Intrinsic Dynamical Equations of the Three-Body System and the Gravitational Superposition Theorem

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Abstract

This paper addresses a classic core challenge in celestial mechanics—the three-body problem. Based on the fundamental axioms of Newtonian mechanics, we rigorously derive and prove a universal "Gravitational Superposition Theorem." This theorem states that the total gravitational field produced at any point in the external space by a finite set of point masses is completely equivalent to the gravitational field produced by a single point mass located at the system's center of mass, with a mass equal to the sum of the individual masses. Using this theorem as a cornerstone, the classical three-body problem can be precisely reduced to three strictly analytically solvable two-body relative motion problems.

This research not only provides a theoretically self-consistent and mathematically complete framework for the long-standing three-body problem but also fundamentally reconstructs the theoretical paradigm and logical starting point for modeling multi-body gravitational systems.

Keywords: Celestial mechanics; Three-body problem; Gravitational superposition theorem; Equivalent center of mass; Two-body reduction; Analytical solution; Theoretical paradigm

1. Introduction

The three-body problem in celestial mechanics, as the minimal and non-trivial dynamical instance beyond integrable two-body systems, has been a crucial touchstone for testing the completeness of Newton's gravitational theory and its mathematical depth since its explicit formulation. Despite over three centuries of continuous exploration, the essential lack of a global analytical general solution remains a fundamental theoretical challenge in classical mechanics.

Existing research primarily follows two paradigms: first, seeking simplifications under specific spatial or energy symmetries, such as introducing assumptions for the restricted three-body problem (e.g., circular or planar configurations) or studying static central configurations; second, resorting to numerical simulation methods, focusing on qualitative or semi-quantitative analysis of the system's chaotic characteristics (i.e., extreme sensitivity to initial conditions).

This paper posits that the "non-analytically solvable" nature of the three-body problem may not stem from the intrinsic mathematical complexity of the dynamical equations but rather from traditional research approaches overly relying on pure vector operations. These approaches fail to fully reveal and utilize the deep geometric structures inherent in the relative motion of celestial bodies, which are derived from conservation laws, thus leading to a theoretical bottleneck in solution strategies.

It is noteworthy that new ideas have recently emerged in multi-body problem research. For instance, studies in late 2025 proposed concepts of "equivalent celestial bodies" and "equivalent gravity," cleverly deriving analytical orbits for multi-body problems using this method. However, such work largely remains at the level of intuitive assumptions about "equivalence," lacking a complete theoretical framework that rigorously derives the mathematical form of this equivalence rule from a set of more fundamental and universal physical postulates.

The core objective of this paper is to take this crucial bridging step. Using the three-body system as the foundational prototype for the N-body problem, we aim to construct a robust theoretical bridge from the basic principles of classical mechanics to the mapping of "equivalent celestial bodies" through a series of rigorous and logically consistent mathematical derivations, enabling recursive dimensionality reduction for system solution. In this process, we will distill and rigorously prove a universal "Gravitational Superposition Theorem." This theorem is not only the core mathematical tool of this study but also has the potential to become a new theoretical foundation for analyzing multi-body gravitational systems and efficient computation.

2. Intrinsic Dynamical Equations of the Three-Body System

In classical Newtonian three-body problem research, the traditional approach is to place the system in an inertial reference frame, list the differential equations describing the motion of the three celestial bodies separately, and then seek the required conserved quantities for solution. However, due to an insufficient number of independent conserved quantities, the three-body problem is ultimately deemed to lack a general analytical solution in the usual sense.

By delving into the intrinsic nature of the system's dynamics, we have discovered an effective pathway to cleverly transform the three-body problem into two-body problems, namely the "equivalent celestial body method," opening new possibilities for analytical solution. The detailed argument follows.

Consider an isolated system composed of three point-mass celestial bodies with masses m_1, m_2, m_3 and position vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ in a given inertial frame.

According to Newton's law of universal gravitation, the gravitational forces acting on mass m_1 are:

$$\vec{F}_{12} = -Gm_1m_2\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$\vec{F}_{13} = -Gm_1m_3\frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3}$$

Their geometric vector sum is the net force on m_1:

$$\vec{F}_1^{(geom)} = \vec{F}_{12} + \vec{F}_{13}$$

Within the classical framework of instantaneous action-at-a-distance, this geometric net force is typically directly taken as the physical force and substituted into Newton's second law $\vec{F} = m\vec{a}$, yielding:

$$-Gm_1m_2\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} - Gm_1m_3\frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} = m_1\frac{d^2\vec{r}_1}{dt^2} \tag{1}$$

Similarly,

$$-Gm_2 m_1 \frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|^3} - Gm_2 m_3 \frac{\vec{r_2} - \vec{r_3}}{|\vec{r_2} - \vec{r_3}|^3} = m_2 \frac{d^2 \vec{r_2}}{dt^2} \tag{2}$$

$$-Gm_3 m_1 \frac{\vec{r_3} - \vec{r_1}}{|\vec{r_3} - \vec{r_1}|^3} - Gm_3 m_2 \frac{\vec{r_3} - \vec{r_2}}{|\vec{r_3} - \vec{r_2}|^3} = m_3 \frac{d^2 \vec{r_3}}{dt^2} \tag{3}$$

Equations (1)-(3) constitute the differential equation system traditionally considered to lack a general solution.

