

## Solving Ultimate $n$ th-Degree Algebraic Equations: Evading the Abel-Ruffini Theorem?

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### ABSTRACT<sup>1</sup>

The present paper proposes a most parsimonious scheme to arrive at solutions for polynomial algebraic (or ODE) equations of an arbitrary degree (order). The former option treats the polynomial as an implied characteristic of a difference/recurrent equation (dubbed an AID, or algebraic-to-difference path). The latter (i.e. ODE) domain, rather than building on the more trivial differential-difference parallelism, embarks on first reducing ODE to algebraic (while making use of, say, Mikusinski-style E-operators) then applying the above procedure (what amounts to a DARF/DARE, or differential to algebraic to recurrent/functional equation path).

*Keywords:* equations polynomial, difference (recursive, functional), prime numbers

### AID Rationale

Consider first a somewhat singular algebraic case as in (A) below:

$$ax^n + bx^{n-k} + c = 0 = x^n + \frac{b}{a} * x^{n-k} + c/a \quad (A)$$

This readily reduces to, or indeed *induces*, a *functional*-equation form (B):

$$x_n = -\frac{b}{a} * x_{n-k} - \frac{c}{a}, \quad k < n \quad (B)$$

Two corner solutions, (C) and (E), could be seen as the *core* versus *extension*, respectively. At any rate, their *arithmetic-weighted averaging* yields just the *radicals*-based solution as per the *quadratic* case—once a *modified* core is tried, say, divided by either 2 or  $n$  (pertaining to ‘corner’ versus ‘interior’ mixing), applying to each instance of a *core* entering the candidate solution, the former indeed amounting to an *overlap* subject to linear tossing. Please note how (D) showcases an  $n$ -invariant [powered] extension with the free term nil,  $c=0$  (*degeneracy*).

$$x_C^1 = (-b/a)^{1/k} * x_0 = -\frac{b}{a} \equiv x_1, \quad x_0 = x^0 = 1 \quad (C)$$

$$x_D = (-b/a)^{1/k} = \lim_{c \rightarrow 0} x_E = x_C \quad (D)$$

$$x_E^n = (-b/a)^{n/k} - \frac{c}{a} * \sum_{s=0}^{\frac{n}{k}} (-b/a)^s \rightarrow x_E = [e^{\frac{n}{k} * \pi i} * \frac{b}{a} - \frac{c}{a} * \frac{(-b/a)^{\frac{n}{k}-1} - 1}{-\frac{b}{a} - 1}]^{1/n} \quad (E)$$

<sup>1</sup> *Mary as the Solution. In memoriam Kirk.*

Whilst the complex case is [almost<sup>2</sup>] ensured unless  $k$  divides  $n$  (and under any [odd]  $n > 4$ , quintic and beyond, for that matter, given the way  $1/n$ , or  $n$ th root, *generalizes the radicals*), a set of two alternate solutions could be conjectured as in (AD1-2):

$$\begin{aligned} x = \frac{x_C + x_E}{n} &= [(-b/a)^{1/k} + \left[ (-b/a)^{n/k} + c/a * \frac{\left(-\frac{b}{a}\right)^{\frac{n}{k}-1} - 1}{\frac{b}{a} + 1} \right]^{1/n}] / n \\ &= \frac{\left(-\frac{b}{a}\right)^{\frac{1}{k}}}{n} * \left( 1 + \left[ 1 - \frac{ac}{b * (a + b)} - \frac{c}{\frac{a + b}{a} \frac{n}{k}} \right]^{1/n} \right) \quad (AD1) \end{aligned}$$

$$x = \frac{x_C + \sum_{s=1}^n x_E(s)}{n} = \frac{\left(-\frac{b}{a}\right)^{\frac{1}{k}}}{n} + 1/n \sum_{s=1}^n \left[ (-b/a)^{s/k} + c/a * \frac{\left(-\frac{b}{a}\right)^{\frac{s}{k}-1} - 1}{\frac{b}{a} + 1} \right]^{\frac{1}{s}} \quad (AD2)$$

