Foundations of mathematics and quantum theory

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Abstract

As shown by Gödel and other mathematicians, foundational problems of classical mathematics (CM) arise because this theory involves the entire infinite set of natural numbers. Therefore, CM must be modified in some way. A problem discussed in a wide literature is how mathematics should be treated: (1) as a purely abstract discipline, independent of nature; or (2) as a discipline that must ultimately describe nature. Most physicists accept only viewpoint (2), while many mathematicians and philosophers adopt viewpoint (1). However, currently approach (1) did not solve the problem of how CM should be modified, and quantum theory (QT) is considered to be the most general theory for describing nature. Therefore, CM must be modified so that it correctly describes QT. As shown in our publications, finite mathematics (FM) satisfies this condition. It involves a finite ring $R_p = (0, 1, ...p - 1)$ where addition, subtraction, and multiplication are performed modulo p. FM does not contain any foundational problems and is a more general theory than CM: the latter is a degenerate special case of the former in the limit $p \to \infty$. The purpose of this paper is to provide a brief overview of our results to make them understandable to a wide audience of mathematicians and physicists.

1 Problem statement

The goal of Hilbert's approach to classical mathematics (CM) is to find a complete and consistent set of axioms such that theorems based on them will make it possible to conclude whether any mathematical statement is true or false. This problem is also formulated as the Entscheidungsproblem, which asks for algorithms that consider statements and answers "Yes" or "No" according to whether the statements are universally valid, i.e., valid in every structure satisfying the

axioms. The incompleteness of CM follows, for example, from Gödel's two incompleteness theorems:

The First Incompleteness Theorem states that no consistent system of axioms whose theorems can be listed by an effective procedure (i.e., an algorithm) is capable of proving all truths about the arithmetic of natural numbers. For any such consistent formal system, there will always be statements about natural numbers that are true, but that are unprovable within the system;

The Second Incompleteness Theorem states that any formal system strong enough to encompass basic arithmetic (like Peano Arithmetic) cannot prove its own consistency.

In addition, Church and Turing [1, 2, 3] proved the insolvability of Hilbert's Entscheidungsproblem in 1936 by demonstrating that no algorithm could decide whether a given statement in first-order logic is provable.

Thus, although many important and beautiful results have been obtained in CM and it has demonstrated its power in many applications, the foundational problem of CM remains unresolved.

As noted above, CM proceeds from axioms and, by definition, axioms are statements accepted as self-evidently true without needing proof. In other words, the validity of axioms is accepted on faith. One might think that in science (unlike religion) there should be no statements that are accepted solely on the basis of faith. However, Hilbert's approach to mathematics begins with axioms, and this approach encounters foundational problems.

The fact that CM is based on axioms about natural numbers seems so natural that any proof here seems completely unnecessary, and it may seem strange that such an approach leads to foundational problems. However, the theory of relativity and quantum theory have taught us that some phenomena of nature can only be described using approaches that seem to contradict common sense. Now we understand that the problems with using our "common sense" to describe certain phenomena are as follows. Our life experience is formed from observing phenomena that are accessible to our observation in everyday life, and we think that we can extrapolate this experience to cases where our perception of the world ceases to be adequate.

A good historical example is the following: in non-relativistic mechanics (NM), there is no limit to speeds, and it describes many data with very high accuracy. However, when speeds approach c, the for-

mulas of NM cease to work, and the formulas of relativistic mechanics (RM) differ significantly from those of NM. This doesn't mean that RT disproves NT, but RT shows that NM works well only when speeds are much less than c, and our everyday experience corresponds precisely to such cases.

Let's now consider our experience with basic arithmetic (BA). It would seem that all of our experience gives no reason to doubt that BA is the correct theory for any natural numbers. For example, we have no doubt, and experience confirms, that $10^3+10^3=2\cdot10^3$ and $10^9+10^9=2\cdot10^9$. However, if $A = 10^{100}$ can we be sure that $10^A+10^A=2\cdot10^A$? In BA, this fact follows from the axioms, and it would seem that such large numbers as 10^A and $2\cdot10^A$ do not exist in nature. However, as we will see below, there are phenomena that cannot be described within the framework of BA.

