A New Polyhedron Obtained by Truncation of Rhombic Dodecahedron

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1. Introduction

A truncated rhombic dodecahedron is a non-uniform convex polyhedron which has 12 congruent rectangular faces, 6 congruent square faces and 8 congruent equilateral-triangular faces, 48 edges (24 small & 24 large edges) and 24 identical vertices (at each of which two rectangular, one square and one triangular faces meet together) lying on a spherical surface of certain radius. In this paper, a new polyhedron (i.e. truncated rhombic dodecahedron) is generated by truncating all 14 vertices of a rhombic dodecahedron [1] from the points of tangency (where an inscribed circle touches four sides of rhombic face) such that its every rhombic face is changed into a rectangular face which has its length $\sqrt{2}$ times the width and all its 24 edges are changed into 24 new identical vertices (as shown in fig-1). This truncated rhombic dodecahedron looks very similar to a rhombicuboctahedron [2,3] but a truncated rhombic dodecahedron has 12 rectangular faces instead of square faces. The number of faces, edges & vertices of a truncated rhombic dodecahedron generated by truncating all the vertices of parent polyhedron (i.e. rhombic dodecahedron) are obtained as follows

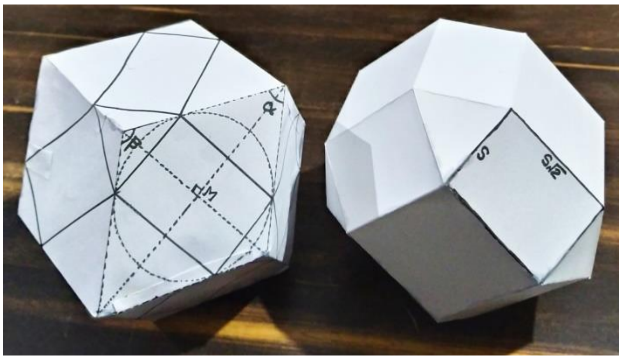

Figure 1: A rhombic dodecahedron (left) is marked to be truncated at each of its 14 vertices to generate 12 rectangular faces, 6 square faces & 8 equilateral triangular faces to form a truncated rhombic dodecahedron (right) with edges s & $s\sqrt{2}$.

$$\text{Number of new rectangular faces} = \text{number of faces in parent solid (rhombic dodecahedron)} = 12$$

$$\text{Number of new square faces} = \text{number of vertices where 4 edges meet in parent solid} = 6$$

$$\text{Number of new triangular faces} = \text{number of vertices where 3 edges meet in parent solid} = 8$$

$$\text{Number of new edges} = (\text{no. of edges in rhombus})(\text{no. of rhombic faces in parent solid}) = 4 \cdot 12 = 48$$

$$\text{Number of new vertices} = \text{number of edges in parent solid} = 24$$

Since a truncated rhombic dodecahedron is obtained directly from a rhombic dodecahedron, its mathematical analysis can be carried out in a systematic and simplified manner by exploiting the known geometrical relations of the parent polyhedron [1]. In the present work, a rhombic dodecahedron of edge length a is taken as the parent solid and truncated at the points of tangency between the inscribed sphere and the rhombic faces, as illustrated in Fig. 1. This truncation generates a truncated rhombic dodecahedron with two distinct edge lengths, namely small & large edge lengths s & $s\sqrt{2}$ respectively. A precise analytical relationship between the edge length a of the parent rhombic dodecahedron and the smaller edge length s of the truncated polyhedron is first established. Using this relationship, closed-form expressions are then derived for the radius of the circumscribed sphere passing through all 24 congruent vertices of the truncated rhombic dodecahedron. Furthermore, analytical formulae are obtained for the normal distances of the rectangular, square, and equilateral triangular faces from the centre of the polyhedron, as well as for its total surface area and enclosed volume.

Employing HCR's Theory of Polygon [4,5], the solid angles subtended at the center by the rectangular, square, and equilateral triangular faces are analytically evaluated. In addition, expressions are derived for the dihedral angles between any two faces meeting at each of the 24 identical vertices and for the solid angle subtended by the truncated rhombic dodecahedron at each of these vertices.

2. Derivation of radius R of circumscribed sphere i.e. passing through all 24 identical vertices of a truncated rhombic dodecahedron

Consider a rhombic dodecahedron having 12 congruent faces each as a rhombus of side a. We know that the midsphere with radius R_{md} touches all 24 edges of rhombic dodecahedron i.e. midsphere touches all four sides of each rhombic face. Consider a rhombic face ABCD touching the midsphere at four distinct points (of tangency) E, F, G & I on the sides AB, BC, CD & AD respectively. Circle passing through the points of tangency E, F, G & I is inscribed by the rhombus ABCD & lies on the surface of midsphere of rhombic dodecahedron. Join these four points of tangency E, F, G & I to get a rectangular face EFGI (as shown in fig-2). Thus we mark a rectangle on each of 12 congruent rhombic faces by joining the points of tangency of the edges & the midsphere and then truncate the rhombic dodecahedron from each of its 14 vertices at the points of tangency to get a truncated rhombic dodecahedron (as shown in above fig-1). From formula derived in 'Mathematical analysis of rhombic dodecahedron', the angles α & β, the lengths of semi major & semi minor diagonals AM & BM of rhombic face ABCD , and the radius R_{md} of midsphere of rhombic dodecahedron with edge length a , are given as

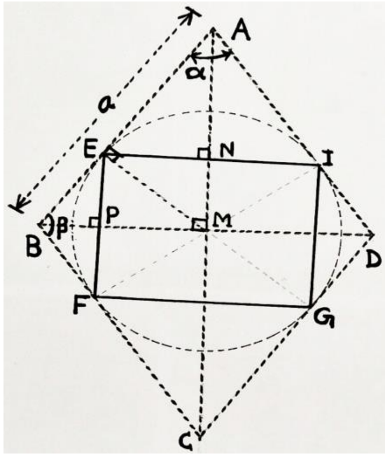

Figure 2: A rhombic face ABCD is changed into a rectangular face EFGI by truncating a rhombic dodecahedron from its vertices at points of tangency E, F, G & I of edges & the mid-sphere.

$$\alpha = 2\cot^{-1}\sqrt{2}, \quad \beta = 2\tan^{-1}\sqrt{2}, \quad AM = a\sqrt{\frac{2}{3}}, \quad BM = \frac{a}{\sqrt{3}}, \quad R_{md} = \frac{2a\sqrt{2}}{3}$$

In right ΔAEM (see fig-2)

$$\cos\frac{\alpha}{2} = \frac{AE}{AM} \quad \Rightarrow \quad AE = AM\cos\frac{\alpha}{2} = a\sqrt{\frac{2}{3}}\cos\left(\frac{2\cot^{-1}\sqrt{2}}{2}\right) = a\sqrt{\frac{2}{3}}\cos\left(\cos^{-1}\sqrt{\frac{2}{3}}\right) = a\sqrt{\frac{2}{3}}\left(\sqrt{\frac{2}{3}}\right) = \frac{2a}{3}$$

$$\Rightarrow \quad BE = AB - AE = a - \frac{2a}{3} = \frac{a}{3}$$

In right ΔANE (see fig-2)

$$\sin\frac{\alpha}{2} = \frac{EN}{AE} \quad \Rightarrow \quad \boldsymbol{EN} = AE\sin\frac{\alpha}{2} = \frac{2a}{3}\sin\left(\frac{2\cot^{-1}\sqrt{2}}{2}\right) = \frac{2a}{3}\sin\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) = \frac{2a}{3}\left(\frac{1}{\sqrt{3}}\right) = \frac{2a}{3\sqrt{3}}$$

$$\Rightarrow \quad \boldsymbol{EI} = 2EN = 2\left(\frac{2a}{3\sqrt{3}}\right) = \frac{4a}{3\sqrt{3}} \qquad \ldots\ldots(\mathrm{I})$$

Similarly, in right ΔBPE (see fig-2 above)

$$\sin\frac{\beta}{2} = \frac{EP}{BE} \quad \Rightarrow \quad \boldsymbol{EP} = BE\sin\frac{\beta}{2} = \frac{a}{3}\sin\left(\frac{2\tan^{-1}\sqrt{2}}{2}\right) = \frac{a}{3}\sin\left(\sin^{-1}\sqrt{\frac{2}{3}}\right) = \frac{a}{3}\left(\sqrt{\frac{2}{3}}\right) = \frac{a}{3}\sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad \boldsymbol{EF} = 2EP = 2\left(\frac{a}{3}\sqrt{\frac{2}{3}}\right) = \frac{2a}{3}\sqrt{\frac{2}{3}} \qquad \ldots\ldots(\mathrm{II})$$

