Deriving Rotational Quaternions Using Pivot Vectors

Russell P. Patera[1]

Abstract

Pivot Vectors are used to derive the orthonormal triad that forms the basis vectors of rotational quaternions. Pivot Vectors are also used to derive Hamilton's rules for quaternion algebra, which forms the foundation of quaternion parameterization of attitude. The concepts of simultaneous rotations and sequential rotations are used with Hamilton's rules to derive the quaternion composition rule for rotations. The quaternion derivation of the rotation composition rule is compared to the Pivot Vector derivation to clarify the respective attitude parameterizations.

Keywords: Hamilton's algebraic rules, Pivot Vector, quaternion, attitude parameters, sequential rotation

Nomenclature

a, b, c, d, e, f	Pivot Vectors
A, B	original axes of rotation
i, j, k	coordinate axis unit vectors and quaternion basis vectors
N	rotation axis and intersection of rotation planes
P, Q	quaternions, including both scalar and vector components
$\mathbf{q}_A$, $\mathbf{q}_B$	vector part of quaternions related to **A** and **B** axes
$\mathbf{Q}_T$, $\mathbf{q}_t$	total quaternion
q	vector component of quaternion
q0	scalar component of quaternion
U(a, α**)**	attitude transformation for **a** - axis, α - angle
θ, ϕ	rotation angle

1. Introduction

An earlier work [1], introduced Pivot Vector, PV, parameterization of rotational kinematics. The PV equation for combining sequential rotations was derived. The quaternion composition rule was also derived, since the quaternion composition rule is equal to the PV composition rule plus a scalar component not present in the PV equation.

There are significant differences in the PV and quaternion formulations of attitude. Quaternions represent rotations as algebraic operations [2], [3], [4], [5], whereas, the PV formulation is based on the geometry of rotations. Quaternions have both scalar and vector components, but PVs have only vector components. Both a quaternion and its negative represent the same rotational transformation. PV pairs are unique, but can be clocking in the rotation plane to produce the same rotation, as long as, the angle between the PVs and the rotation order remain the same. In spite of these differences, there is a close relationship between PVs and rotational quaternions that was investigated in previous work [6].

[1] Russell.P.Patera@gmail.com

This work uses PV Methodology, PVM, to derive Hamilton's rules that form the core of quaternion algebra. Since PVM is based on the geometry of rotation, using them to derive Hamilton's algebraic rules establishes the related geometry. Thus, PVM clarifies the geometry of quaternion algebra.

Hamilton's rules are used to derive the quaternion composition rule. The concept of simultaneous rotations is used to decompose a quaternion into the sum of its basis vector components. The idea of sequential rotations is important because rotations and the associated quaternion products are not commutative. Since multiplication is distributive under addition, the product of two quaternions is the sum of the product of the respective quaternion components. Because quaternion multiplication is not commutative, the products of the respective quaternion components are also noncommutative. The quaternion composition rule was obtained after the application of Hamilton's algebraic rules to the quaternion components.

Section 2 contains a short summary of PVM, since it will be used to derive Hamilton's rules. Section 3 derives the basis vectors and Hamilton's algebraic rules for quaternions. The sequential rotation about two PVs separated by π/2 radians reduces to a single PV orthogonal to the rotation plane. The three mutually orthogonal PVs form the basis vectors of quaternion algebra. Applying PVM to the sequential rotations about the basis vectors results in Hamilton's rules. Section 4 derives the quaternion composition rule using Hamilton's rules and the concepts of simultaneous and sequential rotations. The PV derivation of the quaternion composition rule is presented for comparison. Section 5 provides the conclusion.

