# Mathematical Analysis of a Great Rhombicuboctahedron

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## 1. Introduction

A great rhombicuboctahedron is an Archimedean solid [1] which has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each having equal edge length. It has 72 edges & 48 vertices lying on a spherical surface with a certain radius. It is created/generated by expanding a truncated cube [2] having 8 equilateral triangular faces & 6 regular octagonal faces. Thus by the expansion, each of 12 originally truncated edges changes into a square face, each of 8 triangular faces of the original solid changes into a regular hexagonal face & 6 regular octagonal faces of original solid remain unchanged i.e. octagonal faces are shifted radially. Thus, a solid with 12 squares, 8 hexagonal & 6 octagonal faces, is obtained which is called great rhombicuboctahedron which is an Archimedean solid (as shown in figure 1), thus we have
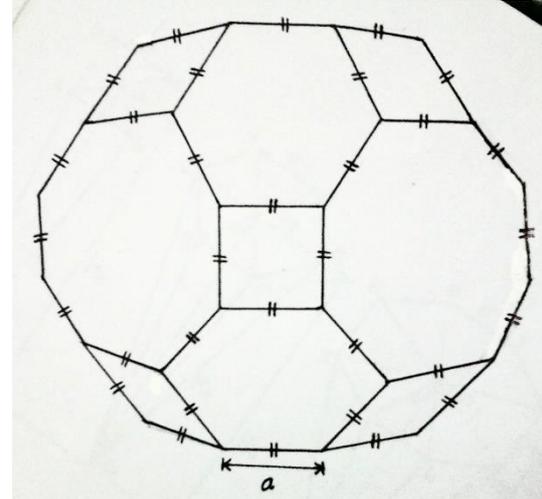


**Figure 1: A great rhombicuboctahedron having 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of equal edge length $a$.**

$$\text{no. of square faces} = 12$$

$$\text{no. of regular hexagonal faces} = 8$$

$$\text{no. of regular octagonal faces} = 6$$

$$\text{no. of edges} = 6(\text{no. of edges in an octagonal face}) + 2(\text{no. of square faces})$$

$$= 6(8) + 2(12) = 48 + 24 = 72$$

$$\text{no. of vertices} = 6(\text{no. of vertices in an octagonal face}) = 6(8) = 48$$

We will apply Theory of Polygon [3,4] to derive a mathematical relationship between radius $R$ of the spherical surface passing through all 48 vertices & the edge length $a$ of a great rhombicuboctahedron for calculating its important parameters such as normal distance of each face, surface area, volume etc.

## 2. Derivation of outer (circumscribed) radius $(R_o)$ of great rhombicuboctahedron

Let $R_o$ be the radius of the spherical surface passing through all 48 vertices of a great rhombicuboctahedron with edge length $a$ & the centre O. Consider a square face ABCD with centre $O_1$, regular hexagonal face EFGHIJ with centre $O_2$ & regular octagonal face KLMNPQRS with centre $O_3$ (see the figure 2 below).

The **normal distance $(H_s)$ of square face** ABCD from the centre O of the great rhombicuboctahedron is calculated as follows

In right $\Delta AO_1O$ (figure 2 below),

$$OO_1 = \sqrt{(OA)^2 - (O_1A)^2} \qquad \left(O_1A = \frac{a}{\sqrt{2}} = \text{circumscribed radius of square}\right)$$

$$\Rightarrow H_s = \sqrt{(R_o)^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{\frac{2R_o{}^2 - a^2}{2}} \qquad \dots\dots\dots\dots\dots(I)$$
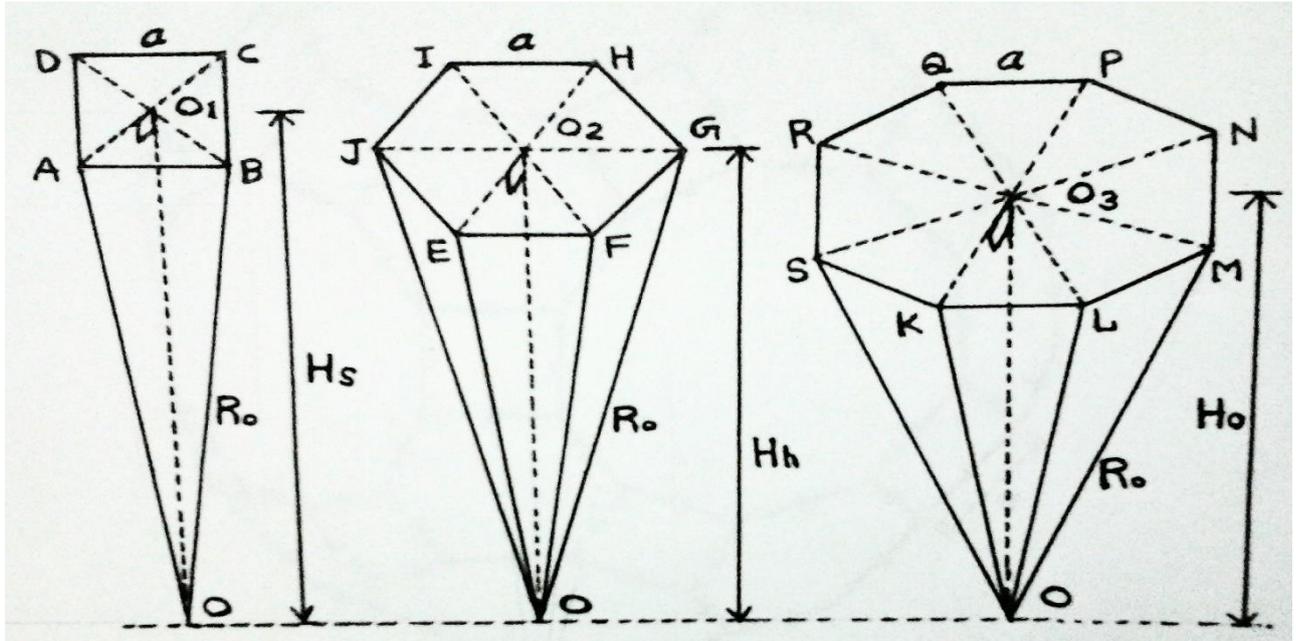
**Figure 2: A square face ABCD, regular hexagonal face EFGHIJ & regular octagonal face KLMNPQRS each of edge length $a$.**