Now, diving the Eq.(I) by Eq.(II) as follows

$$\frac{EI}{EF}=\frac{\frac{4a}{3\sqrt{3}}}{\frac{2a}{3}\sqrt{\frac{2}{3}}}=\sqrt{2}\quad \Rightarrow\ EI=EF\sqrt{2}\ \Rightarrow\ \textbf{Length of rectangular face EFGI}=\sqrt{\mathbf{2}}\times\textbf{Width}$$

If the small side or width EF of the rectangular face EFGI is s i.e. $EF=s$ then the large side or length EI of the rectangular face EFGI is $=EF\sqrt{2}=s\sqrt{2}$. Thus, two unequal edges i.e. length and width of each of 12 congruent rectangular faces of the truncated rhombic dodecahedron are & $s\sqrt{2}$. Now, substituting $EF=s$ in eq.(2) as follows

$$EF=s=\frac{2a}{3}\sqrt{\frac{2}{3}}\qquad\Rightarrow\qquad \boldsymbol{a}=\frac{\mathbf{3}\boldsymbol{s}}{\mathbf{2}}\sqrt{\frac{\mathbf{3}}{\mathbf{2}}}\qquad\qquad \dots\dots\dots\dots(III)$$

Now, the radius R of the circumscribed sphere i.e. passing through all 24 identical vertices of the truncated rhombic dodecahedron with unequal edge lengths s & $s\sqrt{2}$, is given as follows

$$R=\text{Radius } R_{md} \text{ of midsphere of parent rhombic dodecahedron with edge length } a$$

$$R=\frac{2a\sqrt{2}}{3}\qquad\qquad (\text{ from formula derived for a rhombic dodecahedron})$$

$$R=\frac{2\left(\frac{3s}{2}\sqrt{\frac{3}{2}}\right)\sqrt{2}}{3}=s\sqrt{3}\qquad\qquad (\text{ Substituting value of } a \text{ in terms of } s \text{ from eq. (III)})$$

$$\boldsymbol{R}=\boldsymbol{s}\sqrt{\mathbf{3}}\approx\mathbf{1.732050808}\,\boldsymbol{s}\qquad\qquad \dots\dots\dots\dots.(1)$$

2.1. Normal distances H_R, H_S & H_T of rectangular, square & equilateral triangular faces from the centre of a truncated rhombic dodecahedron: Consider a rectangular face EFGI of length and width s & $s\sqrt{2}$ which is at a normal distance $OM=H_R$ from the centre O of truncated rhombic dodecahedron. Join its vertices E, F, G & I & centre M to the centre O (as shown in fig-3)

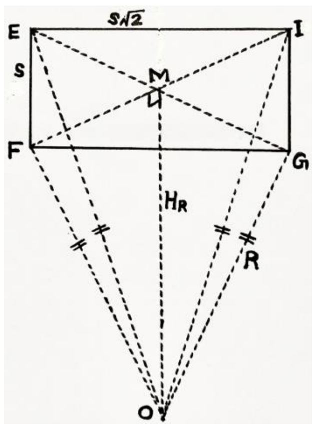

Figure 3: Rectangular face EFGI is at a normal distance H_R from centre O of polyhedron. $OE=OF=OG=OI=R$.

In right ΔOMG (see fig-3), using Pythagorean theorem, we get

$$OM=\sqrt{(OG)^2-(MG)^2}=\sqrt{(R)^2-\left(\frac{EG}{2}\right)^2}=\sqrt{\left(s\sqrt{3}\right)^2-\left(\frac{\sqrt{(s)^2+\left(s\sqrt{2}\right)^2}}{2}\right)^2}$$

$$\boldsymbol{H_R}=\sqrt{3s^2-\frac{3s^2}{4}}=\sqrt{\frac{9s^2}{4}}=\frac{3s}{2}$$

Consider a square face EKLI of side $s\sqrt{2}$ which is at a normal distance $OQ=H_S$ from the centre O of truncated rhombic dodecahedron. Join its vertices E, K, L & I, and centre Q to the centre O (as shown in fig-4 below).

In right ΔOQL (see fig-4 below), using Pythagorean theorem, we get

$$OQ=\sqrt{(OL)^2-(QL)^2}$$

$$H_S=\sqrt{(R)^2-\left(\frac{EL}{2}\right)^2}$$

$$H_S = \sqrt{\left(s\sqrt{3}\right)^2 - \left(\frac{\sqrt{\left(s\sqrt{2}\right)^2 + \left(s\sqrt{2}\right)^2}}{2}\right)^2}$$

$$H_S = \sqrt{3s^2 - \frac{4s^2}{4}}$$

$$H_S = \sqrt{2s^2}$$

$$\boldsymbol{H_S} = s\sqrt{2}$$

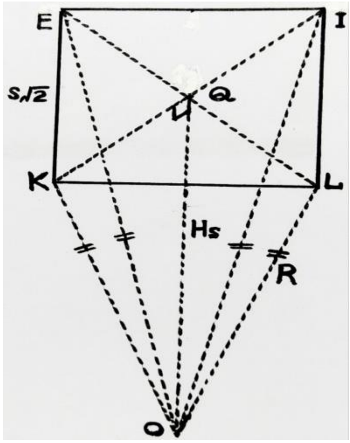

Figure 4: Square face EKLI is at a normal distance H_S from centre O of polyhedron. $OE = OK = OL = OI = R$.

Consider an equilateral triangular face EFJ of side s which is at a normal distance $OT = H_T$ from the centre O of truncated rhombic dodecahedron. Join its vertices E, F, J & centre M to the centre O of truncated rhombic dodecahedron (as shown in fig-5)

In right ΔOTF (see fig-5), using Pythagorean theorem, we get

$$OT = \sqrt{(OF)^2 - (TF)^2}$$

$$H_T = \sqrt{(R)^2 - \left(\frac{s}{\sqrt{3}}\right)^2} \qquad \left(\text{Circum radius of equilateral } \Delta = \frac{\text{side}}{\sqrt{3}}\right)$$

$$H_T = \sqrt{\left(s\sqrt{3}\right)^2 - \frac{s^2}{3}}\ \sqrt{3s^2 - \frac{s^2}{3}} = \sqrt{\frac{8s^2}{3}} = 2s\sqrt{\frac{2}{3}}$$

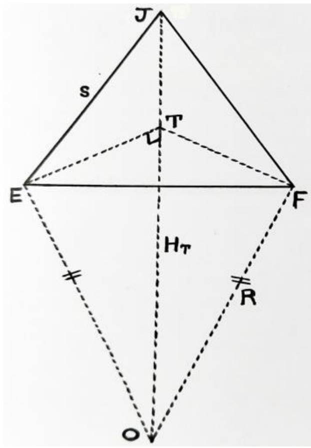

Figure 5: Equilateral triangular face EFJ is at a normal distance H_T from the centre O and $OE = OF = OJ = R$.

Hence, the normal distances H_R , H_S & H_T of rectangular, square & equilateral triangular faces respectively from the centre of truncated rhombic dodecahedron with unequal edges s & $s\sqrt{2}$, are given as follows

$$\boldsymbol{H_R = \frac{3s}{2}} \qquad \ldots\ldots\ldots\ldots(2)$$

$$\boldsymbol{H_S = s\sqrt{2} \approx 1.414213562\ s} \qquad \ldots\ldots\ldots\ldots(3)$$

$$\boldsymbol{H_T = 2s\sqrt{\frac{2}{3}} \approx 1.632993162\ s} \qquad \ldots\ldots\ldots\ldots(4)$$

It is clear from above values of normal distances that the equilateral triangular faces are the farthest from the centre while square faces are the closest to the centre & rectangular faces are at a normal distance between these two. For finite value of small edge length $s \Rightarrow H_S < H_R < H_T < R$.