2. Summary of Pivot Vector Methodology

PVM does not use a rotation about an axis to define a rotational transformation, as do many other attitude parameterizations. Instead, PVM, uses a sequential rotation of π radians about the axes of two PVs, (**a, b**), that are located in the rotation plane, as illustrated in Fig. 1. Each PV extends from the center of a unit sphere to its surface. The vector cross product of the PVs defines the axis of rotation and the angle of rotation is twice the separation angle between the PVs. Thus, the angular separation between the PVs is one half the resulting rotation angle and accounts for the half angle that appears in both PVs and quaternions. The PVs are in the equatorial plane of the sphere with the rotation axis aligned with the North polar axis.

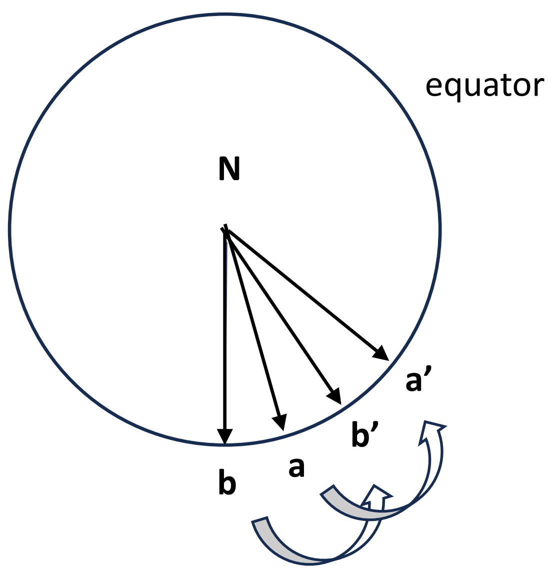

Fig. 1. Sequential rotation of π radians about **a** and **b** result in a rotation about the polar axis by twice the separation angle between **a** and **b**.

PV pairs can be clocked to any location in the rotation plane and still produce the same rotation about the polar axis, as long as the angular separation between them remains the same. Two rotations can be combined by linking the associated PV pairs at the intersection of the respective rotation planes. This is achieved by clocking each pair into the correct position, such that, the 2nd PV of the 1st rotation is aligned with the 1st PV of the second rotation. Since the aligned PVs add to 2π radians, which is equivalent to 0 radians, they effectively cancel each other. Therefore, the combined rotation is defined by the 1st PV of the 1st rotation and the 2nd PV of the 2nd rotation. Linking PV pairs in this manner enable any sequence of rotations to be reduced to a single PV pair representing the combined transformation. The ability of PVM to combine sequential rotation geometrically enable them to derive Hamilton's rules, which involve sequential rotations about the basis vectors used in quaternion algebra.

3. Hamilton's Rules for Quaternions

The transformation produced by a rotation of π radians about PV, **a**, followed by a rotation of π radians about PV, **b**, is shown in eq. (1), where the arguments of **U** are the axis and angle. The cross product term in eq. (1) indicates the direction of the axis, **N**, and the rotation angle, θ, is parameterized in the argument of sin(θ/2). The direction of **N** is independent of the magnitude of θ, as long as **a** and **b** remain in the rotation plane.

$$\mathbf{U}(\mathbf{a},\pi)\,\mathbf{U}(\mathbf{b},\pi) = \mathbf{b}\times\mathbf{a} = \sin\left(\frac{\theta}{2}\right)\mathbf{N} \tag{1}$$

The quaternion, **Q**, equivalent to the PV pair, (**a, b**) is given by eq. (2), as developed in an earlier work [1].

$$\mathbf{Q} = (\mathbf{a}\cdot\mathbf{b}) + (\mathbf{b}\times\mathbf{a}) \tag{2}$$

If we align **a** with the negative **y**-axis and **b** with the **z**-axis in Fig. 2, then sin(θ/2) = 1, θ = π, and **N** is directed along the **x**-axis. Therefore, the PV pair, (**a**, **b**) results in a π rotation about the **x**-axis, as shown in eq. (3).

$$\mathbf{U}(\mathbf{a},\pi)\,\mathbf{U}(\mathbf{b},\pi) = \mathbf{b}\times\mathbf{a} = \mathbf{z}\times(-\mathbf{y}) = \mathbf{y}\times\mathbf{z} = \mathbf{x} = \mathbf{U}(\mathbf{x},\pi) \tag{3}$$