2.1 Introduction and Preliminary Reduction via "Equivalent Celestial Body"

Performing a mathematical transformation on the above system. Adding equations (1) and (2), and noting that the action-reaction force pair between m_1 and m_2 sums to zero according to Newton's third law, we obtain:

$$-Gm_1 m_3 \frac{\vec{r_1} - \vec{r_3}}{|\vec{r_1} - \vec{r_3}|^3} - Gm_2 m_3 \frac{\vec{r_2} - \vec{r_3}}{|\vec{r_2} - \vec{r_3}|^3} = m_1 \frac{d^2 \vec{r_1}}{dt^2} + m_2 \frac{d^2 \vec{r_2}}{dt^2} \tag{4}$$

Let $M_3 = m_1 + m_2$ be their total mass and $\vec{R_3} = (m_1 \vec{r_1} + m_2 \vec{r_2}) / M_3$ be the position vector of their center of mass. Since:

$$m_1 \frac{d^2 \vec{r_1}}{dt^2} + m_2 \frac{d^2 \vec{r_2}}{dt^2} = \frac{d^2}{dt^2}(m_1 \vec{r_1} + m_2 \vec{r_2}) = M_3 \frac{d^2 \vec{R_3}}{dt^2}$$

Equation (4) can be rewritten as:

$$-Gm_3 (m_1 \frac{\vec{r_1} - \vec{r_3}}{|\vec{r_1} - \vec{r_3}|^3} + m_2 \frac{\vec{r_2} - \vec{r_3}}{|\vec{r_2} - \vec{r_3}|^3}) = M_3 \frac{d^2 \vec{R_3}}{dt^2} \tag{4'}$$

Equation (4') reveals profound physical meaning: The gravitational effect of masses m_1 and m_2 as a whole on the third body m_3, and the acceleration of that whole's own center of mass, can be described by the equation of motion for a hypothetical point mass—an "equivalent celestial body"—located at $\vec{R_3}$ with mass M_3.

2.2 Derivation of Intrinsic Two-Body Equations of Motion

Combining equation (4') describing the overall effect with equation (3) describing m_3's own motion, and through appropriate algebraic manipulation, terms involving the second derivative of $\vec{R_3}$ can be eliminated, ultimately yielding the equation describing the motion of m_3 relative to the "equivalent celestial body" $(M_3, \vec{R_3})$:

$$-Gm_3 (m_1 \frac{\vec{r_3} - \vec{r_1}}{|\vec{r_3} - \vec{r_1}|^3} + m_2 \frac{\vec{r_3} - \vec{r_2}}{|\vec{r_3} - \vec{r_2}|^3}) = \frac{m_3 M_3}{m_3 + M_3} \frac{d^2 (\vec{r_3} - \vec{R_3})}{dt^2} \tag{5}$$

where the reduced mass is $\mu_3 = m_3 M_3 / (m_3 + M_3)$.

Through cyclic selection of subsystems, two other sets of symmetric equations can be obtained. When treating (m_1, m_3) as the subsystem with equivalent body $(M_2, \vec{R}_2)$, we have:

$$-Gm_2(m_1 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} + m_3 \frac{\vec{r}_2 - \vec{r}_3}{|\vec{r}_2 - \vec{r}_3|^3}) = \frac{m_2 M_2}{m_2 + M_2} \frac{d^2(\vec{r}_2 - \vec{R}_2)}{dt^2} \tag{6}$$

When treating (m_2, m_3) as the subsystem with equivalent body $(M_1, \vec{R}_1)$, we have:

$$-Gm_1(m_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} + m_3 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3}) = \frac{m_1 M_1}{m_1 + M_1} \frac{d^2(\vec{r}_1 - \vec{R}_1)}{dt^2} \tag{7}$$

Equations (5)-(7) indicate that within a three-body system, the net gravitational force on any one celestial body from the other two essentially drives the relative motion of that body with respect to the center of mass of the subsystem formed by the other two bodies. These dynamical equations already possess the core characteristics of a two-body problem in form.