2.2. Surface Area (A_s) of truncated rhombic dodecahedron: The surface of a truncated rhombic dodecahedron consists of 12 congruent rectangular faces each of length $s\sqrt{2}$ & width s, 6 congruent square faces each with side $s\sqrt{2}$ and 8 congruent equilateral triangular faces each with side s . Therefore, the (total) surface area of truncated rhombic dodecahedron is given as follows

$$A_s = 12(\text{Area of rectangular face}) + 6(\text{Area of square face}) + 8(\text{Area of equi. triangular face})$$

$$= 12(s \cdot s\sqrt{2}) + 6\left(\left(s\sqrt{2}\right)^2\right) + 8\left(\frac{\sqrt{3}}{4}(s)^2\right)$$

$$= 12s^2\sqrt{2} + 12s^2 + 2s^2\sqrt{3}$$

$$= 2s^2\left(6\sqrt{2} + 6 + \sqrt{3}\right)$$

$$\therefore \textbf{Surface area}, \ \boldsymbol{A_s = 2s^2\left(6\sqrt{2} + 6 + \sqrt{3}\right) \approx 32.43466436\, s^2} \qquad \dots\dots\dots\dots(5)$$

2.3. Volume (V) of truncated rhombic dodecahedron: The surface of a truncated rhombic dodecahedron consists of 12 congruent rectangular faces each of length $s\sqrt{2}$ & width s, 6 congruent square faces each with side $s\sqrt{2}$ and 8 congruent equilateral triangular faces each with side s. Thus a solid truncated rhombic dodecahedron can assumed to consisting of 12 congruent right pyramids with rectangular base of length $s\sqrt{2}$ & width s & normal height H_R , 6 congruent right pyramids with square base of side $s\sqrt{2}$ & normal height H_S and 8 congruent right pyramids with equilateral triangular base with side $s\sqrt{2}$ & normal height H_T (as shown in above fig-1). Therefore, the volume of truncated rhombic dodecahedron is given as

$$V = 12(\text{Volume of rectangular right pyramid }) + 6(\text{Volume of square right pyramid }) + 8(\text{Volume of equilateral triangular right pyramid })$$

$$= 12\left(\frac{1}{3}\left(s \cdot s\sqrt{2}\right) \cdot \frac{3s}{2}\right) + 6\left(\frac{1}{3}\left(s\sqrt{2}\right)^2 \cdot s\sqrt{2}\right) + 8\left(\frac{1}{3}\left(\frac{\sqrt{3}}{4}s^2\right) \cdot 2s\sqrt{\frac{2}{3}}\right)$$

$$= 12\left(\frac{s^3}{\sqrt{2}}\right) + 6\left(\frac{2s^3\sqrt{2}}{3}\right) + 8\left(\frac{s^3}{3\sqrt{2}}\right)$$

$$= 6s^3\sqrt{2} + 4s^3\sqrt{2} + \frac{4s^3\sqrt{2}}{3}$$

$$= \frac{34s^3\sqrt{2}}{3}$$

$$\therefore \textbf{Volume}, \ \boldsymbol{V = \frac{34s^3\sqrt{2}}{3} \approx 16.02775371s^3} \qquad \dots\dots\dots\dots(6)$$

2.4. Mean radius (R_m) of truncated rhombic dodecahedron: It is the radius of the sphere having a volume equal to that of a given truncated rhombic dodecahedron with edge lengths s & $s\sqrt{2}$. It is computed as follows

$$\text{volume of sphere with mean radius } \mathrm{R_m} = \text{volume of truncated rhombic dodecahedron}$$

$$\frac{4}{3}\pi(R_m)^3 = \frac{34s^3\sqrt{2}}{3}$$

$$(R_m)^3 = \frac{17s^3}{\pi\sqrt{2}}$$

$$R_m = \left(\frac{17s^3}{\pi\sqrt{2}}\right)^{1/3}$$

$$R_m = s\left(\frac{17}{\pi\sqrt{2}}\right)^{1/3}$$

$$\therefore \text{ Mean radius, } \boldsymbol{R_m = s\left(\frac{17}{\pi\sqrt{2}}\right)^{1/3} \approx 1.564088599\, s} \qquad \ldots\ldots\ldots\ldots(7)$$

3. Dihedral angle between any two faces meeting at a vertex of truncated rhombic dodecahedron

A truncated rhombic dodecahedron has 24 identical vertices at each of which two rectangular, one square & one equilateral triangular faces meet together. We will consider each pair of two faces meeting at a vertex & find the dihedral angle measured internally between these two faces.

3.1. Angle between adjacent rectangular and square faces: Consider two adjacent rectangular and square faces meeting each other at the vertex E which are inclined at a dihedral angle $\boldsymbol{\theta_{RS}}$ (as shown in the fig-6). The line EF shows the small side of rectangular face EFGI (see fig-3 above) & line EK shows the side of square face EKLI (see fig-4 above). Drop the perpendiculars OM & OQ from the centre O to the rectangular & square faces at the centres M & Q which are shown by OM_1 & OQ_1 respectively in the projected view in fig-6.

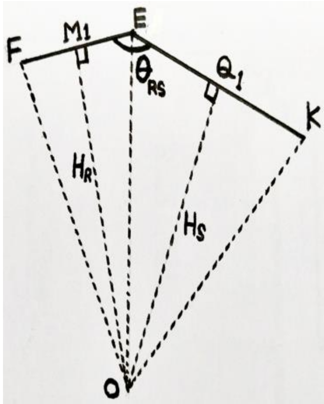

Figure 6: Dihedral angle $\angle FEK = \boldsymbol{\theta_{RS}}$ between a rectangular & a square faces shown by the sides EF & EK ⊥ to the plane of paper.

In right ΔOM_1E (see fig-6),

$$\tan\angle M_1EO = \frac{OM_1}{EM_1} = \frac{H_R}{\left(\frac{EF}{2}\right)} = \frac{\left(\frac{3s}{2}\right)}{\left(\frac{s}{2}\right)} = 3 \qquad (\because\ \mathrm{EF} = s)$$

$$\angle M_1EO = \tan^{-1} 3$$

In right ΔOQ_1E (see fig-6),

$$\tan\angle Q_1EO = \frac{OQ_1}{EQ_1} = \frac{H_S}{\left(\frac{EK}{2}\right)} = \frac{\left(s\sqrt{2}\right)}{\left(\frac{s\sqrt{2}}{2}\right)} = 2 \qquad \left(\because\ \mathrm{EK} = s\sqrt{2}\right)$$

$$\angle Q_1EO = \tan^{-1} 2$$

$$\Rightarrow\ \angle FEK = \angle M_1EO + \angle Q_1EO$$

$$\theta_{RS} = \tan^{-1} 3 + \tan^{-1} 2$$

$$= \pi + \tan^{-1}\left(\frac{3+2}{1-3\cdot 2}\right) \qquad \left(\because \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)\ \forall\, xy > 1\right)$$

$$= \pi + \tan^{-1}(-1)$$

$$= \pi + \left(-\frac{\pi}{4}\right)$$

$$= \frac{3\pi}{4}$$

Hence, the dihedral angle θ_{RS} between any two adjacent rectangular & square faces of a truncated rhombic dodecahedron, is given as follows

$$\boldsymbol{\theta_{RS} = \frac{3\pi}{4} = 135^o} \quad \dots\dots\dots\dots(8)$$

3.2. Angle between adjacent rectangular and triangular faces: Consider two adjacent rectangular and equilateral triangular faces meeting each other at the vertex E which are inclined at a dihedral angle $\boldsymbol{\theta_{RT}}$. The line EI shows the large side of rectangular face EFGI (see fig-3 above) & line EJ_1 shows the altitude from vertex E to the side FJ of equilateral triangular face EFJ (see fig-5 above). Drop the perpendiculars OM & OT from the centre O to these faces which are shown by OM_1 & OT_1 respectively in projected view (as shown in fig-7).

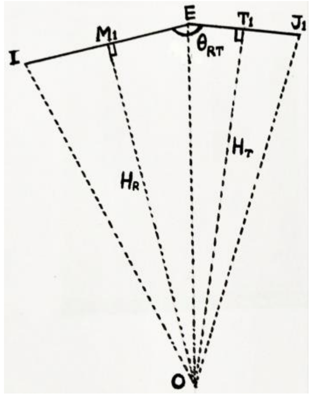

Figure 7: Dihedral angle $\angle IEJ_1 = \theta_{RT}$ between a rectangular & an equilateral triangular faces shown by the side EI & altitude $EJ_1 \perp$ to the plane of paper.