We know that the solid angle ($\omega$) subtended by any regular polygon with each side of length $a$ at any point lying at a distance H on the vertical axis passing through the centre of plane is given by HCR's Theory of Polygon [3,4] as follows

$$\omega = 2\pi - 2n \sin^{-1}\left(\frac{2H \sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2 \cot^2\frac{\pi}{n}}}\right)$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle ($\omega_s$) subtended by each square face (ABCD) at the centre of great rhombicuboctahedron as follows

$$\omega_s = 2\pi - 2 \times 4 \sin^{-1}\left(\frac{2\left(\sqrt{\frac{2R_o{}^2 - a^2}{2}}\right) \sin\frac{\pi}{4}}{\sqrt{4\left(\sqrt{\frac{2R_o{}^2 - a^2}{2}}\right)^2 + a^2 \cot^2\frac{\pi}{4}}}\right)$$

$$= 2\pi - 8 \sin^{-1}\left(\frac{\sqrt{2}\sqrt{2R_o{}^2 - a^2} \times \frac{1}{\sqrt{2}}}{\sqrt{4R_o{}^2 - 2a^2 + a^2(1)^2}}\right) = 2\pi - 8 \sin^{-1}\left(\frac{\sqrt{2R_o{}^2 - a^2}}{\sqrt{4R_o{}^2 - a^2}}\right) = 2\pi - 8 \sin^{-1}\left(\sqrt{\frac{2R_o{}^2 - a^2}{4R_o{}^2 - a^2}}\right)$$

$$\textbf{\textit{Let }} \frac{\boldsymbol{R_o}}{\boldsymbol{a}} = \boldsymbol{x} > \boldsymbol{1} \text{ (any arbitrary variable)}$$

$$\Rightarrow \boldsymbol{\omega_s} = 2\pi - 8 \sin^{-1}\left(\sqrt{\frac{2\left(\frac{R_o}{a}\right)^2 - 1}{4\left(\frac{R_o}{a}\right)^2 - 1}}\right) = \boldsymbol{2\pi - 8 \sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right)} \qquad \dots\dots\dots\dots. (II)$$

Similarly, the **normal distance** $(H_h)$ of **regular hexagonal face** EFGHIJ from the centre O of great rhombicuboctahedron is calculated as follows

In right $\Delta EO_2O$ (figure 2 above),

$$OO_2 = \sqrt{(OE)^2 - (O_2E)^2} \qquad\qquad (O_2E = a = \text{circumscribed radius of regular hexagon})$$

$$\Rightarrow H_h = \sqrt{(R_o)^2 - (a)^2} = \sqrt{R_o{}^2 - a^2} \qquad \ldots\ldots\ldots\ldots\ldots.(III)$$

Hence, by substituting all the corresponding values, the solid angle $(\omega_h)$ subtended by each regular hexagonal face (EFGHIJ) at the centre of great rhombicuboctahedron is given as follows

$$\omega_h = 2\pi - 2 \times 6 \sin^{-1}\left(\frac{2\left(\sqrt{R_o{}^2 - a^2}\right)\sin\frac{\pi}{6}}{\sqrt{4\left(\sqrt{R_o{}^2 - a^2}\right)^2 + a^2 \cot^2\frac{\pi}{6}}}\right)$$

$$= 2\pi - 12 \sin^{-1}\left(\frac{2\sqrt{R_o{}^2 - a^2} \times \frac{1}{2}}{\sqrt{4R_o{}^2 - 4a^2 + a^2\left(\sqrt{3}\right)^2}}\right) = 2\pi - 12 \sin^{-1}\left(\frac{\sqrt{R_o{}^2 - a^2}}{\sqrt{4R_o{}^2 - a^2}}\right)$$

$$\Rightarrow \boldsymbol{\omega_h = 2\pi - 12 \sin^{-1}\left(\sqrt{\frac{R_o{}^2 - a^2}{4R_o{}^2 - a^2}}\right) = 2\pi - 12 \sin^{-1}\left(\sqrt{\frac{x^2 - 1}{4x^2 - 1}}\right)} \qquad \ldots\ldots\ldots\ldots(IV)$$

Similarly, the **normal distance** $(H_o)$ of **regular octagonal face** KLMNPQRS from the centre O of great rhombicuboctahedron is calculated as follows

In right $\Delta KO_3O$ (figure 2 above),

$$\Rightarrow OO_3 = \sqrt{(OE)^2 - (O_3K)^2}$$

$$\boldsymbol{O_3K} = \text{circumscribed radius of regular octagon} = \frac{a}{2}\boldsymbol{\text{cosec}}22.5^o = \frac{a}{2}\sqrt{4 + 2\sqrt{2}}$$

$$\Rightarrow \boldsymbol{H_o} = \sqrt{(R_o)^2 - \left(\frac{a}{2}\sqrt{4 + 2\sqrt{2}}\right)^2} = \sqrt{\frac{4R_o{}^2 - (4 + 2\sqrt{2})a^2}{4}} \qquad \ldots\ldots\ldots\ldots.(V)$$