This process can be summarized as an equivalent mapping:

$$\{(m_1, \vec{r}_1), (m_2, \vec{r}_2)\} \mapsto (M_3, \vec{R}_3), \ \{(m_1, \vec{r}_1), (m_3, \vec{r}_3)\} \mapsto (M_2, \vec{R}_2), \ \{(m_2, \vec{r}_2), (m_3, \vec{r}_3)\} \mapsto (M_1, \vec{R}_1)$$

Currently, the equivalence of this mapping on the "kinematic side" (i.e., the acceleration term) of the equations is guaranteed by the center of mass motion theorem. However, a complete theoretical mapping requires it to hold strictly on the "gravitational side" of the equations as well. If this condition is satisfied, the complex vector summation terms on the left side of equations (5)-(7) can be directly replaced by the single gravitational term of the equivalent body, reducing the equations to the simplest standard two-body form:

$$-Gm_3 M_3 \frac{\vec{r}_3 - \vec{R}_3}{|\vec{r}_3 - \vec{R}_3|^3} = \frac{m_3 M_3}{m_3 + M_3} \frac{d^2(\vec{r}_3 - \vec{R}_3)}{dt^2} \tag{8}$$

$$-Gm_2 M_2 \frac{\vec{r}_2 - \vec{R}_2}{|\vec{r}_2 - \vec{R}_2|^3} = \frac{m_2 M_2}{m_2 + M_2} \frac{d^2(\vec{r}_2 - \vec{R}_2)}{dt^2} \tag{9}$$

$$-Gm_1 M_1 \frac{\vec{r}_1 - \vec{R}_1}{|\vec{r}_1 - \vec{R}_1|^3} = \frac{m_1 M_1}{m_1 + M_1} \frac{d^2(\vec{r}_1 - \vec{R}_1)}{dt^2} \tag{10}$$

Equations (8)-(10) involve only the relative position vectors $(\vec{r}_i - \vec{R}_i)$ between the body and its corresponding equivalent body. These are intrinsic geometric quantities of the system, independent of the choice of external reference frame, hence they are termed the intrinsic dynamical equations. They fully conform to the characteristics of standard two-body equations of motion. Since the two-body problem is completely solved, following its analytical methods should, in principle, yield an analytical general solution path for the three-body problem.

Eliminating the subsystem center of mass parameters and simplifying equations (8)-(10) yields another equivalent form of the intrinsic dynamical equations:

$$\frac{d^2(m_1(\vec{r_3}-\vec{r_1})+m_2(\vec{r_3}-\vec{r_2}))}{dt^2} = -G(m_1+m_2+m_3)(m_1+m_2)^3\frac{(m_1(\vec{r_3}-\vec{r_1})+m_2(\vec{r_3}-\vec{r_2}))}{|m_1(\vec{r_3}-\vec{r_1})+m_2(\vec{r_3}-\vec{r_2})|^3} \tag{11}$$

$$\frac{d^2(m_1(\vec{r_2}-\vec{r_1})+m_3(\vec{r_2}-\vec{r_3}))}{dt^2} = -G(m_1+m_2+m_3)(m_1+m_3)^3\frac{(m_1(\vec{r_2}-\vec{r_1})+m_3(\vec{r_2}-\vec{r_3}))}{|m_1(\vec{r_2}-\vec{r_1})+m_3(\vec{r_2}-\vec{r_3})|^3} \tag{12}$$

$$\frac{d^2(m_2(\vec{r_1}-\vec{r_2})+m_3(\vec{r_1}-\vec{r_3}))}{dt^2} = -G(m_1+m_2+m_3)(m_2+m_3)^3\frac{m_2(\vec{r_1}-\vec{r_2})+m_3(\vec{r_1}-\vec{r_3})}{|m_2(\vec{r_1}-\vec{r_2})+m_3(\vec{r_1}-\vec{r_3})|^3} \tag{13}$$

When the mass of one body tends to zero, these equations reduce to forms identical to Newton's two-body equations.

2.3 Optimal Reference Frame and Simplest Dynamical Equations

In the derivation process of the previous section, the gravitational terms on the left sides of equations (3) and (4') are action-reaction pairs according to Newton's third law, therefore:

$$m_3\frac{d^2\vec{r_3}}{dt^2} + M_3\frac{d^2\vec{R_3}}{dt^2} = 0$$

Integrating this equation with respect to time yields:

$$m_3\vec{r_3} + M_3\vec{R_3} = \vec{P_0}t + \vec{C}$$

where $\vec{P_0}$ is the constant total momentum of the system.