Based on what has been said above about NM and RM, it can be assumed that, similarly, in the set of natural numbers, there is some number p such that BA is confirmed by experiment for numbers much less than p, but must be modified for numbers comparable to or greater than p. In other words, p plays a role similar to the one that c plays in the situation with NM and RM. A natural question arises: is p determined by any theory, and does this number change during the evolution of the universe? We will discuss these issues below.

We assume that BA should now contain not the natural numbers themselves, but these numbers modulo p and all operations with such numbers must also be performed modulo p. Then, if, for example, p is very large, operations with numbers much less than p will be performed according to standard rules and the role of p for such numbers will be imperceptible. It will be noticeable only for numbers comparable to or greater than p.

Now, in complete analogy with finite mathematics (FM), instead of an infinite set of natural numbers, we are dealing with a finite set $R_p = (0, 1, 2, \ldots, p-1)$ where all the numbers are taken modulo p. Unlike the set of all natural numbers, the set R_p automatically becomes a ring because it is obviously possible to introduce addition, multiplication and subtraction into R_p without expanding the set. In the literature, R_p is often denoted Z/p (where Z is the standard ring of integers) but this notation is somewhat misleading. First, it introduces an infinite set into a finite framework. Second, it may suggest that R_p is merely a special case of Z, although Z does not contain modular

operations and is therefore not more general than R_p.

In the *technique* of CM, infinity is treated only as potential infinity. The distinction between potential and actual infinity is well known:

- **Actual infinity** refers to a completed set containing infinitely many elements.

- **Potential infinity** refers to an endless process that never reaches a final completed state but involves only finite sets. Thus, potential infinity appears only as a limit.

For example, the set of all natural numbers represents actual infinity, and—as implied by Gödel's Incompleteness Theorems, there will always be true statements about numbers that cannot be proven or disproven within that system. Nevertheless, CM is based on actual infinity: it begins with the infinite set of natural numbers N and infinite ring of integers Z, and standard textbooks do not even pose the problem of whether Z can be understood as a limit of finite rings. Moreover, in CM the ring Z is extended to fields (e.g., such as reals), involving actual infinities of various cardinalities. In contrast, FM is built solely on finite sets and has no foundational problems. The truth or falsity of any statement can be checked in a finite number of steps. As noted above, typically, FM begins with the ring $R_p = (0, 1, 2, \ldots, p-1)$, where addition, subtraction, and multiplication are defined modulo p.

In this paper, we consider the case where there are two theories, $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ contains a positive natural parameter p, and $\mathcal{B}$ can be defined as the (actual or potential) limit of $\mathcal{A}$ when $p \to \infty$. In [4] we have proposed the following

Definition. Suppose that:

- After taking the limit $p \to \infty$ and obtaining $\mathcal{B}$, $\mathcal{B}$ cannot reproduce all results obtainable in $\mathcal{A}$.

- $\mathcal{A}$ can reproduce any result of $\mathcal{B}$ by choosing an appropriate value of p.

Then $\mathcal{A}$ is more general than $\mathcal{B}$ and $\mathcal{B}$ is a degenerate special case of $\mathcal{A}$.

As mentioned above, CM is based on axioms and has foundational problems. Gödel's works on the incompleteness theorems are written in highly technical terms of mathematical logics. However, the incompleteness is obvious from the principle of verificationism which is accepted in quantum theory. According to this principle, there should be no statements accepted without proof and based on belief in their correctness (i.e., axioms). The theory should contain only those statements that can be verified, where by "verified" physicists mean an experiment involving only a finite number of steps. *From this principle, the statements of Gödel's incompleteness theorems are obvious because if a theory contains the entire set of natural numbers then it is not possible to verify correctness of all operations with them.* On the other hand, since FM works only with a finite number of elements, this mathematics satisfies the principle of verificationism.