For one thing, the hallow-interior (interim-stripped) AD1 appears more in line with the singularity case and the presumption of exactly  $n$  solutions alike—in contrast to the AD2 scenario as if to invoke  $2^n$  candidate solutions. On second thoughts, not only does the latter resemble a HO ODE solution, it likewise points to a striking similarity with a *primality representation* relationship (Shevenyonov 2025) aligning the prime index, or rank, to the powers sum<sup>3</sup> (cf. base exponent versus base 2). What's more, remarkable is the very subtle *powers-index linkage* for the two otherwise disparate areas and their accommodating approaches (algebraic x-powers/functional indices versus 2-powers/prime rank-index). Not least, it could be of interest looking into whether & how these have any bearing on Remizov style (2023) solution<sup>4</sup>, however exceedingly cumbersome any alternative to the approach being attempted herewith would likely have turned out to be *a priori*.

The above could be generalized, or straddled, via a linear tossup as in (1A-B):

$$x_n = x^n = \alpha * x_C^n + \beta * x_E^n = \left( \alpha + \beta - \frac{\beta ac}{b(a + b)} \right) x_C^n - \frac{\beta c}{a + b} \quad (1A)$$

$$x = \left[ \left( \alpha + \beta - \frac{\beta ac}{b(a + b)} \right) x_C^n - \frac{\beta c}{a + b} \right]^{\frac{1}{n}} \quad (1B)$$

<sup>2</sup> Some of the known special quintic solutions may violate the plausibility of sorts, paradoxically positing *real* ones, albeit resulting from sums of *complex* addends. More importantly, they do seem to second our basic intuition as regards the structure & implied set powers of the candidate roots!

<sup>3</sup> Note how the *repetitive* root is denoted similarly:  $\Delta x_n = (c_1 + c_2 n)r^n$  vs  $\Delta p_n = n_k 2^k$  (cf. cross-solution linear dependence)

<sup>4</sup> Pinned as of late October 2025 yet postponed, little had this paper been wagered to urge a ground-breaking rivalry race, as the general programme would still seem a far cry and a long shot. This opening footnote should make the rehashing afterthought.

### A Generalization & the DARF/DARE Pathway

In any event, either of the special-solution candidates would have to be scrutinized by substitution (plugging back in) and, better yet, against a *general*-solution attempt, e.g. by dint of the regular/*standard* difference/recurrence reduction methods as applied to a polynomial *residuale* term. (Please note in passing that both the aforementioned AID solutions are *orduale* in nature (Shevenyonov 2016), meaning *ad hoc* they build on the coefficient & parameter *ratios* rather than values *per se*.)

To begin with, a singular-case degree- $n$  polynomial equation (A-B) could, in a standard way (F1-2), be treated with an eye on the CF & PI as follows<sup>5</sup> (G):

$$f(E)x_n = g(n)a^n \quad (F1)$$

$$\left(E + \frac{b}{a} * E^{-k}\right) x_n = -\frac{c}{a} * 1^n \quad (F2)$$

$$x_n = x^n = CF + PI = (-b/a)^{n/k} * x_0 - \frac{c/a}{1 + \frac{b}{a} * 1^{-k}} = (-b/a)^{n/k} - \frac{c}{a+b} \quad (G1)$$

$$x_{n-k} = (-b/a)^{\frac{n}{k}-1} - \frac{c}{a+b} \quad (G2)$$

$$x = \left[\left(-\frac{b}{a}\right)^{\frac{n}{k}} - \frac{c}{a+b}\right]^{1/n} \text{ vs } x = \left[\left(-\frac{b}{a}\right)^{\frac{n}{k}-1} - \frac{c}{a+b}\right]^{1/(n-k)} \forall k < n \quad (G3)$$

The above does appear to align with the ‘naïve’/unaided, results in (E) and (AD1)—in effect, as an asymptotic ( $n \gg k$ ,  $|b/a| < 1$ ) or special (beta making unity) case. Please note that the (*alpha*, *beta*) tossup,  $x^0$  other than unity (e.g. a finite constant or exogenous function), and a primitive  $n$ th root of -1 could each account for extra solutions potentially. Now, plugging (G1-2) back into (A) suggests a match, whilst its ‘naïve’ generalization (notably finetuning on the structural nuance) may need some calibrating with respect to the weights, or linearization tossup. In principle, the standard DE theory sheds little if any light on whichever solution in (G3) accommodates a functional-polynomial transition—not surprisingly for the two-century-old problem<sup>6</sup>.