In a similar manner, a rotation about the **y**-axis by π radians is given by the PV pair (**c, d**), as shown in Fig. 2. The resulting transformation is given by eq. (4).

$$\mathbf{U}(\mathbf{y},\boldsymbol{\pi}) = \mathbf{U}(\mathbf{c},\pi)\,\mathbf{U}(\mathbf{d},\pi) = \mathbf{d}\times\mathbf{c} = (-\mathbf{x})\times\mathbf{z} = \mathbf{z}\times\mathbf{x} = \mathbf{y} \tag{4}$$

The **x** and **y** rotations can be linked at the **z**-axis, which is the intersection of the respective rotation planes, shown in Fig. 2. Thus, the **x**-axis rotation followed by the **y**-axis rotation yields a π rotation about the **z**-axis, as given by eq. (5). The rotations about **b** and **c** cancel, since **b = c**, which reduces to the PV pair (**a, d**), as shown in eq. (5). This illustrates how rotations can be easily combined using PVM.

$$\mathbf{U}(\mathbf{x},\pi)\,\mathbf{U}(\mathbf{y},\boldsymbol{\pi}) = \mathbf{U}(\mathbf{a},\pi)\,\mathbf{U}(\mathbf{b},\pi)\,\mathbf{U}(\mathbf{c},\pi)\,\mathbf{U}(\mathbf{d},\pi) = \mathbf{U}(\mathbf{a},\pi)\,\mathbf{U}(\mathbf{d},\pi) = \mathbf{U}(\mathbf{z},\pi) \tag{5}$$

In general, the sequential rotations about two PV separated by π/2, results in another PV orthogonal to the original PVs. The three PVs form an orthonormal triad that is used in defining Hamilton's quaternion algebra. The PVs can be cyclically permuted to create eqs. (6-8), similar to the vector cross product operation. Reversing the rotation order yields eqs. (9-11).

$$\mathbf{U}(\mathbf{x},\pi)\,\mathbf{U}(\mathbf{y},\pi) = \mathbf{U}(\mathbf{z},\pi) \tag{6}$$

$$\mathbf{U}(\mathbf{y},\pi)\,\mathbf{U}(\mathbf{z},\pi) = \mathbf{U}(\mathbf{x},\pi) \tag{7}$$

$$\mathbf{U}(\mathbf{z},\pi)\,\mathbf{U}(\mathbf{x},\pi) = \mathbf{U}(\mathbf{y},\pi) \tag{8}$$

$$\mathbf{U}(\mathbf{y},\pi)\,\mathbf{U}(\mathbf{x},\pi) = -\mathbf{U}(\mathbf{z},\pi) \tag{9}$$

