A Family of Analytic π-Series via Modulus-q Expansions

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Abstract

We present a new analytic framework for generating infinite series representations of π. Using the digamma reflection formula and a modulus-q expansion strategy, we construct families of convergent π-series. In particular, we derive a novel modulus-5 weighted series (Khan's π-series), which converges absolutely and exhibits faster convergence than the classical Leibniz series. The approach generalizes naturally to arbitrary moduli $q \geq 3$, yielding entire families of π-series with adjustable convergence behaviour.

1.Introduction

The computation and representation of the mathematical constant π have been central to mathematical analysis since ancient times. While numerous series expansions exist, from the classical Leibniz-Gregory series:

$$\pi = 4 \sum_{k=0}^{\infty} \frac{-(1)^k}{2k+1} \tag{1}$$

to more modern constructions, the search for efficient and elegant representations remains active.

In this paper, we present a unified framework based on the analytic properties of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$. Our main contribution is a systematic method for generating π-series through modulus-q expansions of the digamma reflection formula.

Theorem 1.1 (Main Result). For any integer $q \geq 3$, let $m = \lfloor (q-1)/2 \rfloor$. Then there exist coefficients $\alpha_1, \ldots, \alpha_m$ such that:

$$\pi = q \sum_{k=0}^{\infty} \sum_{r=1}^{\lfloor (q-1)/2 \rfloor} \propto_r \left(\frac{1}{qk+r} - \frac{1}{qk+q-r} \right) \tag{2}$$

The optimal coefficients are explicitly given by:

$$\propto_r = \frac{\cot\left(\frac{\pi r}{q}\right)}{\sum_{j=1}^{\lfloor (q-1)/2 \rfloor} \cot^2\left(\frac{\pi j}{q}\right)} \tag{3}$$

Novelty Statement: While special cases of (2) appear in literature, the systematic generation of π-series through modulus-q expansions with optimally chosen coefficients appears to be new. The explicit modulus-5 case (Khan's series) demonstrates faster convergence than classical alternatives.

2. Preliminaries

We recall fundamental properties of the digamma function [1, 2]

Definition 2.1. The digamma function is defined as $\psi(z) = \frac{d\ln\Gamma(z)}{dz}$,

Lemma 2.2 (Digamma Reflection Formula [1]). For $z \notin \mathbb{Z}$:

$$\psi(1-z) - \psi(z) = \pi\cot(\pi z) \tag{4}$$

Lemma 2.3 (Series Representation [2]). For $z \notin \mathbb{Z} \leq_0$:

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} + \frac{1}{n+z}\right) \tag{5}$$

where γ is the Euler-Mascheroni constant.

3. Modular Series Construction

Theorem 3.1 (Modular Digamma Expansion). For integers $1 \leq p < q$:

$$\pi\cot\left(\pi\frac{p}{q}\right) = q\sum_{k=0}^{\infty}\left(\frac{1}{qk+r} - \frac{1}{qk+q-r}\right) \tag{6}$$

Proof. From Lemma 2.3:

$$\psi(1-z) - \psi(z) = \left[-\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} + \frac{1}{n+1-z}\right)\right] - \left[-\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} + \frac{1}{n+z}\right)\right]$$

$$\Rightarrow \psi(1-z) - \psi(z) = \sum_{n=0}^{\infty}\left(\frac{1}{n+z} + \frac{1}{n+1-z}\right)$$

Substituting z = p/q and gives:

$$\pi\cot\left(\pi\frac{p}{q}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n+p/q} + \frac{1}{n+1-p/q}\right)$$

Multiplying the numerators and denominators of the fractions by q clarifies the structure:

$$\pi\cot\left(\pi\frac{p}{q}\right) = \sum_{n=0}^{\infty}\left[\frac{1}{\frac{qn+p}{q}} + \frac{1}{\frac{qn+(q-p)}{q}}\right] = q\sum_{n=0}^{\infty}\left(\frac{1}{qn+p} - \frac{1}{qn+(q-p)}\right)$$

Letting $r \equiv p \pmod q$ (which, for $1 \le p < q$, simplifies to r = p), we obtain the final form:

$$\pi\cot\left(\pi\frac{p}{q}\right) = q\sum_{k=0}^{\infty}\left(\frac{1}{qk+r} - \frac{1}{qk+q-r}\right)$$

□

4. Modulus-5 Weighted π-Series

4.1 Specialization to Modulus 5

Taking $q = 5$, we obtain two independent modular series from Theorem 3.1.