Let M be a natural number. A natural question is whether Z can be viewed as the limit of R_p as $p \to \infty$ and, if so, how this limit should be defined. In CM, the standard definition of a sequence going to $+\infty$ uses only potential infinity: a sequence (a_n) tends to $+\infty$ as $n \to \infty$ if for every M there exists n_0 such that $a_n \geq M$ for all $n \geq n_0$. By analogy, one may ask whether a proof that $R_p \to Z$ as $p \to \infty$ can be given using only potential infinity. Such a proof is given in our monograph [4], and a simpler version is given in Sec. 2 of this paper. Despite the fundamental nature of this fact, we have not found it elsewhere in mathematical literature.

Another way to prove that $R_p \to Z$ as $p \to \infty$ uses ultraproducts. As shown in [5, 6], infinite fields of characteristic zero and Z can be embedded into ultraproducts of finite fields. This also follows using only rings (see, e.g., Theorem 3.1 in [7]). However, ultraproducts rely on classical results involving actual infinity, including the Łoś's theorem using the axiom of choice. Therefore, ultraproduct-based arguments cannot be used to establish that FM is more general than CM.

The approaches [7, 8, 9] reflect a common viewpoint: structures of characteristic zero are regarded as general, while those of positive characteristic are seen as special cases that sometimes provide useful tools. However, all such arguments depend on actual infinity and thus inherit its foundational issues, as shown by Gödel, Tarski, Church, Turing, and others. Consequently, it becomes important to determine whether the fact that $R_p \to Z$ as $p \to \infty$ can be proved *purely within*

the framework of potential infinity.

The paper is organized as follows. In Sec. 2, a proof of the theorem that $R_p \to Z$ as $p \to \infty$ as $p \to \infty$ is given, and this fact can be proven using only potential infinity. Sec. 3 explains why quantum theory based on FM is more general than standard quantum theory and why FM is more general tham CM. Finally, Sec. 4 is Conclusion.

2 Proof that $R_p \to Z$ at $p \to \infty$ using only potential infinity

Note. To study the limit $p \to \infty$, it suffices to consider values of $p > p_0$ where the specific value of p_0 is irrelevant. In what follows, we define two functions $k = k(p)$ and $n = n(p)$, each taking positive natural values. Any fixed value of p_0 greater than, say, 100 would work, although its explicit value is unimportant.

Because all operations in R_p are carried out modulo p, one may represent R_p as

- $\{0, \pm 1, \pm 2, ..., \pm(p-1)/2)\}$ if p is odd,

- $\{0, \pm 1, \pm 2, ..., \pm(p/2 - 1), p/2\}$ if p is even.

We define the meaning of the statement that $R_p \to Z$ at $p \to \infty$ as follows. Let $k = k(p)$ satisfy $k < (p-1)/2$ if p is odd and $k < p/2 - 1$ if p is even. Let $S = S(k)$ be a set of numbers $(0, \pm 1, \pm 2, ... \pm k)$. Then $S \subset R_p$ and $S \subset Z$. Next, let $n = n(p)$ be a natural number such that:

- For any operation of summation, subtraction and multiplication involving m elements of S where m is any number such that $m \leq n$, the result is the same in both R_p and Z. That is, no modular "wraparound" occurs for any expression small enough compared to p.

- As $p \to \infty$, both $k(p) \to \infty$ and $n(p) \to \infty$.

This means that for the set S and number n, the effects of modular arithmetic never appear. Any experiment restricted to such values cannot distinguish whether the governing arithmetic is that of R_p or Z.

Theorem: $R_p \to Z$ as $p \to \infty$.