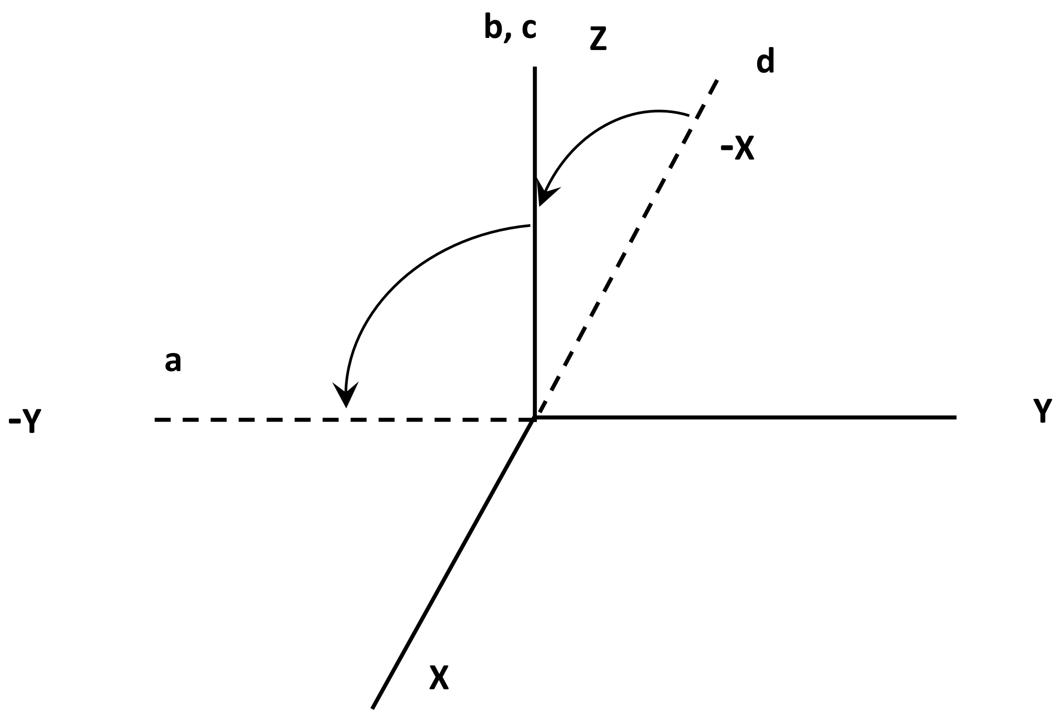

Fig. 2. Pivot Vector geometry for sequential rotations of π radians about the x and y axes.

$$\mathbf{U}(\mathbf{z}, \pi)\ \mathbf{U}(\mathbf{y}, \pi) = -\mathbf{U}(\mathbf{x}, \pi) \tag{10}$$

$$\mathbf{U}(\mathbf{x}, \pi)\ \mathbf{U}(\mathbf{z}, \pi) = -\mathbf{U}(\mathbf{y}, \pi) \tag{11}$$

Eqs. (6-11) are written in compact form shown in eqs. (12-17), where it is assumed that a π rotation is performed about each axis and the rotations are in sequential order.

$$\mathbf{x}\ \mathbf{y} = \mathbf{z} \tag{12}$$

$$\mathbf{y}\ \mathbf{z} = \mathbf{x} \tag{13}$$

$$\mathbf{z}\ \mathbf{x} = \mathbf{y} \tag{14}$$

$$\mathbf{y}\ \mathbf{x} = -\mathbf{z} \tag{15}$$

$$\mathbf{z}\ \mathbf{y} = -\mathbf{x} \tag{16}$$

$$\mathbf{x}\ \mathbf{z} = -\mathbf{y} \tag{17}$$

A rotation about the **z**-axis by π radians is given by the PV pair (**e, f**), as shown in Fig. 3. The resulting transformation is given by eq. (18).

$$\mathbf{U}(\mathbf{z}, \boldsymbol{\pi}) = \mathbf{U}(\mathbf{e}, \pi)\ \mathbf{U}(\mathbf{f}, \pi) = \mathbf{f} \times \mathbf{e} = (\mathbf{y}) \times (-\mathbf{x}) = \mathbf{x} \times \mathbf{y} = \mathbf{z} \tag{18}$$

The sequential rotation about the x, y and z axes is given by eq. (19), since **d** = **e**. Note that **f** = -**a**.