In right ΔOM_1E (see fig-7),

$$\tan\angle M_1EO = \frac{OM_1}{EM_1} = \frac{H_R}{\left(\frac{EI}{2}\right)} = \frac{\left(\frac{3s}{2}\right)}{\left(\frac{s\sqrt{2}}{2}\right)} = \frac{3}{\sqrt{2}} \qquad \left(\because \ \mathrm{EI} = s\sqrt{2}\right)$$

$$\angle M_1EO = \tan^{-1}\frac{3}{\sqrt{2}}$$

In right ΔOT_1E (see fig-7),

$$\tan\angle T_1EO = \frac{OT_1}{ET_1} = \frac{H_T}{\left(\frac{EJ_1}{3}\right)} = \frac{\left(2s\sqrt{\frac{2}{3}}\right)}{\left(\frac{\frac{s\sqrt{3}}{2}}{3}\right)} = 4\sqrt{2} \qquad \left(\because \mathrm{E}J_1 = s\sin 60^o = \frac{s\sqrt{3}}{2}\right)$$

$$\angle T_1EO = \tan^{-1}4\sqrt{2}$$

$$\Rightarrow \angle IEJ_1 = \angle M_1EO + \angle T_1EO$$

$$\theta_{RT} = \tan^{-1}\frac{3}{\sqrt{2}} + \tan^{-1}4\sqrt{2}$$

$$= \pi + \tan^{-1}\left(\frac{\frac{3}{\sqrt{2}} + 4\sqrt{2}}{1 - \frac{3}{\sqrt{2}}\cdot 4\sqrt{2}}\right) \qquad \left(\because \ \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \ \forall \ xy > 1\right)$$

$$= \pi + \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \pi - \tan^{-1}\frac{1}{\sqrt{2}}$$

Hence, the dihedral angle θ_{RT} between any two adjacent rectangular & equilateral triangular faces of a truncated rhombic dodecahedron, is given as follows

$$\boldsymbol{\theta_{RT} = \pi - \tan^{-1}\frac{1}{\sqrt{2}} \approx 144^o 44' 8.2''} \quad \dots\dots\dots\dots(9)$$

3.3. Angle between square and triangular faces with a common vertex but no common edge: Consider a square and an equilateral triangular faces meeting each other at the vertex E which have no common edge and are inclined at a dihedral angle $\boldsymbol{\theta_{ST}}$. The line EL shows the diagonal of square face EKLI (see fig-4 above) & line EF_1 shows the altitude from vertex E to the side FJ of equilateral triangular face EFJ (see fig-5 above). Drop the perpendiculars OQ & OT from the centre O to the centres Q & T of square & triangular faces respectively (as shown in the projected view in fig-8).

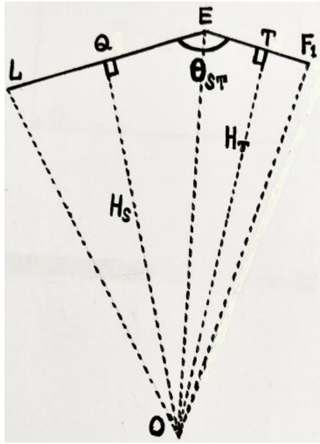

Figure 8: Dihedral angle $\angle LEF_1 = \theta_{ST}$ between a square & an equilateral triangular faces shown by the diagonal EL & altitude $EF_1 \perp$ to the plane of paper.

In right ΔOQE (see fig-8),

$$\tan\angle QEO = \frac{OQ}{QE} = \frac{H_S}{\left(\frac{EL}{2}\right)} = \frac{s\sqrt{2}}{\left(\frac{2s}{2}\right)} = \sqrt{2} \qquad \left(\because\ \mathrm{EL} = \sqrt{2}\left(s\sqrt{2}\right) = 2s\right)$$

$$\angle QEO = \tan^{-1}\sqrt{2}$$

In right ΔOTE (see fig-8),

$$\tan\angle TEO = \frac{OT}{TE} = \frac{H_T}{\left(\frac{2}{3}EF_1\right)} = \frac{\left(2s\sqrt{\frac{2}{3}}\right)}{\left(\frac{2}{3}\frac{s\sqrt{3}}{2}\right)} = 2\sqrt{2} \qquad \left(\because\ \mathrm{E}F_1 = \mathrm{s}\sin 60^o = \frac{s\sqrt{3}}{2}\right)$$

$$\angle TEO = \tan^{-1} 2\sqrt{2}$$

$$\Rightarrow \angle LEF_1 = \angle QEO + \angle TEO$$

$$\theta_{ST} = \tan^{-1}\sqrt{2} + \tan^{-1} 2\sqrt{2}$$

$$= \pi + \tan^{-1}\left(\frac{\sqrt{2} + 2\sqrt{2}}{1 - \sqrt{2}\cdot 2\sqrt{2}}\right) \qquad \left(\because\ \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)\ \forall\, xy > 1\right)$$

$$= \pi + \tan^{-1}\left(-\sqrt{2}\right)$$

$$= \pi - \tan^{-1}\sqrt{2}$$

Hence, the dihedral angle θ_{ST} between square & equilateral triangular faces meeting at the same vertex of truncated rhombic dodecahedron, is given as follows

$$\boldsymbol{\theta_{ST} = \pi - \tan^{-1}\sqrt{2} \approx 125^o 15' 51.8''} \qquad \ldots\ldots\ldots\ldots(10)$$

3.4. Angle between two rectangular faces with a common vertex but no common edge: Consider two congruent rectangular faces meeting each other at the vertex E which are inclined at a dihedral angle $\boldsymbol{\theta_{RR}}$. The line EG shows the diagonal of rectangular face EFGI (see fig-3 above) & line EG' shows the diagonal of another rectangular face EF'G'I'. Drop the perpendiculars OM & OM' from the centre O to the centres M & M' of these rectangular faces with no side common (as shown in the projected in fig-9).

In right ΔOME (see fig-9),

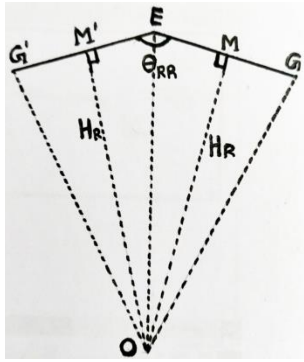

Figure 9: Dihedral angle $\angle G'EG = \theta_{RR}$ between two rectangular faces, meeting at a vertex, shown by diagonals EG & EG'⊥ to the plane of paper.

$$\tan\angle MEO = \frac{OM}{EM} = \frac{H_R}{\left(\frac{EG}{2}\right)} = \frac{\left(\frac{3s}{2}\right)}{\left(\frac{s\sqrt{3}}{2}\right)} = \sqrt{3} \qquad \left(\because \text{EG} = \sqrt{s^2 + \left(s\sqrt{2}\right)^2} = s\sqrt{3}\right)$$

$$\angle MEO = \tan^{-1}\sqrt{3} \qquad \Rightarrow \angle M'EO = \angle MEO = \tan^{-1}\sqrt{3} = \pi/3$$

$$\Rightarrow \angle G'EG = \angle M'EO + \angle MEO$$

$$\theta_{RR} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

Hence, the dihedral angle θ_{RR} between any two rectangular faces meeting at the same vertex of truncated rhombic dodecahedron, is given as follows

$$\boldsymbol{\theta_{RR} = \frac{2\pi}{3} = 120^o} \qquad \dots\dots\dots\dots(11)$$

4. Solid angles ω_R, ω_S & ω_T subtended by rectangular, square and equilateral triangular faces respectively at the centre of truncated rhombic dodecahedron

4.1. Solid angle of rectangular face: The solid angle (ω), subtended by a rectangular plane of length l & width b at any point lying at a distance h on the perpendicular axis passing through the centre, is given by generalized formula [4,5] as follows

$$\boldsymbol{\omega = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2+4h^2)(b^2+4h^2)}}\right)}$$

Substituting the corresponding values in above formula i.e. length $= s\sqrt{2}$, width $b = s$ & normal height $h = H_R = 3s/2$, the solid angle ω_R subtended by the rectangular face EFGI at the centre O of truncated rhombic dodecahedron (as shown in above fig-3), is obtained as follows

$$\omega_R = 4\sin^{-1}\left(\frac{s\sqrt{2}\cdot s}{\sqrt{\left(\left(s\sqrt{2}\right)^2 + 4\left(\frac{3s}{2}\right)^2\right)\left(s^2 + 4\left(\frac{3s}{2}\right)^2\right)}}\right)$$

$$\boldsymbol{\omega_R} = 4\sin^{-1}\left(\frac{s^2\sqrt{2}}{\sqrt{(11s^2)(10s^2)}}\right) = 4\sin^{-1}\left(\frac{s^2\sqrt{2}}{s^2\sqrt{110}}\right) = 4\sin^{-1}\left(\sqrt{\frac{2}{110}}\right) = 4\sin^{-1}\left(\sqrt{\frac{1}{55}}\right)$$