Proof. Define the function $h(p)$ such that $h(p) = (p-1)/2$ if p is odd and $h(p) = p/2 - 1$ if p is even. For every $p > p_0$ there exists a unique natural $n = n(p)$ satisfying $2^{n^2} \leq h(p) < 2^{(n+1)^2}$. Then if $k = k(p) = 2^n$, *Theorem* follows. $\square$

This *Theorem* was originally proved in more elaborate form in Chapter 6.3 of [4] but the above proof is significantly simpler.

The proved *Theorem* shows that:

- a) Any result in Z can be obtained in R_p if p is chosen to be sufficiently large.

- b) In Z one cannot reproduce results in R_p that rely on genuinely modular operations.

Therefore, as follows from *Definition*:

Statement 1: R_p is more general than Z, and Z is a special degenerate case of R_p in the limit $p \to \infty$.

3 Standard Quantum Theory vs. Finite Quantum Theory

We now examine how *Theorem* proved in Sec. 2 contributes to the foundations of mathematics and physics. A key preliminary question is how mathematics itself should be understood:

- (1) As a purely abstract discipline, independent of nature.

- (2) As a discipline that must ultimately describe nature.

Most physicists accept only viewpoint (2), while many mathematicians and philosophers adopt viewpoint (1) which may be called Hilbert's approach.

As noted above, Hilbert's approach to CM is formulated as the Entscheidungsproblem but Hilbert believed that physics should be axiomatized in a similar way to mathematics. His 6th Problem is formulated as follows [10]: "To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics". Gödel's incompleteness theorems, however,

raised deep questions about the foundations of this approach, and, as noted above, Church and Turing proved the insolvability of Hilbert's Entscheidungsproblem [1, 2, 3].

Nevertheless, Hilbert's perspective should not be dismissed. Dirac, for example, emphasized trusting mathematical structure above physical intuition, arguing that mathematically beautiful theories eventually reveal physical meaning. A historical example is the use of Hilbert spaces: studied since the early twentieth century, they were not applied to quantum physics until the 1930s. Under Hilbert's approach, many powerful results have been achieved. However, since it relies on actual infinity, its foundational issues remain unresolved.

We will now consider the viewpoint (2). As shown in the extensive physics literature (see, e.g., Chapter 1.3 of [4]):

Statement 2: Classical (i.e., non-quantum) theory is a special degenerate case of quantum one in the limit $\hbar \to 0$ where $\hbar$ is the Planck constant.

Then the question of whether CM or FM is more general depends on which physical theory—Standard Quantum Theory (SQT) based on CM, or Finite Quantum Theory (FQT) based on FM—is more general.

Let's first discuss some properties of SQT. Here, physical states live in a separable Hilbert space $\mathcal{H}$. In quantum theory (both, SQT and FQT), any system is considered to consist of elementary particles described by irreducible representations (IRs) of a symmetry algebra. In nonrelativistic theory, the symmetry algebra is the Galilei algebra, in relativistic theory — the Poincare algebra, and in de Sitter (dS) and anti-de Sitter (AdS) theories — the dS and AdS algebras, respectively.

In SQT, IRs of these algebras describing elementary particles are infinite-dimensional. The state vector of the entire system is the tensor product of the state vectors for the elementary particles in the system. Therefore, the Hilbert space $\mathcal{H}$ for the entire system is infinite-dimensional, even if the system consists of a single elementary particle.

A known result of the theory of Hilbert spaces is [11]:

Statement 3: A Hilbert space is separable if and only if it admits a countable orthonormal basis $(e_1, e_2, ...e_n, ...)$ and it is always possible to choose a basis such that the norm of each e_j $(j = 1, 2, ...\infty)$ is an integer.