Definition 4.1. For modulus 5, define the series:

- For $z_1 = \frac{1}{5}$, we get

$$\psi\left(\frac{4}{5}\right) - \psi\left(\frac{1}{5}\right) = \pi\cot\left(\frac{\pi}{5}\right) = 5\sum_{k=0}^{\infty}\left(\frac{1}{5k+1} - \frac{1}{5k+4}\right)$$

Define

$$S_1 = \sum_{k=0}^{\infty}\left(\frac{1}{5k+1} - \frac{1}{5k+4}\right) \qquad\qquad 7$$

- For $z_2 = \frac{2}{5}$, we get

$$\psi\left(\frac{3}{5}\right) - \psi\left(\frac{2}{5}\right) = \pi\cot\left(\frac{2\pi}{5}\right) = 5\sum_{k=0}^{\infty}\left(\frac{1}{5k+2} - \frac{1}{5k+3}\right)$$

Define

$$S_2 = \sum_{k=0}^{\infty}\left(\frac{1}{5k+2} - \frac{1}{5k+3}\right) \qquad\qquad 8$$

So we have two **independent modulus-5 alternating series** S_1, S_2.

Lemma 4.2. These series satisfy:

$$\pi\cot\left(\frac{\pi}{5}\right) = 5S_1$$

$$\pi\cot\left(\frac{2\pi}{5}\right) = 5S_2$$

4.2 Weighted linear combination

Theorem 4.3 (Weighted Combination). For any coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$:

$$\sum_{i=1}^{2} \alpha_i\big(\psi(1 - z_i) - \psi(z_i)\big) = \pi \sum_{i=1}^{2} \alpha_i cot(\pi z_i) = 5 \sum_{k=0}^{\infty} \left[\propto_1 \left(\frac{1}{5k+1} - \frac{1}{5k+4} \right) + \propto_2 \left(\frac{1}{5k+2} - \frac{1}{5k+3} \right) \right] \qquad 9$$

If we enforce the normalization condition:

$$\propto_1 cot\left(\frac{\pi}{5}\right) + \propto_2 cot\left(\frac{2\pi}{5}\right) = 1$$

Then:

$$\pi = 5 \sum_{k=0}^{\infty} \left[\propto_1 \left(\frac{1}{5k+1} - \frac{1}{5k+4} \right) + \propto_2 \left(\frac{1}{5k+2} - \frac{1}{5k+3} \right) \right] \qquad 10$$

Proof. From Lemma 4.2, each series satisfies:

$$\pi cot(\pi z_i) = 5 S_i, \quad where\ i = 1,2$$

Therefore, for any coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$:

$$\sum_{i=1}^{2} \alpha_i\big(\psi(1 - z_i) - \psi(z_i)\big) = \sum_{i=1}^{2} \alpha_i\big(cot(\pi z_i)\big)$$

$$= \pi \sum_{i=1}^{2} \alpha_i\big(cot(\pi z_i)\big)$$

$$= 5 \sum_{i=1}^{2} \alpha_i S_i$$

$$= 5 \sum_{k=0}^{\infty} \left[\propto_1 \left(\frac{1}{5k+1} - \frac{1}{5k+4} \right) + \propto_2 \left(\frac{1}{5k+2} - \frac{1}{5k+3} \right) \right]$$

If we impose the normalization condition:

$$\propto_1 cot\left(\frac{\pi}{5}\right) + \propto_2 cot\left(\frac{2\pi}{5}\right) = 1$$

then the left-hand side becomes:

$$\pi \times 1 = \pi$$

yielding equation (13).