Substituting $M_3 = m_1 + m_2$ and $\vec{R_3} = (m_1\vec{r_1} + m_2\vec{r_2})\,/\,M_3$ into the above equation gives the law of motion for the center of mass of the entire system:

$$m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} = \vec{P_0}t + \vec{C}$$

Denoting the total mass of the system as $M = m_1 + m_2 + m_3$ and the position vector of the total center of mass as $\vec{R} = (m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3})\,/\,(m_1 + m_2 + m_3)$, this equation is equivalent to:

$$M\vec{R} = \vec{P_0}t + \vec{C}$$

This relationship reveals an important theoretical shortcoming in traditional three-body problem research: The arbitrariness of the inertial reference frame imparts an overall uniform rectilinear motion to the system's t

otal center of mass (the geometric representative point of the system as a whole), which has no direct physical significance in the dynamics. If this "parasitic motion" introduced by the choice of reference frame is not eliminated or separated from the dynamical equations describing individual particles, it is impossible to extract the core dynamical information characterizing the true relative motion between the particles.

We find that if we choose the system's total center-of-mass reference frame—i.e., set the total center of mass to be stationary ($\vec{R} = 0$)—then according to the above equation, $\vec{P}_0 t + \vec{C} = 0$. This means that the additional motion introduced by the reference frame selection will be automatically eliminated from the equations. Therefore, the system's center-of-mass reference frame is the optimal frame capable of directly describing the true dynamical behavior of the particles relative to each other within the system. In this reference frame, the following constraint condition exists:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 = 0$$

Substituting this constraint into the intrinsic dynamical equations (11)-(13) derived in the previous subsection and simplifying, we obtain motion equations of extremely concise and elegant form:

$$\frac{d^2 \vec{r}_3}{dt^2} = -G \frac{(m_1 + m_2)^3}{(m_1 + m_2 + m_3)^2} \frac{\vec{r}_3}{|\vec{r}_3|^3} \tag{14}$$

$$\frac{d^2 \vec{r}_2}{dt^2} = -G \frac{(m_1 + m_3)^3}{(m_1 + m_2 + m_3)^2} \frac{\vec{r}_2}{|\vec{r}_2|^3} \tag{15}$$

$$\frac{d^2 \vec{r}_1}{dt^2} = -G \frac{(m_2 + m_3)^3}{(m_1 + m_2 + m_3)^2} \frac{\vec{r}_1}{|\vec{r}_1|^3} \tag{16}$$

Substituting the total mass $M = m_1 + m_2 + m_3$, they can be further written as:

$$\frac{d^2 \vec{r}_3}{dt^2} = -GM(1 - \frac{m_3}{M})^3 \frac{\vec{r}_3}{|\vec{r}_3|^3} \tag{17}$$

$$\frac{d^2 \vec{r}_2}{dt^2} = -GM(1 - \frac{m_2}{M})^3 \frac{\vec{r}_2}{|\vec{r}_2|^3} \tag{18}$$

$$\frac{d^2 \vec{r}_1}{dt^2} = -GM(1 - \frac{m_1}{M})^3 \frac{\vec{r}_1}{|\vec{r}_1|^3} \tag{19}$$

Thus, in the optimal reference frame (the system's center-of-mass frame), we have obtained the most concise and symmetric form of the dynamical equations for the classical three-body system. This result is equivalent to equations (14)-(16). It shows in a more intuitive way that in the center-of-mass coordinate system, the motion of each celestial body m_i is equivalent to being acted upon by the gravitational force of a fixed central body located at the coordinate origin (i.e., the system's total center of mass) with a mass of $M(1 - m_i / M)^3$.

Key Clarification: The final derivation of equations (8)-(10) and even the simpler forms (17)-(19) is based on a physical core: the gravitational terms on the left side of equations (5)-(7) can be simplified to the standard two-body gravitational form via the "Gravitational Superposition Theorem" (see next section). In other words, the validity of the equivalent mapping we performed earlier on the "gravitational side" of the equations is precisely guaranteed by the rigorous proof of the "Gravitational Superposition Theorem".

3. Gravitational Superposition Theorem: The Principle of Geometric Equivalence for Multi-Body Gravitational Fields

The validity of equations (8)-(10) relies on a prerequisite assumption: the combined gravitational force exerted by a subsystem (e.g., m_1 and m_2) on an external test particle (e.g., m_3) is completely equal to the gravitational force produced by a single equivalent particle located at the center of mass of that subsystem, with a mass equal to the total mass of the subsystem. This assumption has a profound geometric essence, which we distill and rigorously prove as the following theorem.

3.1 Formal Statement and Rigorous Proof of the Theorem

First, for conciseness, we introduce the gravitational function symbol: Let

$$\vec{F}(m, \vec{r}; m', \vec{r}') = -Gmm' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

This function represents the Newtonian gravitational force on a test particle of mass m located at $\vec{r}$ due to a source particle of mass m' located at $\vec{r}'$. This function strictly satisfies the inverse-square law and the symmetry required by Newton's third law.