Let the complex numbers $(c_1, c_2...)$ be the decomposition coefficients for a vector $x \in \mathcal{H}$ over the basis $(e_1, e_2...)$. The only require-

ment that the coefficients must satisfy is: $\sum_{j=1}^{\infty} |c_j|^2 < \infty$. The known result of the theory of Hilbert spaces is that [11]:

Statement 4: The set of all points $(c_1, c_2, ...)$ with only finitely many nonzero coordinates, each a rational number, is dense in the separable Hilbert space.

This implies that, *with any desired accuracy*, each element of $\mathcal{H}$ can be approximated by a finite linear combination

$$x = \sum_{j=1}^{n} c_j e_j \tag{1}$$

where $c_j = a_j + ib_j$ and all the numbers (a_j, b_j) $(j = 1, 2, ...n)$ are rational.

The next observation is that spaces in quantum theory are projective: for any complex number $c \neq 0$, the elements x and cx describe the same state. The meaning of this statement is that not the probability itself but ratios of different probabilities have a physical meaning. As a consequence, both parts of Eq. (1) can be multiplied by a common denominator of all the nonzero numbers a_j and b_j, and

Statement 5: Each element of a separable projective Hilbert space can be approximated with any desired accuracy by a finite linear combination (1) where all the numbers a_j and b_j are integers, i.e., belong to Z.

This shows that SQT contains an enormous redundancy: although SQT formally uses an uncountable set of complex-valued vectors, any experimental prediction can be approximated using only finitely many coefficients and all the numbers (a_j, b_j) belong to Z. In other words, although the states in SQT formally live in a complex Hilbert space, effectively one can consider that they live in a space over the ring $Z + iZ$.

In SQT, as shown by Dyson [12] (see also Chapter 1.3 in [4] and [13]), it follows even from purely mathematical considerations that:

- Nonrelativistic theory (NT) is a special degenerate case of relativistic one (RT) in the formal limit $c \to \infty$. The quantity c is usually associated with the speed of light but in fact this is only a constant of the theory.

- RT is a special degenerate case of dS and AdS invariant theories in the formal limit $R \to \infty$ where R is the parameter of con-

traction from the dS or AdS Lie algebras to the Poincare Lie algebra.

- In turn, since dS and AdS algebras are semisimple, dS and AdS theories cannot be obtained from more general theories by contraction.

In FQT, no dimensional constants such as kilograms, meters, or seconds appear, since these derive from macroscopic physics. Thus, FQT cannot have Galilei or Poincare symmetry. Here quantum states live not in Hilbert spaces, but in vector spaces over the "complexified finite ring" $R_p + iR_p$. As noted above, states in SQT effectively can be treated as belonging to a space over the $Z + iZ$ and, as shown in Sec. 2, R_p is more general than Z. Therefore, the natural question arises: does this imply that FQT is a more general theory than SQT?

By the Zassenhaus theorem [14], all IRs of the algebras over the rings of nonzero characteristics (modular IRs) are finite-dimensional. Explicit modular IRs of the dS and AdS algebras were constructed in [15, 16]. Therefore, any system containing finitely many elementary particles has a finite-dimensional state space.

A fundamental difference between SQT and FQT concerns particle/antiparticle structure:

- In SQT, IRs contain either only positive-energy states or only negative-energy states. These correspond to particles and antiparticles, and conservation laws (electric charge, baryon number) prevent transitions between them.

- In FQT, each IR necessarily contains both positive and negative energy states [15, 16]. As $p \to \infty$, this IR splits into two separate IRs corresponding to particles and antiparticles in SQT [4, 17].

Thus each FQT IR has greater symmetry than the corresponding pair of IRs in SQT. One might think this contradicts experiment because charge and baryon number appear conserved. But this impression is due to the fact that at the present stage of the universe p is extremely large, so transitions between positive- and negative-energy states are effectively suppressed.