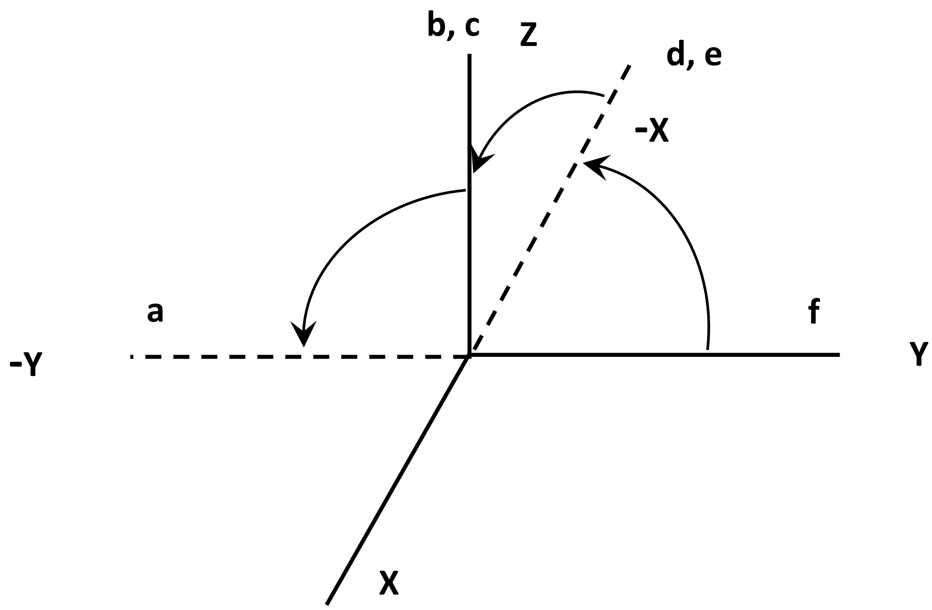

Fig. 3. Pivot Vector geometry for sequential rotations of π radians about the x, y and z axes.

$$\mathbf{U}(\mathbf{x},\pi)\ \mathbf{U}(\mathbf{y},\boldsymbol{\pi})\ \mathbf{U}(\mathbf{z},\boldsymbol{\pi}) = \mathbf{U}(\mathbf{a},\pi)\ \mathbf{U}(\mathbf{b},\pi)\ \mathbf{U}(\mathbf{c},\pi)\ \mathbf{U}(\mathbf{d},\pi)\ \mathbf{U}(\mathbf{e},\pi)\ \mathbf{U}(\mathbf{f},\pi) = \mathbf{U}(\mathbf{a},\pi)\ \mathbf{U}(\mathbf{f},\pi) \tag{19}$$

The rotation angle associated with eq. (19) is zero, but the related quaternion is given by -1, as shown in eq. (20).

$$\mathbf{q_T} = \mathrm{q}_0 + \mathbf{q}\,,\ \text{where}\ \ \mathrm{q}_0 = \mathbf{a}\cdot\mathbf{f} = \mathbf{a}\cdot(-\mathbf{a}) = -1\,,\quad \mathbf{q} = (-\mathbf{a})\times\mathbf{a} = 0$$

$$\mathbf{q_T} = (-1 \quad 0 \quad 0 \quad 0) \tag{20}$$

Eq. (19) can be written more compactly to obtain one of Hamilton's rules, as shown in eq. (21).

$$\mathbf{U}(\mathbf{x},\pi)\ \mathbf{U}(\mathbf{y},\pi)\ \mathbf{U}(\mathbf{z},\pi) = \mathbf{i}\ \mathbf{j}\ \mathbf{k} = -1 \tag{21}$$

Fig. 4 illustrates the sequential rotation about the **x**-axis of π radians followed by a second rotation about the **x**-axis of π radians. In this case, **a** = **-y, b** = **z, c** = **z**, and **d** = **y**, where the 2nd PV of the 1st rotation is aligned with the 1st PV of the 2nd rotation. The combined transformation is shown in eq. (22}, where **d** = **-a**.

$$\mathbf{U}(\mathbf{x},\pi)\ \mathbf{U}(\mathbf{x},\boldsymbol{\pi}) = \mathbf{U}(\mathbf{a},\pi)\ \mathbf{U}(\mathbf{b},\pi)\ \mathbf{U}(\mathbf{c},\pi)\ \mathbf{U}(\mathbf{d},\pi) = \mathbf{U}(\mathbf{a},\pi)\ \mathbf{U}(\mathbf{d},\boldsymbol{\pi}) = \mathbf{U}(\mathbf{a},\pi)\ \mathbf{U}(-\mathbf{a},\pi) \tag{22}$$