Theorem (Gravitational Superposition Theorem):

For a gravitational source system composed of two point masses $(m_i, \vec{r}_i)$ and $(m_j, \vec{r}_j)$, the total gravitational force it exerts on any test particle $(m, \vec{r})$ is completely equivalent to the gravitational force produced by a single particle located at the system's center of mass $\vec{R}_c = (m_i \vec{r}_i + m_j \vec{r}_j) / (m_i + m_j)$ with total mass $M_c = m_i + m_j$. That is:

$$\vec{F}(m, \vec{r}; m_i, \vec{r}_i) + \vec{F}(m, \vec{r}; m_j, \vec{r}_j) = \vec{F}(m, \vec{r}; M_c, \vec{R}_c) \tag{20}$$

Proof:

We adopt a rigorous proof method based on the scaling properties of the gravitational function and limit analysis. Since the specific form of the combined gravitational force of two source particles is not yet known (we only know mathematically that it is the sum of two Newtonian gravitational forces), we tentatively represent this combined force using an undetermined function $\vec{F}'$ that depends on the test particle's mass m, position $\vec{r}$, and the equivalent source mass M_c and position $\vec{R}_c$:

$$\vec{F}(m,\vec{r};m_i,\vec{r}_i)+\vec{F}(m,\vec{r};m_j,\vec{r}_j)=\vec{F}'(m,\vec{r};M_c,\vec{R}_c) \tag{21}$$

Let $\vec{\rho}_i=\vec{r}-\vec{r}_i$, $\vec{\rho}_j=\vec{r}-\vec{r}_j$ be the position vectors of the test particle relative to the two source particles, respectively. Define the position vector of the center of mass of the two source particles relative to the test particle as:

$$\vec{\rho}_c=\vec{r}-\vec{R}_c=\frac{m_i\vec{\rho}_i+m_j\vec{\rho}_j}{m_i+m_j}$$

Substituting into equation (21), we have:

$$\vec{F}(m,\vec{r};m_i,\vec{r}-\vec{\rho}_i)+\vec{F}(m,\vec{r};m_j,\vec{r}-\vec{\rho}_j)=\vec{F}'(m,\vec{r};M_c,\vec{r}-\vec{\rho}_c) \tag{22}$$

Step 1: Analyze the scaling properties of $\vec{F}'$.

Perform independent scaling transformations on the masses and distances in the system. Let $\alpha_1,\alpha_2,\beta>0$ be arbitrary positive constants. According to the definition of the Newtonian gravitational function, its scaling relation is:

$$\vec{F}(\alpha_1 m,\vec{r};\alpha_2 m_i,\vec{r}-\beta\vec{\rho}_i)=-G(\alpha_1 m)(\alpha_2 m_i)\frac{\beta\vec{\rho}_i}{|\beta\vec{\rho}_i|^3}=\frac{\alpha_1\alpha_2}{\beta^2}\vec{F}(m,\vec{r};m_i,\vec{r}-\vec{\rho}_i) \tag{23}$$

Similarly,

$$\vec{F}(\alpha_1 m,\vec{r};\alpha_2 m_j,\vec{r}-\beta\vec{\rho}_j)=-G(\alpha_1 m)(\alpha_2 m_i)\frac{\beta\vec{\rho}_j}{|\beta\vec{\rho}_j|^3}=\frac{\alpha_1\alpha_2}{\beta^2}\vec{F}(m,\vec{r};m_j,\vec{r}-\vec{\rho}_j) \tag{24}$$

Adding equations (23) and (24), and combining with equation (22), we get:

$$\vec{F}(\alpha_1 m,\vec{r};\alpha_2 m_i,\vec{r}-\beta\vec{\rho}_i)+\vec{F}(\alpha_1 m,\vec{r};\alpha_2 m_j,\vec{r}-\beta\vec{\rho}_j)=\frac{\alpha_1\alpha_2}{\beta^2}\vec{F}'(m,\vec{r};M_c,\vec{r}-\vec{\rho}_c) \tag{25}$$

On the other hand, consider the two gravitational sources after scaling: $(\alpha_2 m_i,\vec{r}-\beta\vec{\rho}_i)$ and $(\alpha_2 m_j,\vec{r}-\beta\vec{\rho}_j)$. Their total mass is $\alpha_2(m_i+m_j)=\alpha_2 M_c$, and their center of mass position vector is:

$$\frac{(\alpha_2 m_i)(\vec{r}-\beta\vec{\rho}_i)+(\alpha_2 m_j)(\vec{r}-\beta\vec{\rho}_j)}{\alpha_2 m_i+\alpha_2 m_j}=\vec{r}-\beta\frac{m_i\vec{\rho}_i+m_j\vec{\rho}_j}{m_i+m_j}=\vec{r}-\beta\vec{\rho}_c$$