□

4.3 Optimal Choice of Coefficients

Theorem 4.4 (Optimal Coefficients). The optimal coefficients satisfying with normalization condition are given by:

$$\alpha_1 = \frac{1}{2} \cot\left(\frac{\pi}{5}\right) \ , \alpha_2 = \frac{1}{2} \cot\left(\frac{2\pi}{5}\right)$$

with numerical values:

$$\alpha_1 \approx 0.688190960236 \ , \ \alpha_2 \approx 0.162459848116$$

Proof. Define the vector:

$$c = \left(\cot\left(\frac{\pi}{5}\right), \cot\left(\frac{2\pi}{5}\right)\right) \approx \left(\sqrt{1 + \frac{2}{\sqrt{5}}} \ , \sqrt{1 - \frac{2}{\sqrt{5}}}\right)$$

The normalization condition (12) becomes $\alpha \cdot c = 1$. The optimal choice minimizing $\|\alpha\|_2$ is :

$$\alpha = \frac{c}{\|c\|^2} = \frac{c}{c \cdot c} \qquad\qquad 11$$

Computing the squared norm:

$$\|c\|^2 = \cot^2\left(\frac{\pi}{5}\right) + \cot^2\left(\frac{2\pi}{5}\right)$$

$$\Rightarrow \|c\|^2 = \left(\sqrt{1 + \frac{2}{\sqrt{5}}}\right)^2 + \left(\sqrt{1 - \frac{2}{\sqrt{5}}}\right)^2$$

$$\Rightarrow \|c\|^2 \approx 1.8944271909999 + 0.1055728090001 = 2.0000000000000 \approx 2$$

Thus $\alpha = c/2$, yielding the stated coefficients.

$\square$

4.4 Convergence Analysis

Definition 4.5. Define the partial sums:

$$T_N = \sum_{k=0}^{N-1} \left[5 \, \alpha_1 \left(\frac{1}{5k+1} - \frac{1}{5k+4}\right) + 5 \, \alpha_2 \left(\frac{1}{5k+2} - \frac{1}{5k+3}\right)\right] \qquad 12$$

and the infinite sum:

$$S = 5 \sum_{k=0}^{\infty} \left[\alpha_1 \left(\frac{1}{5k+1} - \frac{1}{5k+4}\right) + \alpha_2 \left(\frac{1}{5k+2} - \frac{1}{5k+3}\right)\right]$$

Theorem 4.6 (Convergence). The series converges absolutely and:

$$\lim_{N \to \infty} T_N = S = \pi$$

Proof. Consider the basic differences:

$$\Delta_r(k) = \frac{1}{5k+r} - \frac{1}{5k+5-r} \; , \; r = 1, 2$$

for large k we expand :

$$\Delta_r(k) = \frac{(5k+5-r)-(5k+r)}{(5k+r)(5k+5-r)} = \frac{5-2r}{(5k+r)(5k+5-r)}$$

Thus for large k

$$\Delta_r(k) = \frac{5-2r}{25k^2} + o\left(\frac{1}{k^3}\right)$$

So $\Delta_r(k) = o\left(\frac{1}{k^2}\right)$. Therefore each series $\sum_{k \geq 0} \Delta_r(k)$ is absolutely convergent (because $\sum \frac{1}{k^2}$ converges).

Now each term in **S** is a linear combination $5 \propto_1 \Delta_1(k) + 5 \propto_2 \Delta_2(k)$, so also $o\left(\frac{1}{k^2}\right)$. Hence the infinite sum **S** converges absolutely; therefore the partial sums T_N converge to **S** as $N \to \infty$

Theorem 4.7 (Error Bound). The tail error satisfies :

$$|S - T_N| \leq \frac{C}{N} \; , C = \frac{|\propto_1| + |\propto_2|}{5}$$

Proof. From the asymptotic expansion:

$$|5 \propto_1 \Delta_1(k) + 5 \propto_2 \Delta_2(k)| \leq \frac{5|\propto_1| + 5|\propto_2|}{5k^2} = \frac{|\propto_1| + |\propto_2|}{5k^2}$$

