Same Effect

Recall from (11) that

$$m^2(\mu) \sim \mu^2 g(\mu) \sim \frac{\mu^2}{\ln(\mu/\Lambda)}. \quad (44)$$

Compare with Eq. (43). Both scale as:

$$\sim \frac{1}{\ln(\mu/\Lambda)}. \quad (45)$$

This means that the same logarithmic RG memory that produces mass also produces entropy. In this sense, mass generation can be viewed as a process leading to information loss.

6. Summarizing the Physical Picture

At $D = 4$:

- RG flow ceases to be structurally stable,
- Marginal modes accumulate,
- UV-IR decoupling fails,

- RG trajectories become history-dependent,
- Entropy grows due to irreversible coarse graining.

This is why:

- Higgs vev,
- Electroweak scale,
- QCD mass gap

all coincide with maximal RG entropy production.

7. Relation to the Arrow of Scale

Define an RG “time” equivalent,

$$\tau = \ln (\mu_0/\mu). \quad (46)$$

Then entropy obeys:

$$\frac{dS}{d\tau} \geq 0, \quad (47)$$

with equality only at hyperbolic fixed points. Thus, the RG bifurcation at $D = 4$ defines a *fundamental arrow of scale*, analogous to the thermodynamic arrow of time.

8. In a Nutshell...

It can be concluded from the above that infrared mass generation is a *single dynamical mechanism* with multiple ramifications in,

- Information theory (entropy),
- Geometry (fractal dimension),
- Dynamics (bifurcation),
- Physics (mass gap).

We close Appendix B with a look at the relationship between the infrared mass generation and the *Kolmogorov–Sinai (KS) Entropy*. We proceed in the following way:

1. Definition of the Kolmogorov–Sinai Entropy

For a dynamical system with phase-space flow T^t , the Kolmogorov–Sinai entropy h_{KS} measures the rate of information loss per unit time under coarse graining.

By Pesin’s theorem:

$$h_{\text{KS}} = \sum_{\lambda_i > 0} \lambda_i \quad (48)$$

where λ_i are the positive Lyapunov exponents of the flow. By definition,

- $h_{\text{KS}} > 0 \Leftrightarrow$ chaos,
- $h_{\text{KS}} = 0 \Leftrightarrow$ regular (integrable) dynamics.

2. RG Flow as a Dynamical System

Recall that the renormalization group defines a flow in coupling space:

$$\frac{dg_i}{d\tau} = \beta_i(\{g\}), \quad \tau \equiv \ln(\mu_0/\mu). \quad (49)$$

which is a *nonlinear autonomous dynamical system*. In this context, the RG “phase space” is the space of couplings $\{g_i\}$.

3. Linear Stability and Lyapunov Exponents

Consider a trajectory $g_i(\tau)$. Linearizing:

$$\frac{d \delta g_i}{d\tau} = \sum_j M_{ij} \delta g_j, \quad M_{ij} = \frac{\partial \beta_i}{\partial g_j}. \quad (50)$$

Lyapunov exponents are defined as:

$$\lambda_i = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \frac{|\delta g_i(\tau)|}{|\delta g_i(0)|}. \quad (51)$$

4. Hyperbolic RG Flow \Rightarrow Zero KS Entropy

Away from $D = 4$, the beta function has the form:

$$\beta(g) = -\epsilon g + b g^2, \quad \epsilon \neq 0. \quad (52)$$

At the fixed point $g_* = \epsilon/b$:

$$\frac{d\beta}{dg} \Big|_{g_*} = +\epsilon. \quad (53)$$

Thus:

- Lyapunov spectrum is discrete,
- No marginal directions,
- RG flow is structurally stable.

Therefore:

$$h_{KS} = 0. \quad (54)$$

(54) signals that, in hyperbolic RG flows, there is no chaos, no entropy production, and the UV–IR decoupling holds.

5. RG Bifurcation at $D = 4$: Onset of Chaos

At $D = 4$ ($\epsilon = 0$):

$$\beta(g) = bg^2, \quad (55)$$

and:

$$\frac{d\beta}{dg} \Big|_{g=0} = 0. \quad (56)$$

This implies:

- Vanishing linear Lyapunov exponents,
- Breakdown of hyperbolicity,
- Emergence of so-called *intermittent dynamics* (Pomeau–Manneville type).

(56) defines a *saddle-node bifurcation* in RG space.

6. Intermittency and KS Entropy

For systems with marginal fixed points, separation of trajectories is sub-exponential:

$$\delta g(\tau) \sim \tau^\alpha, \quad \alpha > 0. \quad (57)$$

The effective Lyapunov exponent becomes:

$$\lambda_{\text{eff}}(\tau) \sim \frac{1}{\tau}. \quad (58)$$

In this case, the KS entropy is nonzero but scale-dependent:

$$h_{\text{KS}}(\tau) \sim \frac{1}{\ln(\mu/\Lambda)}. \quad (59)$$

This is precisely the same logarithmic structure appearing in (44) and (45).

7. Identification with RG Entropy Production

Recall the RG entropy production rate derived earlier:

$$\frac{dS_{\text{RG}}}{d\tau} \sim \frac{1}{\ln(\mu/\Lambda)}. \quad (60)$$

Comparing with Eq. (59) leads to

$$\boxed{\frac{dS_{\text{RG}}}{d\tau} \equiv h_{\text{KS}}^{\text{RG}}} \quad (61)$$

It follows that the RG entropy production rate is identical to the KS entropy of the RG flow.

8. Chaos, Memory, and Mass Generation

From the RG running of the coupling:

$$g(\mu) \sim \frac{1}{\ln(\mu/\Lambda)}. \quad (62)$$

From mass generation:

$$m^2(\mu) \sim \mu^2 g(\mu) \sim \frac{\mu^2}{\ln(\mu/\Lambda)}. \quad (63)$$

From KS entropy:

$$h_{\text{KS}}(\mu) \sim \frac{1}{\ln(\mu/\Lambda)}. \quad (64)$$

Thus:

$$\boxed{m^2(\mu) \propto \mu^2 h_{\text{KS}}(\mu)} \quad (65)$$

(65) tells us that mass generation is proportional to the chaotic information loss induced by RG bifurcations.

9. Physical Interpretation

At the RG bifurcation:

- RG trajectories become chaotic in the KS sense,
- Infinitesimal UV effects carry over to the IR,
- Entropy is produced at a finite rate,
- A mass gap emerges as signature of chaos.

This explains why:

- QCD has a mass gap,
- Higgs vev is nonzero,
- Purely free theories such as QED remain massless (no chaos \rightarrow no KS entropy).

10. Relation to Classical Chaos

This RG chaos is mathematically analogous to:

- Hamiltonian systems at the onset of chaos,
- Kolmogorov–Arnold–Moser (KAM) breakdown,
- Intermittency routes to chaos.

In conclusion, the RG flow satisfies the same axioms as deterministic dynamical systems to which the KS entropy applies.

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