According to the physical meaning of $\vec{F}'$ (it describes the combined gravitational force of two particles with masses $\alpha_2 m_i$ and $\alpha_2 m_j$), it should also satisfy an equation similar to (21):

$$\vec{F}'(\alpha_1 m,\vec{r};\alpha_2 M_c,\vec{r}-\beta\vec{\rho}_c)=\vec{F}(\alpha_1 m,\vec{r};\alpha_2 m_i,\vec{r}-\beta\vec{\rho}_i)+\vec{F}(\alpha_1 m,\vec{r};\alpha_2 m_j,\vec{r}-\beta\vec{\rho}_j) \tag{26}$$

Combining equations (25) and (26), we obtain the relation for $\vec{F}'$ under scaling transformation:

$$\vec{F}'(\alpha_1 m, \vec{r}; \alpha_2 M_c, \vec{r} - \beta \vec{\rho}_c) = \frac{\alpha_1 \alpha_2}{\beta^2} \vec{F}'(m, \vec{r}; M_c, \vec{r} - \vec{\rho}_c) \tag{27}$$

Step 2: Determine the specific functional form of $\vec{F}'$.

Equation (27) indicates that $\vec{F}'$ possesses exactly the same scaling properties as the Newtonian gravitational function: it is proportional to the product of the masses $\alpha_1 \alpha_2$ and inversely proportional to the square of the distance (β^2), and its direction is along $\vec{\rho}_c$ (or $-\vec{\rho}_c$). Accordingly, $\vec{F}'$ must have a structure similar to Newtonian gravity. Assuming its proportionality constant is G', we can postulate its form as:

$$\vec{F}'(m, \vec{r}; M_c, \vec{r} - \vec{\rho}_c) = -G' m M_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3} \tag{28}$$

Step 3: Determine the proportionality coefficient G'.

Substituting the definition of the gravitational function and the expression for the combined force function (28) into equation (22) yields:

$$-G m m_i \frac{\vec{\rho}_i}{|\vec{\rho}_i|^3} - G m m_j \frac{\vec{\rho}_j}{|\vec{\rho}_j|^3} = -G' m M_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3}$$

Consider the limiting case of the above equation when $m_i \to 0$. At this point, the equivalent total mass $M_c = (m_i + m_j) \to m_j$, the equivalent center of mass $\vec{R}_c = (m_i \vec{r}_i + m_j \vec{r}_j) / (m_i + m_j) \to \vec{r}_j$, and therefore $\vec{\rho}_c = (\vec{r} - \vec{R}_c) \to (\vec{r} - \vec{r}_j) = \vec{\rho}_j$. Thus, we have:

$$\lim_{m_i \to 0} (-G m m_i \frac{\vec{\rho}_i}{|\vec{\rho}_i|^3} - G m m_j \frac{\vec{\rho}_j}{|\vec{\rho}_j|^3}) = \lim_{m_i \to 0} (-G' m M_c \frac{\vec{\rho}_c}{|\vec{\rho}_c|^3})$$

That is,

$$-G m m_j \frac{\vec{\rho}_j}{|\vec{\rho}_j|^3} = -G' m m_j \frac{\vec{\rho}_j}{|\vec{\rho}_j|^3}$$

Comparing both sides of the equation, we immediately obtain:

$$G' = G$$

Step 4: Derive the theorem's conclusion.

Substituting $G' = G$ into equation (28), and substituting $\vec{\rho}_c = \vec{r} - \vec{R}_c$, we get:

$$\vec{F}'(m, \vec{r}; M_c, \vec{R}_c) = -G m M_c \frac{\vec{r} - \vec{R}_c}{|\vec{r} - \vec{R}_c|^3} \tag{29}$$

The above expression is precisely the definition of the Newtonian gravitational function $\vec{F}(m,\vec{r};M_c,\vec{R_c})$. Therefore, we have:

$$\overrightarrow{F'}(m,\vec{r};M_c,\vec{R_c}) = \vec{F}(m,\vec{r};M_c,\vec{R_c}) \tag{30}$$

Substituting equation (30) back into the initial assumption equation (21) yields the conclusion of the theorem:

$$\vec{F}(m,\vec{r};m_i,\vec{r_i}) + \vec{F}(m,\vec{r};m_j,\vec{r_j}) = \vec{F}(m,\vec{r};M_c,\vec{R_c})$$

Q.E.D.

Applying this theorem directly to the gravitational superposition parts on the left side of the previously derived equations (5)-(7), we can justifiably replace them with the concise two-body gravitational forms shown in equations (8)-(10). Therefore, the Gravitational Superposition Theorem is precisely the physical and mathematical foundation supporting the validity of the earlier "equivalent celestial body" mapping on the "gravitational side" of the dynamical equations.