Hence, by substituting all the corresponding values, the solid angle $(\omega_o)$ subtended by each regular octagonal face (KLMNPQRS) at the centre of great rhombicuboctahedron is given as follows

$$\omega_o = 2\pi - 2 \times 8 \sin^{-1}\left(\frac{2\left(\sqrt{\frac{4R_o{}^2 - (4 + 2\sqrt{2})a^2}{4}}\right)\sin\frac{\pi}{8}}{\sqrt{4\left(\sqrt{\frac{4R_o{}^2 - (4 + 2\sqrt{2})a^2}{4}}\right)^2 + a^2 \cot^2\frac{\pi}{8}}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\frac{\sqrt{4R_o^2 - (4+2\sqrt{2})a^2}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{\sqrt{4R_o^2 - (4+2\sqrt{2})a^2 + a^2(\sqrt{2}+1)^2}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\sqrt{\frac{4\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)R_o^2 - (4+2\sqrt{2})\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)a^2}{4R_o^2 - (4+2\sqrt{2})a^2 + (3+2\sqrt{2})a^2}}\right) = 2\pi - 16\sin^{-1}\left(\frac{\sqrt{(2-\sqrt{2})R_o^2 - a^2}}{\sqrt{4R_o^2 - a^2}}\right)$$

$$\Rightarrow \boldsymbol{\omega_o} = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})R_o^2 - a^2}{4R_o^2 - a^2}}\right) = \boldsymbol{2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2 - 1}{4x^2 - 1}}\right)} \qquad \ldots\ldots\ldots\ldots\ldots (VI)$$

Since a great rhombicuboctahedron is a closed surface & we know that the total solid angle, subtended by any closed surface at any point lying inside it, is $4\pi\ sr$ (Ste-radian) [5], hence the sum of solid angles subtended by 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces at the centre of the great rhombicuboctahedron must be $4\pi\ sr$. Thus we have

$$12[\omega_s] + 8[\omega_h] + 6[\omega_o] = 4\pi$$

Now, by substituting the values of $\boldsymbol{\omega_s, \omega_h}$ & $\boldsymbol{\omega_o}$ from eq(II), (IV) & (VI) in the above expression we get

$$12\left[2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)\right] + 8\left[2\pi - 12\sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)\right] + 6\left[2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)\right]$$
$$= 4\pi$$

$$\Rightarrow 96\left[\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)\right] = 52\pi - 4\pi = 48\pi$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right) = \frac{48\pi}{96} = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) = \frac{\pi}{2} - \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{1 - \left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)^2} + \sqrt{\frac{x^2-1}{4x^2-1}}\sqrt{1 - \left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)^2}\right) = \cos^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)$$

$$\left(\because \sin^{-1}X + \sin^{-1}Y = \sin^{-1}\left(X\sqrt{1-Y^2} + Y\sqrt{1-X^2}\right) \quad \& \quad \frac{\pi}{2} - \sin^{-1}X = \cos^{-1}X\right)$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{\frac{3x^2}{4x^2-1}}+\sqrt{\frac{x^2-1}{4x^2-1}}\sqrt{\frac{2x^2}{4x^2-1}}\right)=\sin^{-1}\left(\sqrt{1-\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)^2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{x\sqrt{3(2x^2-1)}}{4x^2-1}+\frac{x\sqrt{2(x^2-1)}}{4x^2-1}\right)=\sin^{-1}\left(\sqrt{\frac{(2+\sqrt{2})x^2}{4x^2-1}}\right)=\sin^{-1}\left(x\sqrt{\frac{(2+\sqrt{2})}{4x^2-1}}\right)$$

$$\Rightarrow \frac{x\sqrt{3(2x^2-1)}}{4x^2-1}+\frac{x\sqrt{2(x^2-1)}}{4x^2-1}=x\sqrt{\frac{(2+\sqrt{2})}{4x^2-1}}$$

$$\Rightarrow \sqrt{3(2x^2-1)}+\sqrt{2(x^2-1)}=\sqrt{(2+\sqrt{2})(4x^2-1)}$$

$$\Rightarrow \left(\sqrt{3(2x^2-1)}+\sqrt{2(x^2-1)}\right)^2=\left(\sqrt{(2+\sqrt{2})(4x^2-1)}\right)^2$$

$$\Rightarrow 3(2x^2-1)+2(x^2-1)+2\sqrt{6(x^2-1)(2x^2-1)}=(2+\sqrt{2})(4x^2-1)$$

$$\Rightarrow 4(2+\sqrt{2})x^2-(2+\sqrt{2})-8x^2+5=2\sqrt{6(x^2-1)(2x^2-1)}$$

$$\Rightarrow \left(4\sqrt{2}x^2+(3-\sqrt{2})\right)^2=\left(2\sqrt{6(x^2-1)(2x^2-1)}\right)^2$$

$$\Rightarrow 32x^4+8\sqrt{2}(3-\sqrt{2})x^2+(11-6\sqrt{2})=24(x^2-1)(2x^2-1)=48x^4-72x^2+24$$

$$\Rightarrow 16x^4-8(7+3\sqrt{2})x^2+(13+6\sqrt{2})=0$$

Now, solving the above **biquadratic equation for the values of $x>1$** as follows

$$\Rightarrow x^2=\frac{-\left(-8(7+3\sqrt{2})\right)\pm\sqrt{\left(-8(7+3\sqrt{2})\right)^2-4(16)(13+6\sqrt{2})}}{2\times 16}$$

$$=\frac{8(7+3\sqrt{2})\pm 8\sqrt{67+42\sqrt{2}-(13+6\sqrt{2})}}{32}=\frac{(7+3\sqrt{2})\pm\sqrt{54+36\sqrt{2}}}{4}=\frac{(7+3\sqrt{2})\pm 3\sqrt{6+4\sqrt{2}}}{4}$$

$$=\frac{(7+3\sqrt{2})\pm 3\sqrt{(2+\sqrt{2})^2}}{4}=\frac{(7+3\sqrt{2})\pm 3(2+\sqrt{2})}{4}$$

1. **Taking positive sign, we have**

$$x^2=\frac{(7+3\sqrt{2})+3(2+\sqrt{2})}{4}=\frac{13+6\sqrt{2}}{4} \quad or \quad x=\sqrt{\frac{13+6\sqrt{2}}{4}}=\frac{1}{2}\sqrt{13+6\sqrt{2}}$$

Since, $x>1$ hence, the above value is acceptable.