Thus

$$|S - T_N| \leq \sum_{k=N}^{\infty} \frac{|\propto_1| + |\propto_2|}{5k^2} \leq \frac{|\propto_1| + |\propto_2|}{5} \int_{N-1}^{\infty} \frac{1}{x^2} dx = \frac{|\propto_1| + |\propto_2|}{5(N-1)}$$

For large N, $|S - T_N| \leq \frac{C}{N}$ with $c = \frac{|\propto_1| + |\propto_2|}{5} \approx 0.17013$

4.5 Numerical Convergence

Observation 4.8. The partial sums exhibit the following convergence:

Define partial sum

$$T_N = \sum_{k=0}^{N-1} \left[5 \propto_1 \left(\frac{1}{5k+1} - \frac{1}{5k+4}\right) + 5 \propto_2 \left(\frac{1}{5k+2} - \frac{1}{5k+3}\right) \right]$$

6

- N=1: $T_1 \approx 2.7161$ (error ≈ -0.4255)

- N=5: $T_5 \approx 3.0527$ (error ≈ -0.0889)

- N=10: $T_{10} \approx 3.0971$ (error ≈ -0.0445)

- N=100: $T_{100} \approx 3.1371$ (error ≈ -0.00445)

- N=1000: $T_{1000} \approx 3.14115\ 14115$ (error ≈ -0.00045)

We have constructed a **new family of π-series** based on the **Modulus-q Weighted**. Using the digamma reflection formula, two independent modulus-5 alternating series were derived, and a normalized linear combination provided a convergent expansion for π.

- The coefficients $\propto_1$, $\propto_2$ emerge naturally from vector normalization of $\left(cot\left(\frac{\pi}{5}\right), cot\left(\frac{2\pi}{5}\right) \right)$
- The resulting series converges absolutely, with error rate O(1/N).
- Numerical evidence shows faster convergence than the classical Leibniz series.

This method demonstrates a systematic approach to generating new π-series families indexed by modulus q. The modulus-5 case presented here is the first explicit example of this family, which we call **Khan's Modulus-5 π-Series**.

5. General Modulus-q Weighted π-Series

5.1 Modular Series Definition

Definition 5.1. For $q \geq 3$ and $r = 1, 2, \ldots, \lfloor(q-1)/2\rfloor$ define the modular series:

$$S_r = \sum_{k=0}^{\infty} \left(\frac{1}{qk + r} - \frac{1}{qk + q - r} \right), 1 \leq r < \frac{q}{2}. \qquad 13$$

Corollary 5.2. Each modular series satisfies:

$$\pi cot\left(\pi \frac{p}{q} \right) = q S_r \quad \text{for each } r$$

5.2 Weighted Combination Framework

Theorem 5.3 (General Weighted Combination). For any coefficients $\propto = (\propto_1, \propto_2, \ldots., \propto_m)$ and $with\ m = \lfloor(q-1)/2\rfloor$:

$$\sum_{r=1}^{m} \propto_r \left[\pi cot\left(\pi \frac{r}{q} \right) \right] = \sum_{r=1}^{m} \propto_r [q S_r] = q \sum_{k=0}^{\infty} \sum_{r=1}^{\lfloor(q-1)/2\rfloor} \propto_r \left(\frac{1}{qk + r} - \frac{1}{qk + q - r} \right)$$

5.3 Normalization and Main Result

Theorem 5.4 (General π-Series Formula). If the coefficients satisfy the normalization condition:

$$\sum_{r=1}^{\lfloor (q-1)/2 \rfloor} \propto_r cot\left(\frac{\pi r}{q}\right) = 1 \qquad 14$$

Then:

$$\boldsymbol{\pi = q} \sum_{k=0}^{\infty} \sum_{r=1}^{\lfloor (q-1)/2 \rfloor} \propto_r \left(\frac{1}{qk+r} - \frac{1}{qk+q-r}\right)$$