The rotation angle associated with eq. (22) is zero, but the related quaternion is given by -1, as shown in eq. (23).

$$\mathbf{q_T} = \mathrm{q}_0 + \mathbf{q}\,,\ \text{where}\ \ \mathrm{q}_0 = \mathbf{a}\cdot(-\mathbf{a}) = -1\,,\quad \mathbf{q} = (-\mathbf{a})\times\mathbf{a} = 0 \tag{23}$$

$$\mathbf{q_T} = (-1 \quad 0 \quad 0 \quad 0)$$

Equations similar to eq. (22) can be derived for the **y**-axis and the **z**-axis and can be simplified, as shown in eqs. (24-26).

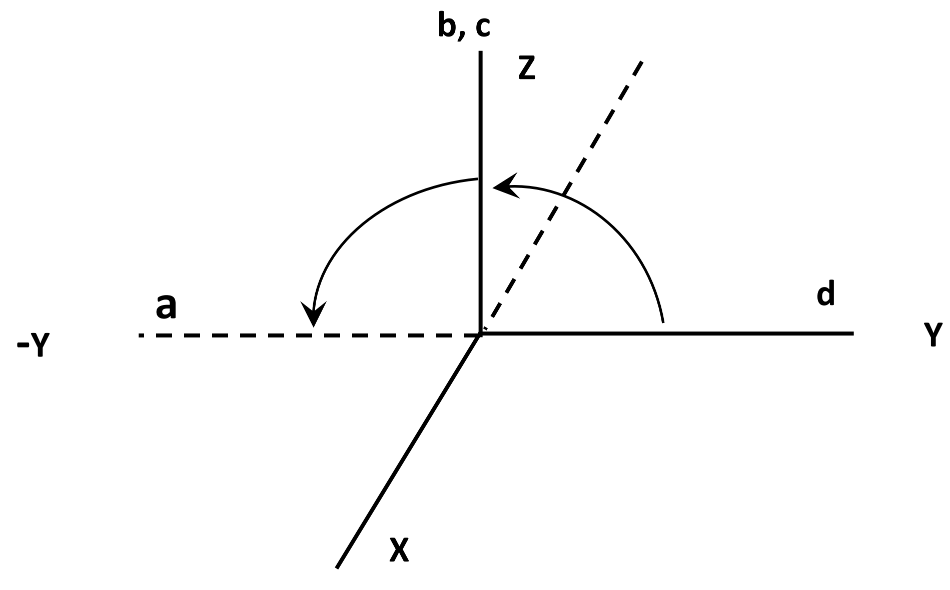

Fig. 4. Pivot Vector geometry for two sequential rotations about the x-axis of π radians.

$$\mathbf{x}\,\mathbf{x} = -1 \tag{24}$$

$$\mathbf{y}\,\mathbf{y} = -1 \tag{25}$$

$$\mathbf{z}\,\mathbf{z} = -1 \tag{26}$$

Eqs. (12-17), eq. (21), and eqs. (24-26) are Hamilton's rules for quaternion algebra. The unit vector axes appear to be quaternion products, but they were also shown to be sequential rotations of π radians about the respective axes.

4. Quaternion composition rule

Hamilton's rules for quaternion algebra can be used to derive the quaternion composition rules, which is used to combine a sequence of two rotations. A rotation of θ radians about unit vector axis, **A**, followed by a rotation of ϕ radians about unit vector axis **B**, involves the sequential product of quaternions, **P** and **Q** , as shown in eq. (27) with **P** and **Q** defined in eqs. (28) and eq. (29), where **i, j, k** represents the **x, y, z** unit vector axes, respectively.