2. **Taking negative sign, we have**

$$x^2 = \frac{(7 + 3\sqrt{2}) - 3(2 + \sqrt{2})}{4} = \frac{1}{4} \quad or \quad x = \sqrt{\frac{1}{4}} = \frac{1}{2} \quad \Rightarrow \quad x < 1 \text{ but } x > 1 \text{ (required condition)}$$

Hence, the above value is discarded, now we have

$$x = \frac{1}{2}\sqrt{13 + 6\sqrt{2}} \quad \Rightarrow \quad \frac{R_o}{a} = x = \frac{1}{2}\sqrt{13 + 6\sqrt{2}} \quad or \quad R_o = \frac{a}{2}\sqrt{13 + 6\sqrt{2}}$$

Hence, the outer (circumscribed) radius $(R_o)$ of a great rhombicuboctahedron with edge length $a$ is given as

$$\boldsymbol{R_o = \frac{a}{2}\sqrt{13 + 6\sqrt{2}} \approx 2.317610913a} \qquad \dots\dots\dots\dots\dots. \quad (VII)$$

### 2.1. Normal distance $(H_s)$ of square faces from the centre of great rhombicuboctahedron

The normal distance $(H_s)$ of each of 12 congruent square faces from the centre O of a great rhombicuboctahedron is given from eq(I) as follows

$$H_s = OO_1 = \sqrt{\frac{2R_o{}^2 - a^2}{2}} = \sqrt{\frac{2\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}{2}} = a\sqrt{\frac{13 + 6\sqrt{2} - 2}{4}} = \frac{a}{2}\sqrt{11 + 6\sqrt{2}} = \frac{a}{2}\sqrt{\left(3 + \sqrt{2}\right)^2}$$

$$= \frac{(3 + \sqrt{2})a}{2}$$

$$\therefore \; \boldsymbol{H_s = \frac{(3 + \sqrt{2})a}{2} \approx 2.207106781a}$$

It's clear that all 12 congruent square faces are at an equal normal distance $H_s$ from the centre of any great rhombicuboctahedron.

### 2.2. Solid angle $(\omega_s)$ subtended by each of the square faces at the centre of great rhombicuboctahedron

The solid angle $(\omega_s)$ subtended by each square face is given from eq(II) as follows

$$\omega_s = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2 - a^2}{4x^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2R_o{}^2 - a^2}{4R_o{}^2 - a^2}}\right) \qquad \left(since, \; x = \frac{R_o}{a}\right)$$

Hence, by substituting the corresponding value of $R_o$ in the above expression, we get

$$\omega_s = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{13 + 6\sqrt{2} - 2}{2(13 + 6\sqrt{2} - 1)}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{11 + 6\sqrt{2}}{12(2 + \sqrt{2})}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{(11 + 6\sqrt{2})(2 - \sqrt{2})}{12(2 + \sqrt{2})(2 - \sqrt{2})}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{10 + \sqrt{2}}{24}}\right)$$

$$\therefore \; \boldsymbol{\omega_s = 2\pi - 8\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{10 + \sqrt{2}}{6}}\right) = 4\sin^{-1}\left(\frac{2 - \sqrt{2}}{12}\right) \approx 0.195339779 \; sr}$$

**2.3. Normal distance $(H_h)$ of regular hexagonal faces from the centre of great rhombicuboctahedron**

The normal distance $(H_h)$ of each of 8 congruent regular hexagonal faces from the centre O of a great rhombicuboctahedron is given from eq(III) as follows

$$H_h = OO_2 = \sqrt{R_o{}^2 - a^2} = \sqrt{\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2} = a\sqrt{\frac{13 + 6\sqrt{2} - 4}{4}} = \frac{a}{2}\sqrt{9 + 6\sqrt{2}} = \frac{a}{2}\sqrt{3\left(1 + \sqrt{2}\right)^2}$$

$$= \frac{\sqrt{3}\left(1 + \sqrt{2}\right)a}{2}$$

$$\therefore\ H_h = \frac{\sqrt{3}\left(1 + \sqrt{2}\right)a}{2} \approx 2.090770275a$$

It's clear that all 8 congruent regular hexagonal faces are at an equal normal distance $H_h$ from the centre of any great rhombicuboctahedron.

**2.4. Solid angle $(\omega_h)$ subtended by each of the regular hexagonal faces at the centre of great rhombicuboctahedron**

The solid angle $(\omega_h)$ subtended by each regular hexagonal face is given from eq(IV) as follows

$$\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{x^2 - a^2}{4x^2 - a^2}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{R_o{}^2 - a^2}{4R_o{}^2 - a^2}}\right) \qquad \left(since,\ x = \frac{R_o}{a}\right)$$

Hence, by substituting the corresponding value of $R_o$ in the above expression, we get

$$\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{13 + 6\sqrt{2} - 4}{4\left(13 + 6\sqrt{2} - 1\right)}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\sqrt{\frac{3 + 2\sqrt{2}}{8\left(2 + \sqrt{2}\right)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(3 + 2\sqrt{2}\right)\left(2 - \sqrt{2}\right)}{8\left(2 + \sqrt{2}\right)\left(2 - \sqrt{2}\right)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{2 + \sqrt{2}}{16}}\right)$$

$$\therefore\ \omega_h = 2\pi - 12\sin^{-1}\left(\frac{\sqrt{2 + \sqrt{2}}}{4}\right) \approx 0.5210126\ sr$$

**2.5. Normal distance $(H_o)$ of regular octagonal faces from the centre of great rhombicuboctahedron**

The normal distance $(H_o)$ of each of 6 congruent regular octagonal faces from the centre O of a great rhombicuboctahedron is given from eq(V) as follows

$$H_o = OO_3 = \sqrt{\frac{4R_o{}^2 - \left(4 + 2\sqrt{2}\right)a^2}{4}} = \sqrt{\frac{4\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - \left(4 + 2\sqrt{2}\right)a^2}{4}} = \frac{a}{2}\sqrt{13 + 6\sqrt{2} - \left(4 + 2\sqrt{2}\right)}$$

$$= \frac{a}{2}\sqrt{9 + 4\sqrt{2}} = \frac{a}{2}\sqrt{\left(1 + 2\sqrt{2}\right)^2} = \frac{\left(1 + 2\sqrt{2}\right)a}{2}$$

$$\therefore\ H_o = \frac{\left(1 + 2\sqrt{2}\right)a}{2} \approx 1.914213562a$$

It's clear that all 6 congruent regular octagonal faces are at an equal normal distance $H_o$ from the centre of any great rhombicuboctahedron.