As shown in [4, 18], FQT is a more general theory than SQT because SQT is a special degenerate case of FQT at $p \to \infty$. As follows from *Definition*, to prove this statement, it is necessary to prove that:

- *Statement 6A: For any result of SQT, one can choose p suffi-ciently large so that FQT reproduces the same result.*

- *Statement 6B: The explanation of some physical phenomena re-quires modular arithmetic.*

As shown in [4, 18], *Statement 6A* follows from *Theorem* in Sec. 2 and from *Statement 5*. For proving *Statement 6B* we consider three exam-ples: gravity, baryon asymmetry of the universe and Dirac's vacuum energy problem.

Quantum gravity within SQT is non-renormalizable and plagued by divergences. In contrast, as shown in Chapter 5.5 of [4], at least the Newtonian gravity emerges in the semiclassical limit of FQT. In this approach, the gravitational constant G is not taken from the outside but depends on p approximately as $1/ln(p)$. Matching this with experiment gives $ln(p) \approx 10^{80}$ or more and therefore p is a huge number of the order of $exp(10^{80})$ or more. One might think that since p is so huge then in practice p can be treated as an infinite number. However, since $ln(p)$ is "only" of the order of 10^{80}, gravity is observable. In the formal limit $p \to \infty$, G becomes zero and gravity disappears. Therefore, in our approach, gravity is a consequence of finiteness of nature.

Let us discuss the following question. In many publications (see e.g., [19] and references therein), arguments are given that our universe works like a computer. Then the number p that determines the laws of physics in our universe is not a fundamental number given by a theory but is a number that is determined by the state of the universe at its present stage. And, since the state of the universe is changing, it is natural to expect that the number p describing physics at different stages of the evolution of the universe will be different at different stages. As noted above, in the situation where p is very large, it may seem that the electric charge and baryon number are conserved quantum numbers. The above result about gravity shows that, at the present stage of the universe, the number p is huge, and this might be a justification of the postulate of modern particle theory that the electric charge and baryon number are strictly conserved quantum numbers.

The paradox with the baryon asymmetry of the universe is for-mulated as follows. According to modern cosmological theories, at early stages of the universe, the numbers of baryons and antibaryons

were the same. Then, as follows from the law of baryon number conservation, those numbers should be the same at the present stage of the universe. However, at this stage, the number of baryons is much greater than the number of antibaryons. The above paradox arises if we assume that the number p was huge not only in the early stages of the universe but always and therefore the laws of conservation of electric charge and baryon number always held true. However, there is no basis for this assumption, and therefore the baryon asymmetry paradox does not arise.

Consider now the Dirac vacuum energy problem which is discussed in practically every textbook on quantum field theory. In its simplified form it can be described as follows. Suppose that the energy spectrum is discrete and n is the quantum number enumerating the states. Let $E(n)$ be the energy in the state n. Consider the electron-positron field. As a result of quantization one gets for the energy operator

$$E = \sum_n E(n)[a(n)^*a(n) - b(n)b(n)^*] \tag{2}$$

where $a(n)$ is the operator of electron annihilation in the state n, $a(n)^*$ is the operator of electron creation in the state n, $b(n)$ is the operator of positron annihilation in the state n and $b(n)^*$ is the operator of positron creation in the state n. It follows from this expression that only anticommutation relations are possible since otherwise the energy of positrons will be negative. However, if anticommutation relations are assumed, it follows from Eq. (2) that

$$E = \{\sum_n E(n)[a(n)^*a(n) + b(n)^*b(n)]\} + E_0 \tag{3}$$

where E_0 is some infinite negative constant. Its presence was a motivation for developing Dirac's hole theory. In the modern approach, it is usually required that the vacuum energy should be zero. This can be obtained by assuming that all operators should be written in the normal form. However, this requirement is not quite consistent since the result of quantization is Eq. (2) where the positron operators are not written in that form.

However, a detailed calculation in Chapter 8.8 of [4] shows that within the framework FQT where the calculations use operations modulo a number, the vacuum energy is zero for particles with the spin $s = 1/2$ in the usual units. This result shows that since the rules of

arithmetic in FM and CM are different, it is possible that quantities which are infinite in standard theory (e.g., the vacuum energy) will be zero in FQT.