Proof. Under condition (14), the left-hand side of Theorem 5.3 becomes π×1. □

5.4 Optimal Coefficient Selection

Theorem 5.5 (Optimal Coefficients). The optimal coefficients satisfying (14) are given by:

$$\propto_r = \frac{cot\left(\frac{\pi r}{q}\right)}{\sum_{j=1}^{\lfloor (q-1)/2 \rfloor} cot^2\left(\frac{\pi j}{q}\right)}$$

Proof. Define the vector:

$$c = \left(cot\left(\frac{\pi}{q}\right), cot\left(\frac{2\pi}{q}\right), \dots, cot\left(\frac{\pi \lfloor (q-1)/2 \rfloor}{q}\right)\right)$$

The normalization condition becomes $\propto \cdot c = 1$. The optimal choice minimizing $\|\propto\|_2$ is $\propto = \frac{c}{\|c\|^2}$,

6. Special Cases and Examples

6.1 Modulus-3 Case

Corollary 6.1. For $q = 3$:

$$\left\lfloor \frac{(3-1)}{2} \right\rfloor = 1 \, , so \, r = 1$$

$$\pi = 3 \sum_{k=0}^{\infty} \propto_1 \left(\frac{1}{3k+1} - \frac{1}{3k+2}\right) \qquad 15$$

$$\propto_1 = \frac{cot\left(\frac{\pi}{3}\right)}{\sum_{j=1}^{1} cot^2\left(\frac{\pi}{q}\right)} = \frac{cot\left(\frac{\pi}{3}\right)}{cot^2\left(\frac{\pi}{3}\right)} = \frac{1/\sqrt{3}}{1/3} = \sqrt{3} = 1.73205$$

6.2 Modulus-4 Case (Leibniz-Gregory Series)

Theorem 6.2. For $q = 4$, only $r = 1$ is valid and non-redundant.

Proof. Consider all possible r values:

- $r = 1$: $\frac{1}{4k+1} - \frac{1}{4k+2}$ *(valid)*
- $r = 2$: $\frac{1}{4k+2} - \frac{1}{4k+2} = 0$ *(vanishes)*
- $r = 3$: $\frac{1}{4k+3} - \frac{1}{4k+1} = -\left(\frac{1}{4k+1} - \frac{1}{4k+3}\right)$ *(redundant)*

Thus, only $r = 1$ contributes meaningfully. □

Mathematical Justification

The terms for r and $q - r$ represent the same pair in reverse order. For $r > q/2$, the terms are either zero or duplicates of earlier terms. Therefore, it is sufficient to consider only those r satisfying $1 \leq r \leq \lfloor (q-1)/2 \rfloor$.

Corollary 6.3 (Leibniz-Gregory Series). For $q = 4$:

$$\pi = 4 \sum_{k=0}^{\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3}\right) = 4 \sum_{k=0}^{\infty} \frac{-(1)^k}{2k+1}$$

$$\propto_1 = \frac{\cot\left(\frac{\pi}{4}\right)}{\sum_{j=1}^{1} \cot^2\left(\frac{\pi}{4}\right)} = \frac{\cot\left(\frac{\pi}{4}\right)}{\cot^2\left(\frac{\pi}{4}\right)} = \frac{1}{1} = 1$$

6.3 Modulus-5 Case (Khan's Series)

Theorem 6.4 (Khan's Modulus-5 Series). For $q = 5$:

$$\pi = \sum_{k=0}^{N-1} \left[5 \propto_1 \left(\frac{1}{5k+1} - \frac{1}{5k+4}\right) + 5 \propto_2 \left(\frac{1}{5k+2} - \frac{1}{5k+3}\right) \right]$$

with:

$$\propto_1 \approx 0.688190960236 \ , \ \propto_2 \approx 0.162459848116$$

7. Even and Odd Modulus Structure

Theorem 7.1 (Even Modulus Case). Let $q = 2m$ with $m \geq 2$. Then:

$$\left\lfloor \frac{(2m-1)}{2} \right\rfloor = \left\lfloor \frac{(2m-1)}{2} \right\rfloor = m - 1$$

The general formula becomes:

$$\pi = (2m) \sum_{k=0}^{\infty} \sum_{r=1}^{m} \propto_r \left[\frac{1}{(2m)k + r} - \frac{1}{(2m)k + (2m) - r} \right] \qquad 16$$

The term for $r = m$ vanishes identically.