Hence, the solid angle ω_R subtended by any of 12 congruent rectangular faces at the centre of a truncated rhombic dodecahedron, is given as follows

$$\boldsymbol{\omega_R = 4\sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\ \text{sr} \approx 0.541007833\ \text{sr}} \qquad \dots\dots\dots\dots(12)$$

4.2. Solid angle of square face: Similarly, substituting the corresponding values in above formula i.e. length $=$ $s\sqrt{2}$, width $b = s\sqrt{2}$ & normal height $h = H_S = s\sqrt{2}$, the solid angle ω_S subtended by the square face EKLI at the centre O of truncated rhombic dodecahedron (as shown in above fig-4), is obtained as follows

$$\omega_S = 4\sin^{-1}\left(\frac{s\sqrt{2}\cdot s\sqrt{2}}{\sqrt{\left(\left(s\sqrt{2}\right)^2+4\left(s\sqrt{2}\right)^2\right)\left(s^2+4\left(s\sqrt{2}\right)^2\right)}}\right)$$

$$\omega_S = 4\sin^{-1}\left(\frac{2s^2}{\sqrt{(10s^2)(10s^2)}}\right) = 4\sin^{-1}\left(\frac{2s^2}{10s^2}\right) = 4\sin^{-1}(0.2)$$

Hence, the solid angle ω_S subtended by any of 6 congruent square faces at the centre of a truncated rhombic dodecahedron, is given as follows

$$\boldsymbol{\omega_S = 4\sin^{-1}(0.2)\ \text{sr} \approx 0.805431683\ \text{sr}} \qquad \dots\dots\dots\dots(13)$$

4.3. Solid angle of equilateral triangular face: We know that the solid angle (ω) subtended by any regular polygonal plane with n no. of sides each of length a at any point lying at a distance H on the vertical axis passing through the centre, is given by generalized formula [4,5] as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2+a^2\cot^2\frac{\pi}{n}}}\right)$$

Substituting the corresponding values in above formula i.e. number of sides $n = 3$ (for regular Δ), length of each side $a = s$, & normal height $H = H_T = 2s\sqrt{2/3}$, the solid angle ω_T subtended by the equilateral triangular face EFJ at the centre O of truncated rhombic dodecahedron (as shown in above fig-5), is obtained as follows

$$\omega_T = 2\pi - 2\times 3\sin^{-1}\left(\frac{2\left(2s\sqrt{\frac{2}{3}}\right)\sin\frac{\pi}{3}}{\sqrt{4\left(2s\sqrt{\frac{2}{3}}\right)^2+s^2\cot^2\frac{\pi}{3}}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{4s\sqrt{\frac{2}{3}}\times\frac{\sqrt{3}}{2}}{\sqrt{\frac{32s^2}{3}+\frac{s^2}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{2s\sqrt{2}}{\sqrt{\frac{33s^2}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{11}}\right)$$

Hence, the solid angle ω_T subtended by any of 8 congruent equilateral triangular faces at the centre of a truncated rhombic dodecahedron, is given as follows

$$\boldsymbol{\omega_T = 2\pi - 6\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right)\ \text{sr} \approx 0.155210814\ \text{sr}} \qquad \dots\dots\dots\dots(14)$$

4.4. Total solid angle: We know that a truncated rhombic dodecahedron has 12 congruent rectangular faces, 6 congruent square faces & 8 congruent equilateral triangular faces. Therefore, the total solid angle subtended by all the faces at the centre of a truncated rhombic dodecahedron, is given as follows

$$\omega = 12(\omega_R) + 6(\omega_S) + 8(\omega_T)$$

$$\omega = 12\left(4\sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\right) + 6(4\sin^{-1}(0.2)) + 8\left(2\pi - 6\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right)\right)$$

$$\omega = 48\sin^{-1}\left(\frac{1}{\sqrt{55}}\right) + 24\sin^{-1}(0.2) + 16\pi - 48\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right)$$

$$\omega = 16\pi + 24\sin^{-1}(0.2) - 48\left(\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\right)$$

$$\omega = 16\pi + 24\sin^{-1}\left(\frac{1}{5}\right) - 48\left(\sin^{-1}\left(\sqrt{\frac{8}{11}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\right)$$

Using formula: $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \quad \forall\ |x|,|y| \in [0,1]$,

$$\omega = 16\pi + 48\left(\frac{1}{2}\sin^{-1}\left(\frac{1}{5}\right)\right) - 48\left(\sin^{-1}\left(\sqrt{\frac{8}{11}}\sqrt{\frac{54}{55}} - \sqrt{\frac{3}{11}}\frac{1}{\sqrt{55}}\right)\right)$$

Using formula: $\frac{1}{2}\sin^{-1}x = \sin^{-1}\left(\sqrt{\frac{1-\sqrt{1-x^2}}{2}}\right) \quad \forall\ |x| \in [0,1]$,

$$\omega = 16\pi + 48\left(\sin^{-1}\left(\sqrt{\frac{1-\sqrt{1-\left(\frac{1}{5}\right)^2}}{2}}\right)\right) - 48\left(\sin^{-1}\left(\frac{12\sqrt{3}}{11\sqrt{5}} - \frac{\sqrt{3}}{11\sqrt{5}}\right)\right)$$

$$\omega = 16\pi + 48\sin^{-1}\left(\sqrt{\frac{1-\frac{2\sqrt{6}}{5}}{2}}\right) - 48\left(\sin^{-1}\left(\frac{12\sqrt{3}-\sqrt{3}}{11\sqrt{5}}\right)\right)$$

$$\omega = 16\pi + 48\sin^{-1}\left(\sqrt{\frac{5-2\sqrt{6}}{10}}\right) - 48\left(\sin^{-1}\left(\frac{11\sqrt{3}}{11\sqrt{5}}\right)\right)$$

$$\omega = 16\pi + 48\sin^{-1}\left(\sqrt{\frac{\left(\sqrt{3}-\sqrt{2}\right)^2}{10}}\right) - 48\left(\sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right)$$

$$\omega = 16\pi + 48\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}\right) - 48\sin^{-1}\left(\sqrt{\frac{3}{5}}\right)$$

$$\omega = 16\pi - 48\left(\sin^{-1}\left(\sqrt{\frac{3}{5}}\right) - \sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}\right)\right)$$

$$\omega = 16\pi - 48\left(\sin^{-1}\left(\sqrt{\frac{3}{5}}\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{10}}\right) - \sqrt{\frac{2}{5}}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}\right)\right)\right)$$

$$\omega = 16\pi - 48\left(\sin^{-1}\left(\frac{\sqrt{3}(\sqrt{3}+\sqrt{2})}{5\sqrt{2}} - \frac{\sqrt{2}(\sqrt{3}-\sqrt{2})}{5\sqrt{2}}\right)\right)$$

$$\omega = 16\pi - 48\sin^{-1}\left(\frac{3+\sqrt{6}}{5\sqrt{2}} - \frac{\sqrt{6}-2}{5\sqrt{2}}\right)$$

$$\omega = 16\pi - 48\sin^{-1}\left(\frac{3+\sqrt{6}-\sqrt{6}+2}{5\sqrt{2}}\right)$$

$$\omega = 16\pi - 48\sin^{-1}\left(\frac{5}{5\sqrt{2}}\right)$$

$$\omega = 16\pi - 48\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\omega = 16\pi - 48\left(\frac{\pi}{4}\right)$$

$$\omega = 16\pi - 12\pi$$

$$\boldsymbol{\omega = 4\pi}\ sr$$

Above result shows that the solid angle subtended by a truncated rhombic dodecahedron at its centre is $4\pi\ sr$ It is true that the solid angle, subtended by any closed surface at any point inside it, is always $4\pi\ sr$ [6].

A truncated rhombic dodecahedron has total 24 identical vertices at each of which two rectangular, one square & one equilateral triangular faces meet together. Thus, one of 24 identical vertices will be considered to compute solid angle subtended by a truncated rhombic dodecahedron at the same vertex.

5. Solid angle subtended by truncated rhombic dodecahedron at any of its 24 identical vertices

Consider any of 24 identical vertices say vertex P of truncated rhombic dodecahedron. Join the end points A, B, C & D of the edges AP, BP, CP & DP, meeting at vertex P, to get a (plane) trapezium ABCD of sides $AB = s,\ BC = AD = s\sqrt{3}$ & $CD = 2s$ (as shown in fig-10).