In [4, 18] we gave other examples when FQT can solve problems that SQT cannot. Therefore, the above arguments show that

Statement 7: FQT is a more general theory than SQT.

In turn, from the viewpoint (2), it follows from *Statement 7* that

Statement 8: Finite Mathematics is a more general theory than Classical Mathematics.

In conclusion of this section, let us discuss the following question. As noted above, in CM, the ring Z is generalized to the case of various fields in which four operations are possible: addition, subtraction, multiplication, and division. However, when generalizing CM to FM, we considered only the ring R_p and its complex extension. In FM, division may seem unnatural. For example, in the Galois field F_p, where p is a prime, $1/2$ is a large number $(p + 1)/2$ if p is large. However, the main question is whether it is necessary to have division in FQT.

SQT is essentially based on the concept of infinitesimals introduced by Newton and Leibniz more than 300 years ago. This concept was in the spirit of the experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts. However, now we know about the existence of elementary particles. Even the name "elementary particle" suggests that such a particle cannot be subdivided into parts. For example, the energies of electrons in modern accelerators are millions of times greater than the electron rest energy, and such electrons experience many collisions with other particles. If the electron could be divided into parts, this would have been discovered long ago. So, in physics, division has limited applicability, since when we reach the level of elementary particles, further division is no longer possible. Standard macroscopic theory and standard geometry (the concepts of continuous lines and surfaces) can work well only in the approximation when sizes of atoms are neglected. It seems rather strange that, although most physicists understand this, they nevertheless consider the concept of infinitesimals not as only approximate but as fundamental.

4 Conclusion

The proof of *Theorem* in Sec. 2 using only potential (not actual) infinity is much simpler than in Chapter 6.3 of [4]. It makes natural the conclusion of Sec. 3 that Finite Mathematics is more general than Classical Mathematics and Finite Quantum Theory is more general than Standard Quantum Theory. As a consequence:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the concepts of limit, infinitesimals and continuity are needed only in calculations describing nature approximately.

It is clear that the ultimate quantum theory can only be based on mathematics free of foundational problems. As noted above, finite mathematics indeed does not have such problems. At the same time, in the approach of Cantor, Hilbert and other mathematicians the following questions arise:

- A) If we accept that actual infinity is necessary, then how does this correspond to the works of Gödel and other mathematicians, that if we start from the entire infinite series of natural numbers, then problems arise in the foundation of mathematics?

- B) In this approach, there is the concept of infinitesimals but, as noted in the preceding section, it is not clear whether this concept is compatible with the existence of elementary particles.

So, although in the approach of Cantor, Hilbert and other mathematicians, many strong and beautiful results have been obtained, the question arises how to reconcile this approach with A) and B). The answer to this question is currently unknown. However, it is clear that the problem exists, and this could be a good incentive for further research in mathematics and physics.

As noted in Sec. 1, the following historical analogy can be given here. In nonrelativistic theory, many strong results have been obtained, but here there is no limit on the magnitude of speed. There is such a limit in relativistic theory where the magnitude cannot exceed c. The relativistic theory does not refute the nonrelativistic one, but indicates that the latter is applicable when speeds are much less than c, and the former should be applied when the problem involves speeds comparable to c. Note that in SQT, there is no limit on the magnitude

of angular momentum but in FQT, where all the quantities are taken modulo p, the magnitude of any angular momentum cannot exceed p.

Thus, the main results of this paper are as follows.

- While classical mathematics (CM) has foundational problems and standard quantum theory (SQT) contains divergences;

- Finite mathematics (FM) and finite quantum theory (FQT) do not have such problems and CM and SQT are special degenerate cases of FM and FQT, respectively, in the limit $p \to \infty$ where p is the characteristic of the ring used in FM and FQT. Therefore, for any result of CM or SQT it is possible to find a value of p at which this result is reproduced in FM or FQT, respectively.