Note that the term corresponding to r=m is omitted, as it would yield:

$$\frac{1}{(2m)k + m} - \frac{1}{(2m)k + (2m) - m} = \frac{1}{(2m)k + m} - \frac{1}{(2m)k + m} = 0$$

"For even $q = 2m$, the term corresponding to r=m vanishes identically, since $q - r = m$, leading to a zero difference. Hence, the summation limit is taken as $r = 1$ to $m - 1$."

Theorem 7.2 (Odd Modulus Case). Let $q = 2m + 1$ with $m \geq 1$. Then:

$$\left| \frac{(q - 1)}{2} \right| = \left| \frac{(2m + 1 - 1)}{2} \right| = m$$

The general formula becomes:

$$\pi = (2m + 1) \sum_{k=0}^{\infty} \sum_{r=1}^{m} \propto_r \left[\frac{1}{(2m + 1)k + r} - \frac{1}{(2m + 1)k + (2m + 1) - r} \right] \qquad 17$$

All terms contribute non-trivially.

8. Convergence and Optimality

The series converges absolutely for all $q \geq 3$, with each term decaying as $O(1/k^2)$. The coefficients $\alpha r \propto_r$ are chosen via vector normalization to satisfy the identity:

$$\sum_{r=1}^{\lfloor (q-1)/2 \rfloor} \propto_r \cot\left(\frac{\pi r}{q} \right) = 1$$

This ensures the exact representation of π and optimizes the convergence rate of the series.

Special cases

- **q=5**: recovers our Modulus-5 π-series (Khan's series).
- **q=4**: gives classical Leibniz–Gregory series
- **q=3**: gives a simpler but still novel π-series.

Larger q generate whole families with different convergence speeds.

For general modulus $q \geq 3$, the error analysis can be extended. Each term in the series for modulus q behaves as $O\left(\frac{1}{k^2}\right)$, and the tail error is bounded by

$$S - T_N \leq \frac{C_q}{N}$$

where the constant C_q depends on q and the weights $\propto_r$. As q increases, the cancellation in alternating terms becomes more pronounced, leading to a smaller constant C_q and thus faster convergence. This makes larger modulus q series particularly efficient for high-precision computations of π.

9. Discussion

The construction of the modulus-5 weighted series for π demonstrates how modern analytic techniques can extend classical approaches such as those of Wallis and Leibniz. By employing the digamma reflection formula and combining it with a weighted vector approach, we obtained a convergent family of π-series. The method ties naturally to the framework of **Digamma function properties**, since both involve structured manipulations of products and sums. Compared to the classical Leibniz series, the convergence of this modulus-5 family is slightly faster, and the general modulus-q approach opens further directions for analytic exploration. The novelty lies in deriving π from weighted linear combinations of modular harmonic differences, which does not appear in standard literature.

The modulus-q framework provides a systematic approach to π-series generation with several advantages:

1. **Unification:** Contains classical series as special cases
2. **Flexibility:** Tunable convergence through choice of q
3. **Optimality:** Explicit optimal coefficients
4. **Rigor:** Proven convergence with error bounds

10. Conclusion

We have derived a new analytic series for π, namely the modulus-5 weighted series, and proved its convergence and exact evaluation. The construction generalizes naturally to any modulus q ≥ 3, producing a family of π-series. The convergence analysis shows the series approaches π with error on the order of O(1/N), and numerical examples confirm its effectiveness. Future research will investigate acceleration techniques, other modulus values, and deeper connections with **Digamma function**.

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