**2.6. Solid angle $(\omega_o)$ subtended by each of the regular octagonal faces at the centre of great rhombicuboctahedron**

The solid angle $(\omega_o)$ subtended by each regular octagonal face is given from eq(VI) as follows

$$\omega_o = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2 - 1}{4x^2 - 1}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})R_o{}^2 - a^2}{4R_o{}^2 - a^2}}\right) \quad \left(\because x = \frac{R_o}{a}\right)$$

Hence, by substituting the corresponding value of $R_o$ in the above expression, we get

$$\omega_o = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^2 - a^2}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})(13 + 6\sqrt{2}) - 4}{4(13 + 6\sqrt{2} - 1)}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\sqrt{\frac{10 - \sqrt{2}}{24(2 + \sqrt{2})}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(10 - \sqrt{2})(2 - \sqrt{2})}{24(2 + \sqrt{2})(2 - \sqrt{2})}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\sqrt{\frac{22 - 12\sqrt{2}}{48}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(3 - \sqrt{2})^2}{24}}\right) = 2\pi - 16\sin^{-1}\left(\frac{3 - \sqrt{2}}{2\sqrt{6}}\right)$$

$$\therefore \; \boldsymbol{\omega_o = 2\pi - 16\sin^{-1}\left(\frac{3 - \sqrt{2}}{2\sqrt{6}}\right) \approx 1.009032076 \; sr}$$

It's clear from the above results that the solid angle subtended by each of 6 regular octagonal faces is greater than the solid angle subtended by each of 12 square faces & each of 8 regular hexagonal faces at the centre of any great rhombicuboctahedron.

It's also clear from the above results that $H_s > H_h > H_o$ i.e. the normal distance $(H_s)$ of square faces is greater than the normal distance $H_h$ of the regular hexagonal faces & the normal distance $H_o$ of the regular octagonal faces from the centre of a great rhombicuboctahedron i.e. regular octagonal faces are closer to the centre as compared to the square & regular hexagonal faces in any great rhombicuboctahedron.

**3. Important parameters of a great rhombicuboctahedron**

1.  **Inner (inscribed) radius $(R_i)$:** It is the radius of the largest sphere inscribed (trapped inside) in a great rhombicuboctahedron. The largest inscribed sphere always touches all 6 congruent regular octagonal faces but does not touch any of 12 congruent square & any of 8 congruent regular hexagonal faces at all since all 6 octagonal faces are closest to the centre in all the faces. Thus, inner radius is always equal to the normal distance $(H_o)$ of regular octagonal faces from the centre of a great rhombicuboctahedron & is given as follows

$$R_i = H_o = \frac{(1 + 2\sqrt{2})a}{2} \approx 1.914213562a$$

Hence, the **volume of inscribed sphere** is given as

$$V_{inscribed} = \frac{4}{3}\pi(R_i)^3 = \frac{4}{3}\pi\left(\frac{(1+2\sqrt{2})a}{2}\right)^3 \approx 29.38054016a^3$$

2. **Outer (circumscribed) radius** $(R_o)$: It is the radius of the smallest sphere circumscribing a great rhombicuboctahedron or it's the radius of a spherical surface passing through all 48 vertices of a great rhombicuboctahedron. It is from the eq(VII) as follows

$$R_o = \frac{a}{2}\sqrt{13 + 6\sqrt{2}} \approx 2.317610913a$$

Hence, the **volume of circumscribed sphere** is given as

$$V_{circumscribed} = \frac{4}{3}\pi(R_o)^3 = \frac{4}{3}\pi\left(\frac{a}{2}\sqrt{13 + 6\sqrt{2}}\right)^3 = 52.14470211a^3$$

3. **Surface area** $(A_s)$: We know that a great rhombicuboctahedron has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of edge length $a$. Hence, its surface area is given as follows

$$A_s = 12(area\ of\ square) + 8(area\ of\ regular\ hexagon) + 6(area\ of\ regular\ octagon)$$

We know that **area of any regular n-polygon** with each side of length $a$ is given as

$$A = \frac{1}{4}na^2 cot\frac{\pi}{n}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 12 \times \left(\frac{1}{4} \times 4a^2 cot\frac{\pi}{4}\right) + 8 \times \left(\frac{1}{4} \times 6a^2 cot\frac{\pi}{6}\right) + 6 \times \left(\frac{1}{4} \times 8a^2 cot\frac{\pi}{8}\right)$$

$$= 12a^2 + 12\sqrt{3}a^2 + 12(1 + \sqrt{2})a^2 = 12(2 + \sqrt{2} + \sqrt{3})a^2$$

$$A_s = 12(2 + \sqrt{2} + \sqrt{3})a^2 \approx 61.75517244a^2$$

4. **Volume** $(V)$: We know that a great rhombicuboctahedron with edge length $a$ has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces. Hence, the volume (V) of the great rhombicuboctahedron is the sum of volumes of all its elementary right pyramids with square base, regular hexagonal base & regular octagonal base (face) (see figure 2 above) & is given as follows

$$V = 12(volume\ of\ right\ pyramid\ with\ square\ base)$$
$$+ 8(volume\ of\ right\ pyramid\ with\ regular\ hexagonal\ base)$$
$$+ 6(volume\ of\ right\ pyramid\ with\ regular\ octagonal\ base)$$

$$= 12\left(\frac{1}{3}(area\ of\ square) \times H_s\right) + 8\left(\frac{1}{3}(area\ of\ regular\ hexagon) \times H_h\right)$$
$$+ 6\left(\frac{1}{3}(area\ of\ regular\ octagon) \times H_o\right)$$