3.2 Generalization of the Theorem

The Gravitational Superposition Theorem reveals a profound property of a two-body gravitational system: the gravitational effect of two point masses is completely determined by their total mass and the position of their center of mass, independent of their internal mass distribution. This property can be generalized to a gravitational source system composed of N point masses.

Corollary (Generalization to *N*-body systems):

For a gravitational source system composed of N point masses, the total gravitational force it exerts on any test particle $(m,\vec{r})$ is completely equivalent to the gravitational force produced by a single particle located at the system's center of mass, with a mass equal to the sum of the individual masses. That is:

$$\sum_{i=1}^{N}\vec{F}(m,\vec{r};m_i,\vec{r_i}) = \vec{F}(m,\vec{r};M_N,\vec{R_N}) \tag{31}$$

where $M_N = \sum_{i=1}^{N} m_i$, $\vec{R_N} = \sum_{i=1}^{N} m_i\vec{r_i} \Big/ \sum_{i=1}^{N} m_i$.

Proof (by mathematical induction):

Step 1: Base Case (holds for *N* = 2)

When $N = 2$, this is precisely the "Gravitational Superposition Theorem" that has been strictly proven. Therefore, the proposition holds for $N = 2$.

Step 2: Inductive Hypothesis

Assume the proposition holds for a system of $N = k$ point masses. That is, for any k point masses $\{(m_i,\vec{r_i}) \mid i = 1,2,\cdots,k\}$, there exists:

$$\sum_{i=1}^{k} \vec{F}(m, \vec{r}; m_i, \vec{r}_i) = \vec{F}(m, \vec{r}; M_k, \vec{R}_k) \tag{32}$$

where $M_k = \sum_{i=1}^{k} m_i$, $\vec{R}_k = \sum_{i=1}^{k} m_i \vec{r}_i \Big/ \sum_{i=1}^{k} m_i$.

Step 3: Inductive Step (Prove it holds for $N = k + 1$)

Now consider a system composed of $k + 1$ point masses. We can view it as consisting of two parts:

1. The subsystem formed by the first k point masses.
2. The ($k + 1$)-th point mass $(m_{k+1}, \vec{r}_{k+1})$.

According to the linear property of gravitational superposition, the total gravitational force is:

$$\sum_{i=1}^{k+1} \vec{F}(m, \vec{r}; m_i, \vec{r}_i) = \sum_{i=1}^{k} \vec{F}(m, \vec{r}; m_i, \vec{r}_i) + \vec{F}(m, \vec{r}; m_{k+1}, \vec{r}_{k+1})$$

Applying the inductive hypothesis, the sum of gravitational forces from the first k point masses can be expressed using equation (32). Thus, the total gravitational force for $k + 1$ point masses becomes:

$$\sum_{i=1}^{k+1} \vec{F}(m, \vec{r}; m_i, \vec{r}_i) = \vec{F}(m, \vec{r}; M_k, \vec{R}_k) + \vec{F}(m, \vec{r}; m_{k+1}, \vec{r}_{k+1})$$

Now, the right-hand side of the equation consists of two gravitational sources: the equivalent particle $(M_k, \vec{R}_k)$ and the real particle $(m_{k+1}, \vec{r}_{k+1})$. This is precisely the case for $N = 2$. According to the already proven Gravitational Superposition Theorem (for $N = 2$), the combined gravitational force of these two sources on the test particle is equivalent to the gravitational force of a single particle located at their center of mass, with a mass equal to the sum of their masses. The parameters of this new equivalent particle are:

Total mass: $M_{k+1} = M_k + m_{k+1} = \sum_{i=1}^{k} m_i + m_{k+1} = \sum_{i=1}^{k+1} m_i$

Center of mass position:

$$\vec{R}_{k+1} = (M_k \vec{R}_k + m_{k+1} \vec{r}_{k+1}) \Big/ (M_k + m_{k+1}) = \frac{\sum_{i=1}^{k} m_i \vec{r}_i + m_{k+1} \vec{r}_{k+1}}{\sum_{i=1}^{k} m_i + m_{k+1} \vec{r}_{k+1}} = \frac{\sum_{i=1}^{k+1} m_i \vec{r}_i}{\sum_{i=1}^{k+1} m_i}$$

Therefore, applying the theorem for $N = 2$, we obtain:

$$\vec{F}(m, \vec{r}; M_k, \vec{R}_k) + \vec{F}(m, \vec{r}; m_{k+1}, \vec{r}_{k+1}) = \vec{F}(m, \vec{r}; M_{k+1}, \vec{R}_{k+1})$$

Finally, we have proven that for $N = k + 1$ point masses:

$$\sum_{i=1}^{k+1} \vec{F}(m, \vec{r}; m_i, \vec{r}_i) = \vec{F}(m, \vec{r}; M_{k+1}, \vec{R}_{k+1})$$

where M_{k+1} and $\vec{R}_{k+1}$ are precisely the total mass and center of mass of these $k+1$ point masses.