$$\left(\alpha + \beta - \frac{\beta ac}{b(a+b)}\right) x_c^n - \frac{\beta c}{a+b} + \frac{b}{a} * \left[\left(\alpha + \beta - \frac{\beta ac}{b(a+b)}\right) x_c^{n-k} - \frac{\beta c}{a+b}\right] + \frac{c}{a} = 0 \quad (AA)$$

$$x_c^{n-k} = x_c^n * (-a/b)$$

<sup>5</sup> Please note that, while close formally, the *complimentary function* versus *particular integral* need not reduce to our *core* versus *extension*, respectively, as these dichotomies have been motivated in distinct conceptual ways, as implied throughout.

<sup>6</sup> An exercise in humility would be warranted in just how ‘Dare’ & ‘Darf,’ both connote the Germanic cognates for enterprises more daring/audacious if reckless than accomplished.

$$(\alpha + \beta)x_C^n * \left(1 - \frac{a}{b}\right) = \frac{c}{a} * (\beta - 1), \quad \frac{\alpha + \beta}{1 - \beta} = \frac{c}{(b - a)} * \left(-\frac{a}{b}\right)^{\frac{n}{k}-1} \quad (AB)$$

Hence (1A) can now be rearranged as (1C),

$$x_n = c * \left[ \frac{\beta}{a + b} * \left( \left(-\frac{b}{a}\right)^{\frac{n}{k}-1} - 1 \right) + \frac{\frac{(1 - \beta)}{(b - a)}}{\left(-\frac{b}{a}\right)^{\frac{n}{k}-1}} \right] \quad (1C)$$

Under beta near-unity, (1C) mimics the standard solution (G1) rather closely. Alternatively, the calibrating factors could be endogenized as follows (AC), *phi* posing a finite exogenous function or constant:

$$\alpha = \varphi\beta \quad (AC)$$

$$\alpha[x_C^n] + \beta[x_E^n] = 0, \quad [x_C^n] \equiv ax_C^n + bx_C^{n-k}, \quad [x_E^n] \equiv ax_E^n + bx_E^{n-k}$$

As one possibility, insofar as alpha & beta prove arbitrary or cancel out at zero polynomial-value, the phi could be [inversely] related with *n*.

Now, consider how a slight generalization (G4) posits a linkage across all possible solutions, in particular lending merit to linear tossups (1A) & reconciling the corner (AD1) versus interior (AD2) representations:

$$x_{n-s} = x_C^{(n-s)/k} + c * \frac{\left(-\frac{b}{a}\right)^{\frac{n-s}{k}} - 1}{\frac{b}{a} + 1} = x_C^n * \varphi_s + c * \frac{x_C^n * \varphi_s - 1}{\frac{b}{a} + 1} \quad \forall s < n$$

$$\forall s, t: \Delta x_n \equiv x_{n-s} - x_{n-t} = x_n * \Delta\varphi \quad (G4)$$

(Whilst of similar *posterior* natures, the phi's in G4 vs AC are generally distinct *a priori*.) Now, when it comes to further generalizations, consider a full-fledged polynomial as treated by standard difference methods. Pick an arbitrary pair of terms for CF then treat the residue as the PI function (H0-2):

$$\sum_{k=0}^n a_k x^k = 0, \quad \forall s, t: a_s x^s + a_t x^t \equiv R(s, t) = -(a_s x^s + a_t x^t) * 1^n$$

$$CF: x_s = \left(-\frac{a_t}{a_s}\right)^{s/t} \quad PI: \frac{g(n)}{f(1)} = -\frac{(a_s x^s + a_t x^t)}{a_s 1^s + a_t 1^t} \equiv -[\gamma x^s + (1 - \gamma)x^t] \quad (H0)$$