$$\mathbf{Q_T} = \mathbf{P}\,\mathbf{Q} = \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\mathbf{A}\right]\left[\cos\left(\frac{\Phi}{2}\right) + \sin\left(\frac{\Phi}{2}\right)\mathbf{B}\right] \tag{27}$$

$$\mathbf{P} = \mathrm{q_A}(0) + \mathbf{q_A} = \cos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right)(\mathbf{A_X}\,\mathbf{i} + \mathbf{A_Y}\,\mathbf{j} + \mathbf{A_Z}\,\mathbf{k}) \tag{28}$$

$$\mathbf{Q} = \mathrm{q_B}(0) + \mathbf{q_B} = \cos\left(\frac{\Phi}{2}\right) + sin\left(\frac{\Phi}{2}\right)(\mathbf{B_X}\,\mathbf{i} + \mathbf{B_Y}\,\mathbf{j} + \mathbf{B_Z}\,\mathbf{k}) \tag{29}$$

Since multiplication is distributive over addition, the product in eq. (27) can be simplified to eq. (30).

$$\mathbf{Q_T} = \mathbf{P}\,\mathbf{Q} = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) + \cos\left(\frac{\Phi}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathbf{A} + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)\mathbf{B} + \sin\left(\frac{\Phi}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathbf{A}\,\mathbf{B} \tag{30}$$

The vector part of each quaternion can be broken into its i, j, k components because the components are simultaneous rotations, as shown in eq. (28) and eq. (29). Since the last term in eq. (30) contains the sequential rotation of **A** and **B,** the order of the respective products, $\mathbf{A_i}\,\mathbf{B_j}$, must be maintained. After expanding the product of **A B** and using Hamilton's rules to simplify the result, the last term in eq. (30) becomes eq. (31).

$$\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)\mathbf{A}\,\mathbf{B} = -\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)(\mathbf{A}\cdot\mathbf{B}) + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)(\mathbf{A}\times\mathbf{B}) \tag{31}$$

Using eqs. (28-31), the combined transformation for the product of quaternions **P** and **Q** is given in eq. (32).

$$\mathbf{Q_T} = \mathbf{P}\,\mathbf{Q} = \mathrm{q_A}(0)\,\mathrm{q_B}(0) - (\mathbf{q_A}\cdot\mathbf{q_B}) + \mathrm{q_B}(0)\,\mathbf{q_A} + \mathrm{q_A}(0)\,\mathbf{q_B} + (\mathbf{q_A}\times\mathbf{q_B}) \tag{32}$$

Eq. (32) agrees with the composition rule derived using PVs in an earlier work [1].

The PV method to derive eq. (32) is summarized, as follows. The intersection of the **A**-axis and **B**-axis rotation planes is obtained by the unitized cross product shown in eq. (33).

$$\mathbf{N} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} \tag{33}$$

The resulting vector, **N**, in eq. (29) is aligned with the 2nd PV of the 1st rotation and the 1st PV of the 2nd rotation. The 1st PV of the 1st rotation, **a**, is constructed, such that **N x a** results in a rotation of θ about axis **A**, as shown in eq. (34). This process ensures that the PV pair, (**a, N**) is in the correct clocking location in the rotation plane.

$$\mathbf{a} = \cos\left(\frac{\theta}{2}\right)\mathbf{N} + \sin\left(\frac{\theta}{2}\right)(\mathbf{A}\times\mathbf{N}) \tag{34}$$

In a similar manner, the 2nd PV of the 2nd rotation, **b**, is constructed such that **b x N** results in a rotation of φ about axis **B**, as shown in eq. (35). This ensures that the PV pair (**N, b**) is in the proper location in the rotation plane.

$$\mathbf{b} = \cos\left(\frac{\Phi}{2}\right)\mathbf{N} + \sin\left(\frac{\Phi}{2}\right)(\mathbf{N}\times\mathbf{B}) \tag{35}$$

The combined rotation is given by the sequential rotations associated with the PV pairs (**a, N)** and **(N, b),** as shown in eq. (36), where the π rotations about **N** add to zero rotation.