Join the foot point Q of perpendicular PQ drawn from vertex P to the plane of trapezium ABCD, to the vertices A, B, C & D . Drop the perpendiculars MN & QE from midpoint M & foot point Q to the sides CD & BC respectively of trapezium ABCD (as shown in the fig-11)

We have found out that dihedral angle between square & equilateral triangular faces is $\angle MPN = \pi - \tan^{-1}\sqrt{2}$. The perpendicular PQ dropped from the vertex P to the plane of trapezium ABCD will fall at the point Q lying on the line MN (as shown in the fig-11).

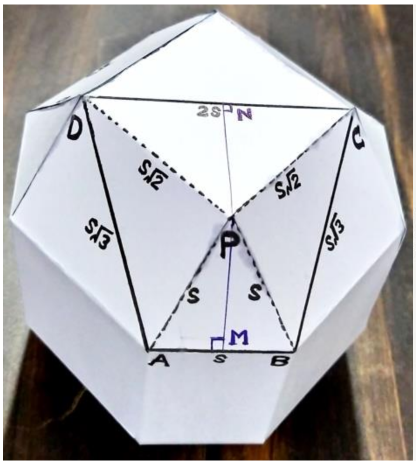

Figure 10: A trapezium ABCD is formed by joining the endpoints A, B, C & D of edges AP, BP, CP & DP meeting at the vertex P of given polyhedron.

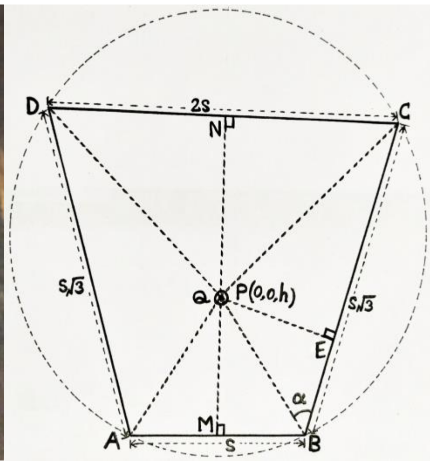

Figure 11: Point Q is the foot of perpendicular PQ drawn from the vertex P to the plane of trapezium ABCD. Point P is lying at a normal height h from the point foot Q (⊥ to the plane of paper).

In ΔMPN (see fig-12 below), using cosine formula as follows

$$\cos\angle MPN = \frac{(PM)^2 + (PN)^2 - (MN)^2}{2(PM)(PN)}$$

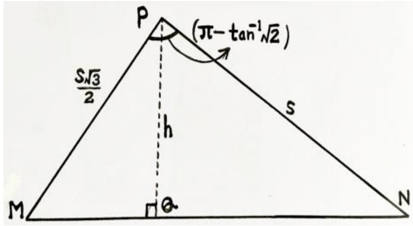

Figure 12: Perpendicular PQ dropped from vertex P to the plane of trapezium ABCD falls at the point Q on the line MN.

$$\cos(\pi - \tan^{-1}\sqrt{2}) = \frac{\left(\frac{s\sqrt{3}}{2}\right)^2 + (s)^2 - (MN)^2}{2\left(\frac{s\sqrt{3}}{2}\right)(s)}$$

$$\Rightarrow -\cos(\tan^{-1}\sqrt{2}) = \frac{\frac{3s^2}{4} + s^2 - MN^2}{s^2\sqrt{3}}$$

$$-\cos\left(\cos^{-1}\frac{1}{\sqrt{3}}\right) = \frac{\frac{7s^2}{4} - MN^2}{s^2\sqrt{3}} \quad \Rightarrow -\frac{1}{\sqrt{3}} = \frac{\frac{7s^2}{4} - MN^2}{s^2\sqrt{3}} \quad \Rightarrow -s^2 = \frac{7s^2}{4} - MN^2$$

$$\Rightarrow \ MN^2 = \frac{7s^2}{4} + s^2 \ \ \Rightarrow \ MN^2 = \frac{11s^2}{4} \quad \Rightarrow \ \boldsymbol{MN} = \frac{\boldsymbol{s\sqrt{11}}}{\boldsymbol{2}}$$

Now, the area of ΔMPN (see fig-12), is given as follows

$$\frac{1}{2}(MN)(PQ) = \frac{1}{2}(PM)(PN)\sin(\pi - \tan^{-1}\sqrt{2})$$

$$\left(\frac{s\sqrt{11}}{2}\right)(PQ) = \left(\frac{s\sqrt{3}}{2}\right)(s)\sin(\tan^{-1}\sqrt{2})$$

$$\sqrt{11}PQ = s\sqrt{3}\sin\left(\sin^{-1}\sqrt{\frac{2}{3}}\right) \quad \Rightarrow \quad \boldsymbol{PQ} = \frac{s\sqrt{3}}{\sqrt{11}}\sqrt{\frac{2}{3}} = \boldsymbol{s}\sqrt{\frac{\boldsymbol{2}}{\boldsymbol{11}}}$$

Using Pythagorean theorem in right ΔPQM (see above fig-12 above) , we get

$$\boldsymbol{MQ} = \sqrt{(PM)^2 - (PQ)^2} = \sqrt{\left(\frac{s\sqrt{3}}{2}\right)^2 - \left(s\sqrt{\frac{2}{11}}\right)^2} = \sqrt{\frac{3s^2}{4} - \frac{2s^2}{11}} = \sqrt{\frac{25s^2}{44}} = \frac{\boldsymbol{5s}}{\boldsymbol{2\sqrt{11}}}$$

$$\Rightarrow \quad \boldsymbol{QN} = MN - MQ = \frac{s\sqrt{11}}{2} - \frac{5s}{2\sqrt{11}} = \frac{\boldsymbol{3s}}{\boldsymbol{\sqrt{11}}}$$

Using Pythagorean theorem in right ΔQMB (see above fig-11), we get

$$\boldsymbol{BQ} = \sqrt{(MQ)^2 + (MB)^2} = \sqrt{\left(\frac{5s}{2\sqrt{11}}\right)^2 + \left(\frac{s}{2}\right)^2} = \sqrt{\frac{25s^2}{44} + \frac{s^2}{4}} = \sqrt{\frac{36s^2}{44}} = \frac{\boldsymbol{3s}}{\boldsymbol{\sqrt{11}}}$$

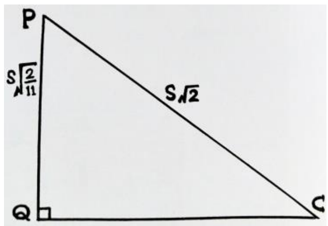

Figure 13: Right ΔPQC is obtained by dropping ⊥ to the plane of trapezium ABCD.

Using Pythagorean theorem in right ΔPQC (above fig-13), we get

$$\boldsymbol{QC} = \sqrt{(PC)^2 - (PQ)^2} = \sqrt{(s\sqrt{2})^2 - \left(s\sqrt{\frac{2}{11}}\right)^2} = \sqrt{2s^2 - \frac{2s^2}{11}} = \sqrt{\frac{20s^2}{11}} = \boldsymbol{2s}\sqrt{\frac{\boldsymbol{5}}{\boldsymbol{11}}}$$

In ΔQBC (see fig-11 above), using cosine formula as follows

$$\cos\alpha=\frac{(BQ)^2+(BC)^2-(QC)^2}{2(BQ)(BC)}=\frac{\left(\frac{3s}{\sqrt{11}}\right)^2+\left(s\sqrt{3}\right)^2-\left(2s\sqrt{\frac{5}{11}}\right)^2}{2\left(\frac{3s}{\sqrt{11}}\right)\left(s\sqrt{3}\right)}=\frac{\frac{9s^2}{11}+3s^2-\frac{20s^2}{11}}{\frac{6s^2\sqrt{3}}{\sqrt{11}}}$$

$$\boldsymbol{\cos\alpha}=\frac{\frac{9s^2+33s^2-20s^2}{11}}{\frac{6s^2\sqrt{3}}{\sqrt{11}}}=\frac{\frac{22s^2}{11}}{\frac{6s^2\sqrt{3}}{\sqrt{11}}}=\frac{\sqrt{11}}{3\sqrt{3}}=\boldsymbol{\frac{1}{3}\sqrt{\frac{11}{3}}}$$

In right ΔBEQ (see above fig-11),

$$\cos\alpha=\frac{BE}{BQ}\quad\Rightarrow\ \boldsymbol{BE}=BQ\ \cos\alpha=\frac{3s}{\sqrt{11}}\cdot\frac{1}{3}\sqrt{\frac{11}{3}}=\boldsymbol{\frac{s}{\sqrt{3}}}$$

$$\Rightarrow\ \boldsymbol{EC}=BC-BE=s\sqrt{3}-\frac{s}{\sqrt{3}}=\boldsymbol{\frac{2s}{\sqrt{3}}}$$