$$= 12\left(\frac{1}{3}\left(\frac{1}{4} \times 4a^2 cot\frac{\pi}{4}\right) \times \frac{(3 + \sqrt{2})a}{2}\right) + 8\left(\frac{1}{3}\left(\frac{1}{4} \times 6a^2 cot\frac{\pi}{6}\right) \times \frac{\sqrt{3}(1 + \sqrt{2})a}{2}\right)$$
$$+ 6\left(\frac{1}{3}\left(\frac{1}{4} \times 8a^2 cot\frac{\pi}{8}\right) \times \frac{(1 + 2\sqrt{2})a}{2}\right)$$

$$= 2(3 + \sqrt{2})a^3 + 6(1 + \sqrt{2})a^3 + 2(1 + \sqrt{2})(1 + 2\sqrt{2})a^3 = (22 + 14\sqrt{2})a^3$$

$$V = (22 + 14\sqrt{2})a^3 \approx 41.79898987a^3$$

5. **Mean radius ($R_m$):** It is the radius of the sphere having a volume equal to that of a great rhombicuboctahedron. It is calculated as follows

volume of sphere with mean radius $R_m$ = volume of the great rhombicuboctahedron

$$\frac{4}{3}\pi(R_m)^3 = (22 + 14\sqrt{2})a^3 \implies (R_m)^3 = \frac{3(11 + 7\sqrt{2})a^3}{2\pi} \quad or \quad R_m = a\left(\frac{3(11 + 7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}}$$

$$R_m = a\left(\frac{3(11 + 7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}} \approx 2.15290926a$$

It's clear from above results that $R_i < R_m < R_o$

6. **Dihedral angles between the adjacent faces:** In order to calculate dihedral angles between the different adjacent faces with a common edge in a great rhombicuboctahedron, let's consider one-by-one all three pairs of adjacent faces with a common edge as given below.

a. **Angle between square face & regular hexagonal face:** Draw the perpendiculars $OO_1$ & $OO_2$ from the centre O of great rhombicuboctahedron to the square face & the regular hexagonal face which have a common edge (See figure 3). We know that the inscribed radius ($r_i$) of any regular n-gon with each side $a$ is given as follows



**Figure 3:** Square face with centre $O_1$ & regular hexagonal face with centre $O_2$ with a common edge (denoted by point T) normal to the plane of paper.

$$r_i = \text{inscribed radius of any regular n} - \text{gon} = \frac{a}{2}\cot\frac{\pi}{n}$$

$$\therefore \ O_1T = \text{inscribed radius of square} = \frac{a}{2}\cot\frac{\pi}{4} = \frac{a}{2} \ \&$$

$$\therefore \ O_2T = \text{inscribed radius of regular hexagon} = \frac{a}{2}\cot\frac{\pi}{6} = \frac{a\sqrt{3}}{2}$$

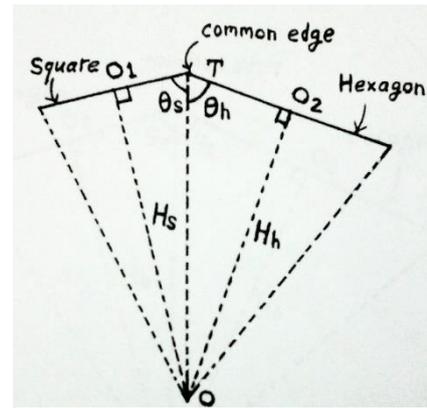In right $\Delta OO_1T$ (Fig. 3),

$$tan\theta_s = \frac{OO_1}{O_1T} = \frac{H_s}{\left(\frac{a}{2}\right)} = \frac{\left(\frac{(3 + \sqrt{2})a}{2}\right)}{\left(\frac{a}{2}\right)} = (3 + \sqrt{2})$$

$$\therefore \ \theta_s = \tan^{-1}(3 + \sqrt{2}) \approx 77.23561032^o \qquad \dots\dots\dots\dots\dots (VIII)$$

In right $\Delta OO_2T$ (Fig. 3),

$$tan\theta_h = \frac{OO_2}{O_2T} = \frac{H_h}{\left(\frac{a\sqrt{3}}{2}\right)} = \frac{\left(\frac{\sqrt{3}(1 + \sqrt{2})a}{2}\right)}{\left(\frac{a\sqrt{3}}{2}\right)} = (1 + \sqrt{2})$$

$$\therefore \ \theta_h = \tan^{-1}(1 + \sqrt{2}) = 67.5^o \qquad \dots\dots\dots\dots\dots (IX)$$

$$\Rightarrow \theta_s + \theta_h = \tan^{-1}(3 + \sqrt{2}) + \tan^{-1}(1 + \sqrt{2}) = \tan^{-1}\left(\frac{(3 + \sqrt{2}) + (1 + \sqrt{2})}{1 - (3 + \sqrt{2})(1 + \sqrt{2})}\right) = \tan^{-1}\left(\frac{4 + 2\sqrt{2}}{1 - (5 + 4\sqrt{2})}\right)$$

$$= \tan^{-1}\left(\frac{-(4 + 2\sqrt{2})}{4 + 4\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{4 + 2\sqrt{2}}{4 + 4\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{1 + \sqrt{2}}{2 + \sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{(1 + \sqrt{2})(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})}\right)$$

$$= \pi - \tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Hence, dihedral angle between the square face & the regular hexagonal face is given as

$$\boldsymbol{\theta_s + \theta_h = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 144.7356103^o}$$

b.  **Angle between square face & regular octagonal face:** Draw the perpendiculars $OO_1$ & $OO_3$ from the centre O of great rhombicuboctahedron to the square face & the regular octagonal face which have a common edge (See figure 4).
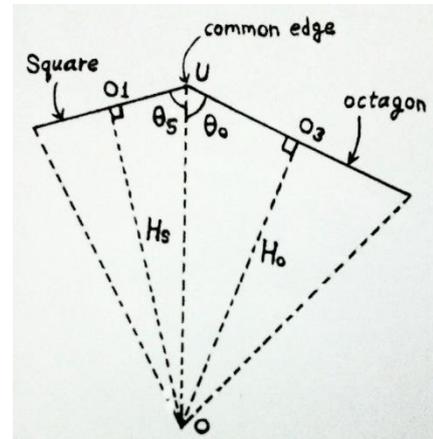
$$O_3U = inscribed\ radius\ of\ regular\ octagon = \frac{a}{2}\cot\frac{\pi}{8} = \frac{a(1 + \sqrt{2})}{2}$$