Technically, one adds the above parts to arrive at an arbitrary *n*th value (H0). However, one could seek some more-elegant solutions, bearing in mind this should apply to *any* index-powers, corner & special ones included. Therefore, consider (H1-2):

$$x_n = \left(-\frac{a_1}{a_n}\right)^{n/1} - [\gamma x_n + (1 - \gamma)x^1], \quad (H1)$$

$$x_n = (1 - \gamma) * (x_{CF}^n + x_{CF}^t) = \left( \frac{a_t}{a_n + a_t} \right) * \left[ \left( -\frac{a_t}{a_n} \right)^{\frac{n}{t}} - 1 \right] \quad (H2)$$

Whereas these do not seem to work very well in the present form, (H0) could further be reworked as a *two-stage standard* solution:

$$x_n + \gamma x_s + (1 - \gamma)x_t = \left( -\frac{a_t}{a_s} \right)^{n/t} \quad (H3)$$

A simplistic quadratic case is ever attainable under, say,  $s=n-1, t=n-2$ :

$$[E^2 + \gamma E + (1 - \gamma)]x_t = 0, \quad F(n) = \left[ \left( -\frac{a_t}{a_{t+1}} \right)^{1/t} \right]^n$$

$$E = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - (1 - \gamma)}, \quad \gamma = \frac{a_{t+1}}{a_t + a_{t+1}}$$

$$CF = c_1 * \left( -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - (1 - \gamma)} \right)^t + c_2 * \left( -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - (1 - \gamma)} \right)^t$$

$$PI = \frac{\left( -\frac{a_t}{a_{t+1}} \right)}{\left[ \left( -\frac{a_t}{a_{t+1}} \right)^{\frac{1}{t}} - c_1 * \left( -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - (1 - \gamma)} \right)^t \right] * \left[ \left( -\frac{a_t}{a_{t+1}} \right)^{\frac{1}{t}} - c_2 * \left( -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - (1 - \gamma)} \right)^t \right]}$$

$$x = (CF + PI)^{1/t}$$

### Afterthoughts

This awkward a reduced form could have been simplified from the outset, per  $n=t$ :

$$(1 - \gamma)(x_t + x_s) = -\frac{a_t}{a_s} \leftrightarrow x_t = -x_s - \frac{a_s + a_t}{a_s} \quad (H4)$$

The (H4) structural form now allows for a reduction to a closed form (Ia-b) by methods ‘naïve’ & standard alike, as attempted throughout:

$$x_t = \begin{cases} x_0 e^{\frac{t}{s}\pi i} - \frac{a_s + a_t}{2a_s} \\ \left( (-1)^{\frac{t}{s}-1} - \frac{a_s + a_t}{a_s} * \frac{(-1)^{\frac{t}{s}-1} - 1}{-1 - 1} \right) = x_0 e^{\frac{t}{s}\pi i} \left( 2 + \frac{a_t}{a_s} \right) - \frac{a_s + a_t}{2a_s} \end{cases} \quad (Ia)$$

$$x = \begin{cases} [x_0 e^{\frac{n}{s}\pi i} - \frac{a_s + a_n}{2a_s}]^{1/n}, \quad s < n \\ \left[ \left( (-1)^{\frac{n}{s}-1} - \frac{a_s + a_n}{a_s} * \frac{(-1)^{\frac{n}{s}-1} - 1}{-1 - 1} \right) \right]^{1/n} = [x_0 e^{\frac{n}{s}\pi i} \left( 2 + \frac{a_n}{a_s} \right) - \frac{a_s + a_n}{2a_s}]^{1/n} \end{cases} \quad (Ib)$$

The latter alternate, as before, tends to the former under  $t \gg s$ . In other words, a tentative exposition as above suggests a *general*-case solution *mimics* a special, or degenerate, case counterpart! Again, it could be rewarding to compare & contrast the patterns and procedures being proposed herewith against the recent results by Remizov (2023 [2026]).

#### References

Remizov, I.D. (2023) Chernoff approximations as a way of finding the resolvent operator with applications to finding the solution of linear ODE with variable coefficients, *arXiv preprint 2301.06765*

Shevenyonov, A.V. (2025) Primality has a yielding stricture, *vixra preprint 2512.0128v1*

Shevenyonov, A. (2016) The ordinalcy & residuality foundations: And now for greater simplicity, *vixra preprint 1610.0357*