The question now arises of how to use the above results to construct the ultimate particle theory. As noted in [4, 17], the main difficulty in such a construction is the following. Our physical intuition comes from the existing theory, which contains particles and antiparticles and laws of conservation of electric charge and baryon number. As explained in [4, 17], such results arise because the symmetry algebras used in particle physics (Galilei, Poincare, and AdS) have the property that in their IRs, the energies of particles can only be either positive or negative, and there are no IRs in which particles have states with different energy signs. At the same time, in FM, each IR necessarily contains both positive and negative energies (see e.g., [4, 15, 16]). It is clear that the case when there is one IR uniting positive and negative energies has a higher symmetry than the case when this IR splits into two independent IRs with positive and negative energies. In the former case, there are no strict concepts such as particle-antiparticle, proton-antiproton, electron-positron, and so on. These concepts can only be approximate when the characteristic of the ring in FM is very large. Therefore, the challenge of constructing the ultimate particle theory is to construct a theory without the assumption that p is anomalously large, and such a theory will be based on new physical concepts.

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References

[1] A. Church, *An unsolvable problem of elementary number theory.* American Journal of Mathematics, **58**, 345–363 (1936).

[2] A. Church, *A note on the Entscheidungsproblem.* Journal of Symbolic Logic, **1**, 40–41 (1936).

[3] A. Turing, *On Computable Numbers, with an Application to the Entscheidungsproblem.* Proceedings of the London Mathematical Society, **2**, 42, (230–265), (1936–7).

[4] F. Lev, *Finite mathematics as the foundation of classical mathematics and quantum theory. With application to gravity and particle theory.* ISBN 978-3-030-61101-9. Springer, Cham, Switzerland (2020).

[5] J.P. Serre, *How to Use Finite Fields for Problems Concerning Infinite Fields.* arXiv:0903.0517 (2009).

[6] T. Tao, *Infinite Fields, Finite Fields, and the Ax-Grothendieck Theorem.* Available online: https://terrytao.wordpress.com/2009/03/07/infinite-fields-finite-fields-and-the-ax-grothendieck-theorem. (2009).

[7] J. Turner, *Ultraproducts in Algebra.* Available online: http://math.uchicago.edu (2017).

[8] C.C. Chang and H.J. Keisler, *Model Theory.* North-Holland Press: Amsterdam (1990).

[9] H. Schoutens, *The Use of Ultraproducts in Commutative Algebra.* Springer: New York (2007).

[10] D. Hilbert, *Mathematical Problems.* Bulletin of the American Mathematical Society. **8**, 437–479 (1902).

[11] A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis.* Dover Publication Inc., New York (1975).

[12] F.G. Dyson, *Missed Opportunities.* Bull. Amer. Math. Soc. **78**, 635-652 (1972).

[13] F. Lev, *de Sitter Symmetry and Quantum Theory.* Phys. Rev. **D85**, paper 065003 (2012).

[14] H. Zassenhaus, *The representations of Lie Algebras of Prime Characteristic.* Proc. Glasgow Math. Assoc. **2**, 1–36 (1954).

[15] F. Lev, *Modular Representations as a Possible Basis of Finite Physics.*, J. Math. Phys **30**, 1985-1998 (1989).

[16] F. Lev, *Finiteness of Physics and Its Possible Consequences.*, J. Math. Phys. **34**, 490-527 (1993).

[17] F. Lev, *Main Problems in Constructing Quantum Theory Based on Finite Mathematics.* Mathematics **12**, paper 3707 (2024).

[18] F. Lev, *Finite mathematics as the most general (fundamental) mathematics.* Symmetry **16**, paper 1340 (2024).

[19] S. Wolfram, *A Class of Models with the Potential to Represent Fundamental Physics.* Complex Systems **29**, 107-536 (2020).