In right $\Delta OO_3U$ (Fig. 4),

$$tan\theta_o = \frac{OO_3}{O_3U} = \frac{H_o}{\left(\frac{a(1 + \sqrt{2})}{2}\right)} = \frac{\left(\frac{(1 + 2\sqrt{2})a}{2}\right)}{\left(\frac{a(1 + \sqrt{2})}{2}\right)} = \frac{(1 + 2\sqrt{2})}{(1 + \sqrt{2})}$$

$$= \frac{(1 + 2\sqrt{2})(\sqrt{2} - 1)}{(\sqrt{2} + 1)(\sqrt{2} - 1)} = \frac{3 - \sqrt{2}}{(2 - 1)} = 3 - \sqrt{2}$$

**Figure 4: Square face with centre $O_1$ & regular octagonal face with centre $O_3$ with a common edge (denoted by point U) normal to the plane of paper.**

$$\therefore \ \ \boldsymbol{\theta_o = \tan^{-1}(3 - \sqrt{2}) \approx 57.76438969^o} \qquad \ldots \ldots \ldots \ldots \ldots (X)$$

$$\Rightarrow \theta_s + \theta_o = \tan^{-1}(3 + \sqrt{2}) + \tan^{-1}(3 - \sqrt{2}) = \tan^{-1}\left(\frac{(3 + \sqrt{2}) + (3 - \sqrt{2})}{1 - (3 + \sqrt{2})(3 - \sqrt{2})}\right) = \tan^{-1}\left(\frac{6}{1 - 7}\right)$$

$$= \tan^{-1}\left(\frac{-6}{6}\right) = \tan^{-1}(-1) = \pi - \tan^{-1}(1) = 180^o - 45^o = 135^o$$

Hence, dihedral angle between the square face & the regular octagonal face is given as

$$\boldsymbol{\theta_s + \theta_o = 135^o}$$

c.  **Angle between regular hexagonal face & regular octagonal face:** Draw the perpendiculars $OO_2$ & $OO_3$ from the centre O of great rhombicuboctahedron to the regular hexagonal face & the regular octagonal face which have a common edge (see figure 5 below). Now from eq(IX) & (X), we get

$$\theta_h + \theta_o = \tan^{-1}(1 + \sqrt{2}) + \tan^{-1}(3 - \sqrt{2}) = \tan^{-1}\left(\frac{(1 + \sqrt{2}) + (3 - \sqrt{2})}{1 - (1 + \sqrt{2})(3 - \sqrt{2})}\right)$$

$$= \tan^{-1}\left(\frac{4}{1-\left(1+2\sqrt{2}\right)}\right) = \tan^{-1}\left(\frac{-4}{2\sqrt{2}}\right) = \tan^{-1}\left(-\sqrt{2}\right) = \pi - \tan^{-1}\left(\sqrt{2}\right)$$

Hence, dihedral angle between the regular hexagonal face & the regular octagonal face is given as

$$\boldsymbol{\theta_h + \theta_o = \pi - \tan^{-1}\left(\sqrt{2}\right) \approx 125.2643897^o}$$



**Figure 5: Regular hexagonal face with centre $O_2$ & regular octagonal face with centre $O_3$ with a common edge (denoted by point V) normal to the plane of paper.**

## 4. Construction of a solid great rhombicuboctahedron

In order to construct a solid great rhombicuboctahedron with edge length $a$ there are two methods.

**4.1. Construction from elementary right pyramids:** In this method, first we construct all elementary right pyramids as follows

**Step 1:** Construct 12 congruent right pyramids with square base of side length $a$ & normal height ($H_s$)

$$H_s = \frac{(3+\sqrt{2})a}{2} \approx 2.207106781a$$

**Step 2:** Construct 8 congruent right pyramids with regular hexagonal base of side length $a$ & normal height ($H_h$)

$$H_h = \frac{\sqrt{3}\left(1+\sqrt{2}\right)a}{2} \approx 2.090770275a$$

**Step 3:** Construct 6 congruent right pyramids with regular octagonal base of side length $a$ & normal height ($H_o$)

$$H_o = \frac{\left(1+2\sqrt{2}\right)a}{2} \approx 1.914213562a$$

**Step 4:** Now, paste/bond by joining all these elementary right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that 4 edges of each square base (face) coincide with the edges of 2 regular hexagonal bases & 2 regular octagonal bases (faces). Thus a solid great rhombicuboctahedron, with 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of edge length $a$, is obtained.

**4.2. Facing a solid sphere:** It is a method of facing, first we select a **blank as a solid sphere** of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of a great rhombicuboctahedron. Then, we perform the facing operations on the solid sphere to generate 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of equal edge length.

**4.2.1. Edge length:** Let there be a blank as a solid sphere with a diameter D. Then the edge length $a$, of a great rhombicuboctahedron of the maximum volume to be produced, can be co-related with the diameter D by **relation of outer radius $(R_o)$ with edge length $(a)$ of the great rhombicuboctahedron** as follows

$$R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}}$$

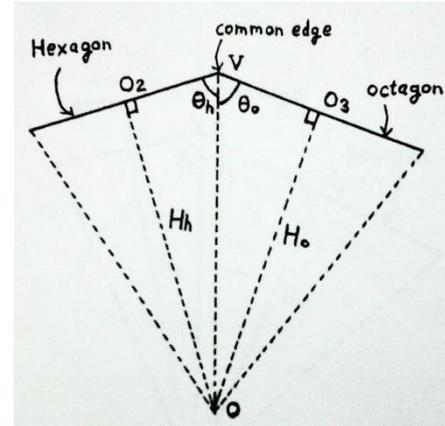Now, substituting $R_o = {}^D\!/_2$ in the above expression, we have

$$\frac{D}{2} = \frac{a}{2}\sqrt{13 + 6\sqrt{2}} \quad or \quad a = \frac{D}{\sqrt{13 + 6\sqrt{2}}}$$

$$\boldsymbol{a = \frac{D}{\sqrt{13 + 6\sqrt{2}}} \approx 0.215739405D}$$

Above relation is very useful for determining the edge length $a$ of a great rhombicuboctahedron to be produced from a solid sphere with known diameter D for manufacturing purpose.

