

# From the Kinematics of Elliptical Motion to the Law of Gravitation

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## Abstract

The law of force governing the motion of a material point along an elliptical orbit is derived using a purely kinematic approach. Starting from the differential equations of motion in a Cartesian coordinate system, an angular equation of motion is obtained that directly leads to Kepler's second law. It is shown that the acceleration is directed toward the focus of the ellipse and is inversely proportional to the square of the distance. The derived expressions are applied to the Earth–Moon system. Forces calculated using Newton's second law are compared with those given by Newton's law of universal gravitation. The relative discrepancy does not exceed 1.1%, confirming their equivalence.

## 1. Introduction

The description of rectilinear motion can be achieved using elementary algebraic relations between position, velocity, and acceleration. In contrast, the analysis of curvilinear motion of a material point requires the use of differential equations of motion. The specific method of obtaining these equations is not unique and depends on the chosen description of motion.

The aim of this work is to derive the form of the force responsible for elliptical motion solely from the kinematics of the trajectory, without any prior assumption about the nature of the interaction.

## 2. Equations of Motion

Consider the motion of a point mass ( $m$ ) along an ellipse, one focus of which is located at the origin of a fixed Cartesian coordinate system. Let the force ( $Q$ ) acting on the particle be directed toward the focus and form an angle  $\varphi(t)$  with the ( $x$ )-axis. The equations of motion have the form:

$$m\ddot{x} = -Q\cos(\varphi(t)) \quad (1)$$

$$m\ddot{y} = -Q\sin(\varphi(t)) \quad (2)$$

Eliminating ( $Q$ ), we obtain the relation:

$$\ddot{y} = \frac{\ddot{x}}{\cos(\varphi(t))} \sin(\varphi(t)) \quad (3)$$

## 3. Kinematic Description of the Ellipse

In polar coordinates with the focus at the origin, the trajectory is given by:

$$\begin{cases} x = r(\varphi(t))\cos(\varphi(t)) \\ y = r(\varphi(t))\sin(\varphi(t)) \end{cases} \quad (4)$$

where

$$r(\varphi(t)) = \frac{a(1-e^2)}{1+e\cos(\varphi(t))} = \frac{a(1-e^2)}{1+e\cos\varphi} \quad (5)$$

Differentiating twice with respect to time and substituting into the kinematic relation, we arrive at the angular equation of motion:

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} \quad (6)$$

Integration yields:

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{C}{r^2} \quad (7)$$

where ( $C$ ) is a constant of integration.

#### 4. Kepler's Laws

The obtained relation corresponds to the law of areas:

$$\frac{dA}{dt} = \frac{1}{2}r^2 \frac{d\varphi}{dt} = \text{const} \quad (8)$$

Thus, Kepler's second law follows directly from the kinematics of elliptical motion. Kepler's first law is satisfied by construction of the trajectory.

#### 5. Acceleration and Central Force

The magnitude of the acceleration is found as:

$$a = \frac{C^2(1+e\cos\varphi)^2}{a^3(1-e^2)^3} \quad (9)$$

The acceleration is always directed toward the focus of the ellipse and is inversely proportional to the square of the distance, which indicates a central force of the Newtonian type.

#### 6. Application to the Earth–Moon System

The Earth–Moon system is considered as a two-body problem, with both bodies moving around their common center of mass. Orbital parameters are taken from standard astronomical data. The semi-major axis and eccentricity determine the geometry of the relative orbit, while the period determines the areal constant.

The derived expressions are applied to the Earth–Moon system using standard astronomical parameters. Accelerations and forces at perigee and apogee are calculated for both bodies and compared with the forces obtained from Newton’s law of universal gravitation.

The relative discrepancy between the results does not exceed 1.1%.

## 7. Conclusions

The presented derivation is intentionally restricted to classical mechanics and avoids variational or Hamiltonian formalisms. This makes the argument transparent and suitable for direct physical interpretation, particularly in the context of orbital mechanics and celestial dynamics.

The law of the central force governing elliptical motion can be derived directly from kinematics.

The resulting acceleration is directed toward the focus and is inversely proportional to the square of the distance.

Kepler’s second law follows naturally from the angular equation of motion.

Numerical estimates for the Earth–Moon system confirm the equivalence between the derived force and Newtonian gravitation.

## References

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## Appendix A: Numerical Estimates for the Earth–Moon System

The following numerical values were used in the calculations:

Gravitational constant:  $G = 6.67430 \cdot 10^{-11} \text{ m}^3 \cdot \text{c}^{-2} \cdot \text{kg}^{-1}$

Mass of the Moon:  $m_1 = 7.3477 \cdot 10^{22} \text{ kg}$

Mass of the Earth:  $m_2 = 5.9726 \cdot 10^{24} \text{ kg}$

Perigee distance:  $r_{p1} = 3.63104 \cdot 10^8 \text{ m}$

Apogee distance:  $r_{a1} = 4.05696 \cdot 10^8 \text{ m}$

Orbital eccentricity: 0.0549

Orbital period:  $T = 2,360,591.5 \text{ s}$

Semi-major axis of the relative orbit:  $a_1 = \frac{r_{p1} + r_{a1}}{2} = 384400000 = 3.844 \cdot 10^8 \text{ m}$

The areal constant is determined from the expression:  $C_1 = \frac{2\pi a_1^2 \sqrt{1-e^2}}{T}$

Accelerations at perigee and apogee follow from the formula:  $a = \frac{C^2(1+e \cdot \cos \varphi)^2}{a^3(1-e^2)^3}$

The corresponding forces are obtained using Newton's second law and compared with the gravitational force:  $F = G \frac{m_1 m_2}{r^2}$

The relative discrepancy between the two approaches does not exceed 1.1%, which lies within the expected accuracy given the adopted orbital parameters.