Using Pythagorean theorem in right ΔBEQ (see above fig-11) , we get

$$\boldsymbol{QE}=\sqrt{(BQ)^2-(BE)^2}=\sqrt{\left(\frac{3s}{\sqrt{11}}\right)^2-\left(\frac{s}{\sqrt{3}}\right)^2}=\sqrt{\frac{9s^2}{11}-\frac{s^2}{3}}=\sqrt{\frac{16s^2}{33}}=\boldsymbol{\frac{4s}{\sqrt{33}}}$$

We know from HCR's Theory of Polygon that the solid angle (ω), subtended by a right triangle OGH having perpendicular p & base b at any point P at a normal distance h on the vertical axis passing through the vertex O (as shown in the fig-14), is given by Standard Formula-1 [4,5] as follows

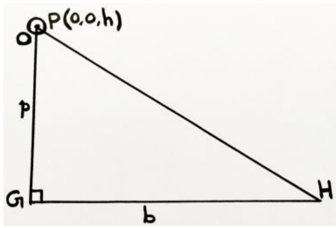

Figure-14: Point P lies at normal height h from vertex O of right ΔOGH (⊥ to plane of paper).

$$\omega=\sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right)-\sin^{-1}\left(\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right)$$

Now, the solid angle $\omega_{\Delta QMB}$ subtended by right ΔQMB at the vertex P (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard-1 formula i.e. base $b=MB=\frac{s}{2}$, perpendicular $p=MQ=\frac{5s}{2\sqrt{11}}$ & normal height $h=PQ=s\sqrt{\frac{2}{11}}$ as follows

$$\omega_{\Delta QMB}=\sin^{-1}\left(\frac{\frac{s}{2}}{\sqrt{\left(\frac{s}{2}\right)^2+\left(\frac{5s}{2\sqrt{11}}\right)^2}}\right)-\sin^{-1}\left(\left(\frac{\frac{s}{2}}{\sqrt{\left(\frac{s}{2}\right)^2+\left(\frac{5s}{2\sqrt{11}}\right)^2}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\sqrt{\left(s\sqrt{\frac{2}{11}}\right)^2+\left(\frac{5s}{2\sqrt{11}}\right)^2}}\right)\right)$$

$$\omega_{\Delta QMB}=\sin^{-1}\left(\frac{\frac{s}{2}}{\frac{3s}{\sqrt{11}}}\right)-\sin^{-1}\left(\left(\frac{\frac{s}{2}}{\frac{3s}{\sqrt{11}}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\frac{s\sqrt{3}}{2}}\right)\right)=\sin^{-1}\left(\frac{\sqrt{11}}{6}\right)-\sin^{-1}\left(\left(\frac{\sqrt{11}}{6}\right)\left(2\sqrt{\frac{2}{33}}\right)\right)$$

$$\omega_{\Delta QMB} = \sin^{-1}\left(\frac{\sqrt{11}}{6}\right) - \sin^{-1}\left(\frac{1}{3}\sqrt{\frac{2}{3}}\right) \quad \dots\dots\dots\dots(15)$$

Similarly, the solid angle $\omega_{\Delta BEQ}$ subtended by right ΔBEQ at the vertex P (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base $b = BE = \frac{s}{\sqrt{3}}$, perpendicular $p = QE = \frac{4s}{\sqrt{33}}$ & normal height $h = PQ = s\sqrt{\frac{2}{11}}$ as follows

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{\frac{s}{\sqrt{3}}}{\sqrt{\left(\frac{s}{\sqrt{3}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right) - \sin^{-1}\left(\left(\frac{\frac{s}{\sqrt{3}}}{\sqrt{\left(\frac{s}{\sqrt{3}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\sqrt{\left(s\sqrt{\frac{2}{11}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right)\right)$$

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{\frac{s}{\sqrt{3}}}{\frac{3s}{\sqrt{11}}}\right) - \sin^{-1}\left(\left(\frac{\frac{s}{\sqrt{3}}}{\frac{3s}{\sqrt{11}}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{s\sqrt{\frac{2}{3}}}\right)\right) = \sin^{-1}\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right)\left(\sqrt{\frac{3}{11}}\right)\right)$$

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{3}\right) \quad \dots\dots\dots\dots(16)$$

Similarly, the solid angle $\omega_{\Delta CEQ}$ subtended by right ΔCEQ at the vertex P (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base $b = EC = \frac{2s}{\sqrt{3}}$, perpendicular $p = QE = \frac{4s}{\sqrt{33}}$ & normal height $h = PQ = s\sqrt{\frac{2}{11}}$ as follows

$$\omega_{\Delta CEQ} = \sin^{-1}\left(\frac{\frac{2s}{\sqrt{3}}}{\sqrt{\left(\frac{2s}{\sqrt{3}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right) - \sin^{-1}\left(\left(\frac{\frac{2s}{\sqrt{3}}}{\sqrt{\left(\frac{2s}{\sqrt{3}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\sqrt{\left(s\sqrt{\frac{2}{11}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right)\right)$$

$$\omega_{\Delta CEQ} = \sin^{-1}\left(\frac{\frac{2s}{\sqrt{3}}}{2s\sqrt{\frac{5}{11}}}\right) - \sin^{-1}\left(\left(\frac{\frac{2s}{\sqrt{3}}}{2s\sqrt{\frac{5}{11}}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{s\sqrt{\frac{2}{3}}}\right)\right) = \sin^{-1}\left(\sqrt{\frac{11}{15}}\right) - \sin^{-1}\left(\left(\sqrt{\frac{11}{15}}\right)\left(\sqrt{\frac{3}{11}}\right)\right)$$

$$\omega_{\Delta CEQ} = \sin^{-1}\left(\sqrt{\frac{11}{15}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) \quad \dots\dots\dots\dots(17)$$

Similarly, the solid angle $\omega_{\Delta QNC}$ subtended by right ΔQNC at the vertex P (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard-1 formula i.e. base $b = NC = s$, perpendicular $p = QN = \frac{3s}{\sqrt{11}}$ & normal height $h = PQ = s\sqrt{\frac{2}{11}}$ as follows

$$\omega_{\Delta QNC}=\sin^{-1}\left(\frac{s}{\sqrt{(s)^2+\left(\frac{3s}{\sqrt{11}}\right)^2}}\right)-\sin^{-1}\left(\left(\frac{s}{\sqrt{(s)^2+\left(\frac{3s}{\sqrt{11}}\right)^2}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\sqrt{\left(s\sqrt{\frac{2}{11}}\right)^2+\left(\frac{3s}{\sqrt{11}}\right)^2}}\right)\right)$$

$$\omega_{\Delta QNC}=\sin^{-1}\left(\frac{s}{2s\sqrt{\frac{5}{11}}}\right)-\sin^{-1}\left(\left(\frac{s}{2s\sqrt{\frac{5}{11}}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{s}\right)\right)=\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)-\sin^{-1}\left(\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)\left(\sqrt{\frac{2}{11}}\right)\right)$$

$$\omega_{\Delta QNC}=\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\qquad \dots\dots\dots\dots(18)$$

Now, according to HCR's Theory of Polygon [4,5], the solid angle ω_{MBCN} subtended by the trapezium MBCN at the vertex P (see above fig-11) is the algebraic sum of solid angles subtended by the elementary right triangles $\Delta QMB, \Delta BEQ, \Delta CEQ$ & ΔQNC which are given from (15), (16), (17) and (18) as follows

$$\omega_{MBCN}=\omega_{\Delta QMB}+\omega_{\Delta BEQ}+\omega_{\Delta CEQ}+\omega_{\Delta QNC}$$

Substituting the corresponding values of solid angles (derived above) as follows

$$\omega_{MBCN}=\left(\sin^{-1}\left(\frac{\sqrt{11}}{6}\right)-\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{2}{3}}\right)\right)+\left(\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right)-\sin^{-1}\left(\frac{1}{3}\right)\right)$$
$$+\left(\sin^{-1}\left(\sqrt{\frac{11}{15}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)+\left(\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right)$$

$$=\left(\sin^{-1}\left(\frac{\sqrt{11}}{6}\right)+\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right)\right)-\left(\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{2}{3}}\right)+\sin^{-1}\left(\frac{1}{3}\right)\right)+\left(\sin^{-1}\left(\sqrt{\frac{11}{15}}\right)+\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)\right)$$
$$-\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right)$$