### 4.2.2. Maximum volume of great rhombicuboctahedron produced from a solid sphere is given as follows

$$V_{max} = \left(22 + 14\sqrt{2}\right)a^3 = \left(22 + 14\sqrt{2}\right)\left(\frac{D}{\sqrt{13 + 6\sqrt{2}}}\right)^3 = \frac{\left(22 + 14\sqrt{2}\right)D^3}{\left(13 + 6\sqrt{2}\right)\sqrt{13 + 6\sqrt{2}}}$$

$$= \frac{\left(22 + 14\sqrt{2}\right)\left(13 - 6\sqrt{2}\right)D^3}{97\sqrt{13 + 6\sqrt{2}}} = \frac{\left(118 + 50\sqrt{2}\right)D^3}{97\sqrt{13 + 6\sqrt{2}}}$$

$$\boldsymbol{V_{max} = \frac{2\left(59 + 25\sqrt{2}\right)D^3}{97\sqrt{13 + 6\sqrt{2}}} \approx 0.419714736D^3}$$

### 4.2.3. Minimum volume of material removed is given as

$$\left(V_{removed}\right)_{min} = \left(volume\ of\ parent\ sphere\ with\ diameter\ D\right) \\ - \left(volume\ of\ great\ rhombicuboctahedron\right)$$

$$= \frac{\pi}{6}D^3 - \frac{2\left(59 + 25\sqrt{2}\right)D^3}{97\sqrt{13 + 6\sqrt{2}}} = \left(\frac{\pi}{6} - \frac{2\left(59 + 25\sqrt{2}\right)D^3}{97\sqrt{13 + 6\sqrt{2}}}\right)D^3$$

$$\boldsymbol{\left(V_{removed}\right)_{min} = \left(\frac{\pi}{6} - \frac{2\left(59 + 25\sqrt{2}\right)}{97\sqrt{13 + 6\sqrt{2}}}\right)D^3 \approx 0.103884038D^3}$$

### 4.2.4. Percentage $(\%)$ of minimum volume of material removed

$$\% \ \boldsymbol{of\ V_{removed}} = \frac{minimum\ volume\ removed}{total\ volume\ of\ sphere} \times 100$$

$$= \frac{\left(\frac{\pi}{6} - \frac{2\left(59 + 25\sqrt{2}\right)}{97\sqrt{13 + 6\sqrt{2}}}\right)D^3}{\frac{\pi}{6}D^3} \times 100 = \left(\boldsymbol{1} - \frac{12\left(59 + 25\sqrt{2}\right)}{97\pi\sqrt{13 + 6\sqrt{2}}}\right) \times \boldsymbol{100} \approx \boldsymbol{19.84\%}$$

It's obvious that when a great rhombicuboctahedron of the maximum volume is produced from a solid sphere then about 19.84% volume of material is removed as scraps. Thus, we can select optimum diameter of blank as a solid sphere to produce a solid great rhombicuboctahedron of the maximum volume (or with maximum desired edge length).

**Conclusions:** Let there be any great rhombicuboctahedron having 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each with edge length $a$ then all its important parameters are calculated/determined as tabulated below.

| Congruent polygonal faces | No. of faces | Normal distance of each face from the centre of the great rhombicuboctahedron | Solid angle subtended by each face at the centre of the great rhombicuboctahedron |
|---|---|---|---|
| Square | 12 | $\dfrac{(3+\sqrt{2})a}{2} \approx 2.207106781a$ | $4\sin^{-1}\left(\dfrac{2-\sqrt{2}}{12}\right) \approx 0.195339779\ sr$ |
| Regular hexagon | 8 | $\dfrac{\sqrt{3}(1+\sqrt{2})a}{2} \approx 2.090770275a$ | $2\pi - 12\sin^{-1}\left(\dfrac{\sqrt{2+\sqrt{2}}}{4}\right) \approx 0.5210126\ sr$ |
| Regular octagon | 6 | $\dfrac{(1+2\sqrt{2})a}{2} \approx 1.914213562a$ | $2\pi - 16\sin^{-1}\left(\dfrac{3-\sqrt{2}}{2\sqrt{6}}\right) \approx 1.009032076\ sr$ |

| | |
|---|---|
| **Inner (inscribed) radius ($R_i$)** | $R_i = \dfrac{(1+2\sqrt{2})a}{2} \approx 1.914213562a$ |
| **Outer (circumscribed) radius ($R_o$)** | $R_o = \dfrac{a}{2}\sqrt{13+6\sqrt{2}} \approx 2.317610913a$ |
| **Mean radius ($R_m$)** | $R_m = a\left(\dfrac{3(11+7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}} \approx 2.15290926a$ |
| **Surface area ($A_s$)** | $A_s = 12(2+\sqrt{2}+\sqrt{3})a^2 \approx 61.75517244a^2$ |
| **Volume ($V$)** | $V = (22+14\sqrt{2})a^3 \approx 41.79898987a^3$ |

**Table for the dihedral angles between the adjacent faces of a great rhombicuboctahedron**

| Pair of the adjacent faces with a common edge | Square & regular hexagon | Square & regular octagon | Regular hexagon & regular octagon |
|---|---|---|---|
| **Dihedral angle of the corresponding pair (of the adjacent faces)** | $\theta_s + \theta_h = \pi - \tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ $\approx 144.7356103^o$ | $\theta_s + \theta_o = 135^o$ | $\theta_h + \theta_o = \pi - \tan^{-1}(\sqrt{2})$ $\approx 125.2643897^o$ |

**Note:** Above articles had been developed *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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