$$\mathbf{U}(\mathbf{a},\pi)\,\mathbf{U}(\mathbf{N},\pi)\,\mathbf{U}(\mathbf{N},\pi)\,\mathbf{U}(\mathbf{b},\pi) = \mathbf{U}(\mathbf{a},\pi)\,\mathbf{U}(\mathbf{b},\pi) = \mathbf{b}\times\mathbf{a} \tag{36}$$

Evaluating the cross product of **b x a** in eq. (36), using eq. (34) and eq. (35) yields the composition rule for PVM, as shown in eq. (37).

$$\mathbf{b}\times\mathbf{a} = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\Phi}{2}\right)\mathbf{A} + \sin\left(\frac{\Phi}{2}\right)\cos\left(\frac{\theta}{2}\right)\mathbf{B} + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)(\mathbf{A}\times\mathbf{B}) \tag{37}$$

Since the associated quaternion, $\mathbf{Q_T}$ includes a scalar part, shown in eq. (38), the related quaternion composition rule is given in eq. (39).

$$(\mathbf{a}\cdot\mathbf{b}) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)(\mathbf{A}\cdot\mathbf{B}) \tag{38}$$

$$\mathbf{Q_T} = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)(\mathbf{A}\cdot\mathbf{B}) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\Phi}{2}\right)\mathbf{A} + \sin\left(\frac{\Phi}{2}\right)\cos\left(\frac{\theta}{2}\right)\mathbf{B} + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)(\mathbf{A}\times\mathbf{B}) \tag{39}$$

Using the definition of the quaternions in eq. (28) and eq. (29), eq. (39) becomes eq. (40), which is in agreement with eq. (32). Thus, quaternion algebra and PVM geometry result in the same rotation composition equation.

$$\mathbf{Q_T} = \mathbf{P}\,\mathbf{Q} = \mathrm{q_A}(0)\,\mathrm{q_B}(0) - (\mathbf{q_A}\cdot\mathbf{q_B}) + \mathrm{q_B}(0)\,\mathbf{q_A} + \mathrm{q_A}(0)\,\mathbf{q_B} + (\mathbf{q_A}\times\mathbf{q_B}) \tag{40}$$

5. Summary

Pivot Vector Methodology was used to derive Hamilton's algebraic rules, which form the foundation of quaternion parameterization of attitude. The sequential rotation of π radians about two orthogonal PVs results in a single PV along the rotation axis. The original PV pair plus the third PV form an orthonormal triad that is the basis vectors of quaternion parameterization. The basis vectors obey the rules of cross product multiplication, which become part of Hamilton's algebraic rules. The remaining Hamilton rules were obtained using PVM to link the associated sequential rotations represented by Hamilton's basis vectors. The quaternion composition equation was found by considering two sequential rotations as the product of the associated quaternions with each quaternion composed of simultaneous rotations along the basis vectors. The multiplication was distributed among the individual components and Hamilton's rules were applied to the product of the basis vectors, each of which represents a sequential rotation. The results produced the quaternion composition rule for rotations. The Pivot Vector derivation of the quaternion composition rule was included to clarify the differences in the respective attitude formulations. The geometry of the Pivot Vector method aids in understanding Hamilton's algebraic rules and the quaternion parameterization of attitude.

References

1. Patera, R. P., 2017. New fundamental parameters for attitude representation, Advances in Space Research 60, 557-570.

2. Shuster, M. D., 1993. A survey of attitude representations, Journal of the Astronautical Science, 41 (4), 439-517.

3. Kuipers, J. B.,1999. Quaternions and Rotation Sequences. Princeton University Press.

4. Wyse-Gallifent, J., 2021. MATH431: Quaternions, https://math.umd.edu/~immortal/MATH431/book/ch_quaternions.pdf

5. Rose, D., 2015. https://danceswithcode.net/engineeringnotes/quaternions/quaternions.html

6. Patera, R., P., 2025. The Relationship Between Pivot Vectors and Rotation Quaternions, https://vixra.org/abs/2505.0089, https://doi.org/10.13140/RG.2.2.29294.93765.