Using formula: $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\quad\forall\ |x|,|y|\in[0,1]$,

$$=\left(\sin^{-1}\left(\frac{\sqrt{11}}{6}\cdot\frac{4}{3\sqrt{3}}+\frac{5}{6}\cdot\frac{1}{3}\sqrt{\frac{11}{3}}\right)\right)-\left(\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{2}{3}}\cdot\frac{2\sqrt{2}}{3}+\frac{5}{3\sqrt{3}}\cdot\frac{1}{3}\right)\right)$$
$$+\left(\pi-\sin^{-1}\left(\sqrt{\frac{11}{15}}\cdot\frac{3}{2\sqrt{5}}+\frac{2}{\sqrt{15}}\cdot\frac{1}{2}\sqrt{\frac{11}{5}}\right)\right)-\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\cdot\frac{3}{\sqrt{10}}+\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{10}}\right)\right)$$

$$=\sin^{-1}\left(\frac{4\sqrt{11}}{18\sqrt{3}}+\frac{5\sqrt{11}}{18\sqrt{3}}\right)-\sin^{-1}\left(\frac{4}{9\sqrt{3}}+\frac{5}{9\sqrt{3}}\right)+\left(\pi-\sin^{-1}\left(\frac{3\sqrt{11}}{10\sqrt{3}}+\frac{2\sqrt{11}}{10\sqrt{3}}\right)\right)-\sin^{-1}\left(\frac{3}{5\sqrt{2}}+\frac{2}{5\sqrt{2}}\right)$$

$$=\sin^{-1}\left(\frac{9\sqrt{11}}{18\sqrt{3}}\right)-\sin^{-1}\left(\frac{9}{9\sqrt{3}}\right)+\left(\pi-\sin^{-1}\left(\frac{5\sqrt{11}}{10\sqrt{3}}\right)\right)-\sin^{-1}\left(\frac{5}{5\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) + \left(\pi - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{3}}\right)\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) + \pi - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{3}}\right) - \frac{\pi}{4}$$

$$= \frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \Rightarrow \boldsymbol{\omega_{MBCN} = \frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)}$$

Thus, using symmetry in trapezium ABCD (see above fig-11), the solid angle ω_{ABCD} subtended by the trapezium ABCD at the vertex P of truncated rhombic dodecahedron will be twice the solid angle ω_{MBCN} subtended by the trapezium MBCN at the vertex P, as follows

$$\omega_{ABCD} = 2\omega_{MBCN} = 2\left(\frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{3\pi}{2} - 2\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{3\pi}{2} - \sin^{-1}\left(2\cdot\frac{1}{\sqrt{3}}\cdot\sqrt{\frac{2}{3}}\right)$$

$$= \frac{3\pi}{2} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \pi + \left(\frac{\pi}{2} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right) = \pi + \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \pi + \sin^{-1}\left(\frac{1}{3}\right)$$

It's worth noticing that the solid angle ω_V subtended by truncated rhombic dodecahedron at its vertex P will be equal to the solid angle ω_{ABCD} subtended by the trapezium ABCD at the vertex P.

Hence, the solid angles ω_V subtended by a truncated rhombic dodecahedron at any of its 24 identical vertices (at each of which two rectangular, one square & one regular triangular faces meet), is given as follows

$$\boldsymbol{\omega_V = \pi + \sin^{-1}\left(\frac{1}{3}\right)\ \mathrm{sr} \approx 3.481429563\ \mathrm{sr}} \qquad \ldots\ldots\ldots\ldots(19)$$

6. Paper model of a truncated rhombic dodecahedron

In order to make the paper model of a truncated rhombic dodecahedron having 12 congruent rectangular faces, 6 congruent square faces & 8 congruent equilateral triangular faces, it first requires the net of 26 faces to be drawn on a paper sheet

Step 1: Prepare a net of 26 faces out of which there are 12 congruent rectangular faces each with length $s\sqrt{2}$ & width s, 6 congruent square faces each with side $s\sqrt{2}$ & 8 congruent equilateral triangular faces each with side s on the plain sheet of paper (as shown in fig-15(a)).

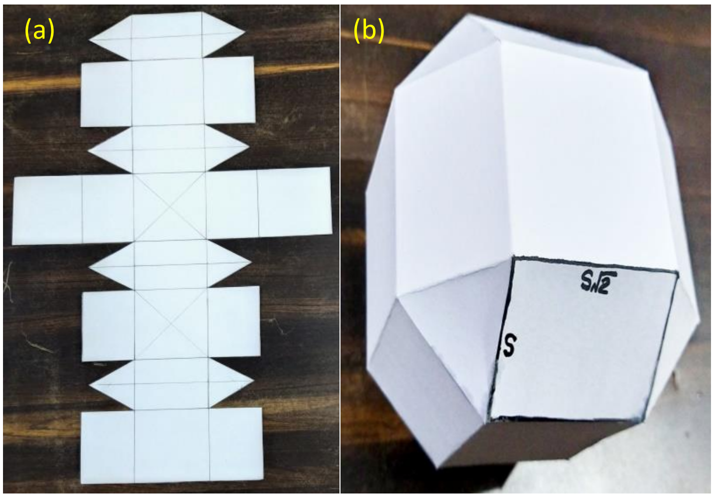

Figure 15: A net of 12 congruent rectangular, 6 congruent square & 8 congruent equilateral triangular faces (a), is folded to conform to the shape of a truncated rhombic dodecahedron (b).

Step 2: Fold each of 26 faces about its common (junction) edge such that the open edges of faces overlap one another & thus the net conforms to a closed surface. Glue the faces at the coincident edges to retain the shape of a truncated rhombic dodecahedron (as shown in fig-15(b)).

Summary: Let there be a truncated rhombic dodecahedron having 12 congruent rectangular faces each with length $s\sqrt{2}$ & width s, 6 congruent square faces each with side $s\sqrt{2}$ and 8 congruent equilateral triangular faces each with side s, 48 edges and 24 identical vertices then all its important parameters are determined as tabulated below.

Radius(R) of circumscribed sphere passing through all 24 vertices	$s\sqrt{3} \approx 1.732050808\ s$
Normal distances H_R, H_S & H_T of rectangular, square & regular triangular faces from the centre of truncated rhombic dodecahedron	$H_R = \frac{3s}{2}$, $H_S = s\sqrt{2} \approx 1.414213562$ s & $H_T = 2s\sqrt{\frac{2}{3}} \approx 1.632993162$ s
Surface area (A_s)	$A_s = 2s^2(6\sqrt{2} + 6 + \sqrt{3}) \approx 32.43466436\ s^2$
Volume (V)	$V = \frac{34s^3\sqrt{2}}{3} \approx 16.02775371s^3$
Mean radius (R_m) or radius of sphere having volume equal to that of the truncated rhombic dodecahedron	$R_m = s\left(\frac{17}{\pi\sqrt{2}}\right)^{1/3} \approx 1.564088599\ s$
Dihedral angle θ_{RS} between any two adjacent rectangular & square faces	$\theta_{RS} = \frac{3\pi}{4} = 135^o$
Dihedral angle θ_{RT} between any two adjacent rectangular & square faces	$\theta_{RT} = \pi - \tan^{-1}\frac{1}{\sqrt{2}} \approx 144^o 44' 8.2''$
Dihedral angle θ_{ST} between square & equilateral triangular faces with a common vertex but no common edge	$\theta_{ST} = \pi - \tan^{-1}\sqrt{2} \approx 125^o 15' 51.8''$
Dihedral angle θ_{RR} between any two rectangular faces with a common vertex but no common edge	$\theta_{RR} = \frac{2\pi}{3} = 120^o$
Solid angles ω_R, ω_S & ω_T subtended by rectangular, square & equilateral triangular faces at the centre of truncated rhombic dodecahedron	$\omega_R = 4\sin^{-1}\left(\frac{1}{\sqrt{55}}\right)$ sr ≈ 0.541 sr , $\omega_S = 4\sin^{-1}(0.2)$ sr ≈ 0.805 sr $\omega_T = 2\pi - 6\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right)$ sr ≈ 0.155210814 sr
Solid angle ω_V subtended by truncated rhombic dodecahedron at its vertex	$\omega_V = \pi + \sin^{-1}\left(\frac{1}{3}\right)$ sr ≈ 3.481429563 sr

Note: Above articles had been derived & illustrated by Mr H.C. Rajpoot (M Tech, Production Engineering)

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