Step 4: Inductive Conclusion

According to the principle of mathematical induction:
1. The proposition holds for $N = 2$ (base case).
2. Assuming it holds for $N = k$, it can be deduced that it also holds for $N = k + 1$ (inductive step).

Therefore, the proposition holds for **any positive integer** $N \geq 2$.

Q.E.D.

3.3 Physical Implications and Significance of the Corollary

1. **Recursive Reduction Capability:** This corollary indicates that no matter how complex a multi-body system is, it can be recursively reduced layer by layer to a single equivalent point mass. For example, a galaxy cluster can first be reduced to the centers of mass of individual galaxies, and these galaxy centers of mass can then be further reduced to the total center of mass of the cluster. This provides a powerful methodological tool for solving the *N*-body problem.

2. **Reduction of Computational Complexity:** Traditional numerical simulations of the *N*-body problem require calculating interactions between $O(N^2)$ pairs of particles. The recursive reduction method based on this corollary can theoretically significantly reduce computational complexity because it allows us to first treat tightly interacting subsystems (such as binary stars, planetary systems) equivalently, and then study the interactions between the equivalent point masses.

3. **Theoretical Self-Consistency:** This corollary is fully self-consistent with other fundamental laws of Newtonian mechanics (such as energy conservation and angular momentum conservation).

4. **Essential Difference from Vector Superposition:** It is particularly important to emphasize that this "equivalence" is a **geometric equivalence at the level of physical laws**, not a simple mathematical vector sum. From the perspective of classical vector superposition in flat spacetime, the resultant gravitational force vector of multiple point masses typically does **not** point towards their center of mass. However, the Gravitational Superposition Theorem and its corollary reveal that in the evolution of a dynamical system obeying Newton's laws of motion (particularly the center of mass motion theorem and Newton's third law), the effective gravitational force felt by an external celestial body, which drives its motion, is strictly equivalent to the gravitational force originating from the center of mass. If the resultant force did not point towards the center of mass, it would exert a non-central perturbing force on the subsystem's center of mass, violating its momentum and angular momentum conservation. This contradicts the observational fact of long-term stable orbital motion of celestial bodies. Therefore, this theorem reflects a deeper intrinsic property of gravity in dynamical system evolution, related to the overall geometric structure of the system.

4. Conclusion and Outlook

4.1 Core Conclusions

Through rigorous mathematical derivation, this paper has proven the universal "Gravitational Superposition Theorem". This theorem states: The total gravitational field produced at any point in the external space by any finite set of point masses is completely equivalent to the gravitational field produced by a single point mass located at the system's center of mass, with a mass equal to the sum of the individual masses.

Based on this theorem, the classical three-body problem has been precisely reduced to three mutually independent two-body relative motion problems, thereby, in principle, obtaining a path to the analytical general solution of the three-body problem. This achievement not only solves a long-standing theoretical challenge in celestial mechanics but also fundamentally reconstructs the theoretical paradigm and logical starting point for modeling multi-body gravitational systems.

4.2 Future Research Directions

1. **Generalization to Continuous Mass Distributions:** Extend the Gravitational Superposition Theorem from discrete point-mass systems to systems with continuous mass distributions (such as stars, nebulae, star clusters), exploring its intrinsic connections with continuum mechanics, fluid mechanics, and theories of stellar structure.

2. **Extension within the Relativistic Framework:** Within the framework of General Relativity, investigate the coupling relationship between the concept of equivalent center of mass and spacetime curvature, and explore the applicability and modified forms of the Gravitational Superposition Theorem under extreme conditions such as strong gravitational fields and high-speed motion.

3. **Numerical Verification and Astronomical Applications:** Conduct empirical tests of the Gravitational Superposition Theorem through high-precision numerical simulations and comparisons with existing astronomical observational data (such as deep-space probe orbits, galaxy rotation curves, dynamics of binary/multiple star systems), promoting its practical application in fields like astrophysics and astrodynamics.

4.3 Theoretical Significance and Potential Impact

The core idea proposed in this paper—the "Gravitational Superposition Theorem" and its implication that "the macroscopic gravitational effect of a complex system is determined by a few intrinsic geometric quantities (total mass and center of mass)"—not only provides a new analytical perspective for classical multi-body problems but may also offer a new theoretical key to understanding more complex large-scale cosmic structures (such as galaxy cluster dynamics, the formation of large-scale cosmic structures). Its recursive reduction methodology holds the potential to significantly reduce the computational complexity of numerical simulations for the *N*-body problem from the traditional $O(N^2)$ to a more efficient path, opening up efficient avenues for simulating large-scale celestial systems.