Mathematical Analysis and Modeling of Pyramidal Containers, Right Pyramids, and Related Polyhedra

Harish Chandra Rajpoot   10 Sept 2019

Department of Mechanical Engineering, Indian Institute of Technology Delhi, New Delhi 110016, India

1. Introduction: In this paper, we aim to derive the generalized formula to compute all the important parameters like V-cut angle using HCR's Theorem, edge length of open end, dihedral angle using HCR's Corollary, surface area, and volume of a pyramidal flat container with a regular polygonal base, right pyramids, and polyhedral [1]. We will apply these generalized formulas to compute important parameters & make paper models of the pyramidal flat containers with square, regular pentagonal, hexagonal, heptagonal & octagonal bases, right pyramids [2,3,4] and non-uniform polyhedra [5,6,7]. The concept of making pyramidal flat containers, right pyramids, and polyhedra (bi-pyramids) is mainly based on the rotation or folding of two co-planar planes about their intersecting straight edges (as discussed in detail and formulated in HCR's Theorem).

In brief, the procedure of making a pyramidal flat container, right pyramid or polyhedron using a sheet (of desired thickness) of paper, polymer, metal or alloy which can be easily cut, bent & butt-joined at the mating lateral edges, is based on the following steps

1. Making a drawing on a thin sheet of paper (we can also use a sheet of polymer, metal, or alloy)
2. Cutting and removing undesired parts from the sheet drawing to get a blank
3. Bending lateral faces (trapezoidal in case of flat container and triangular in case of a pyramid) about the edges of a regular polygonal base
4. Joining the mating edges of lateral faces (gluing in case of paper, welding in case of polymer, metal or alloy)

In the above procedure, the most important step is to make a drawing accurately & precisely on the sheet of desired material & thickness. Preferably sheet of paper is most suitable for making an accurate drawing. We will discuss the procedure & the drawing in details in later stages.

2. Pyramidal flat container with regular n-gonal base: Let's make a right pyramidal flat container of slant height l (i.e. distance between parallel sides of a lateral trapezoidal face), with a regular polygonal base with n no. of sides each of length a such that each lateral trapezoidal face is inclined at an angle θ with the plane of base.

Let's draw a circle of radius r circumscribing the regular n-gonal base $A_1A_2A_3 \ldots \ldots A_{n-1}A_n$ of each side a. Draw a bigger concentric circle of radius R & extend the straight lines joining the vertices of inner regular n-gon to the centre O so that these lines intersect outer big circle at points $B_1, B_2, B_3 \ldots \ldots B_{n-1}$ & B_n. After joining these points (vertices) we get another big regular n-gon $B_1B_2B_3 \ldots \ldots B_{n-1}B_n$ (as shown in the fig-1).
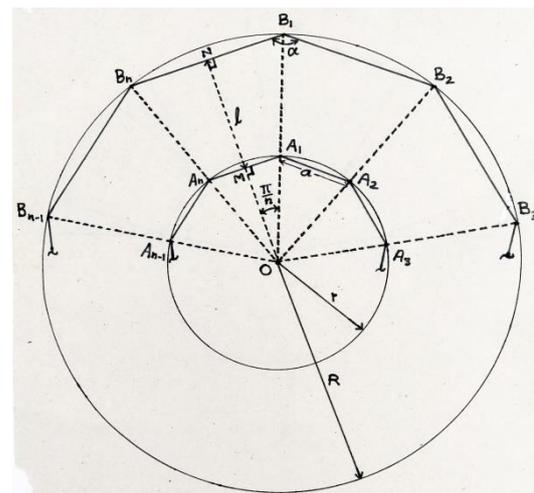
Now, interior angle (α) of regular n-gon,

Figure-1: Inner and Outer circles of radii r & R circumscribe the inner regular n-gonal base & outer regular n-gon. $MN = $ slant height $= l$.

$$\alpha = \frac{(n-2)\pi}{n} \quad \& \quad \angle A_1OM = \angle B_1ON = \frac{\angle A_1OA_n}{2} = \frac{\left(\frac{2\pi}{n}\right)}{2} = \frac{\pi}{n}$$

Drop a perpendicular from con-centre O to the parallel sides A_nA_1 & B_nB_1.

In right ΔOMA_1 (see fig-1 above)

$$\sin \angle A_1OM = \frac{MA_1}{OA_1} \Rightarrow \sin\frac{\pi}{n} = \frac{a/2}{r} \Rightarrow \boldsymbol{r = \frac{a}{2} cosec\frac{\pi}{n}} \quad \&$$

$$\tan\frac{\pi}{n} = \frac{\frac{a}{2}}{OM} \Rightarrow OM = \frac{a}{2}cot\frac{\pi}{n}$$

$$\therefore \boldsymbol{ON = OM + MN} = \frac{a}{2}cot\frac{\pi}{n} + l \qquad (\text{since}, \quad MN = \text{slant height} = l\ )$$

Similarly, in right ΔONB_1 (see fig-1 above)

$$\cos \angle B_1ON = \frac{ON}{OB_1} \qquad \Rightarrow \quad \cos\frac{\pi}{n} = \frac{\left(\frac{a}{2}cot\frac{\pi}{n} + l\right)}{R}$$

$$\boldsymbol{R = \frac{a}{2} cosec\frac{\pi}{n} + l\, sec\frac{\pi}{n}} \quad (\text{value of R gives the size of circular sheet to make desired container })$$

Since each trapezoidal lateral face of pyramidal flat container is inclined at angle θ with the plane of regular n-gonal base hence we need to cut V-part (included angle δ) symmetrically from each vertex of inner polygon such that δ is bisected by radial dotted lines (as shown by shaded area in the figure-2).

3. Derivation of V-cut angle (δ): V-cut angle δ, required to cut remove V-parts and rotate two co-planar planes (i.e. trapezoidal faces) $A_1B_1B_nA_n$ & $A_1B_1B_2A_2$ (initially intersecting at common edge A_1B_1) through an angle θ about their straight edges A_nA_1 & A_1A_2 respectively intersecting each other at an angle α so that their new edges A_1P & A_1Q respectively coincide at $A_1A'_1$, is given by HCR's Theorem [1] as follows

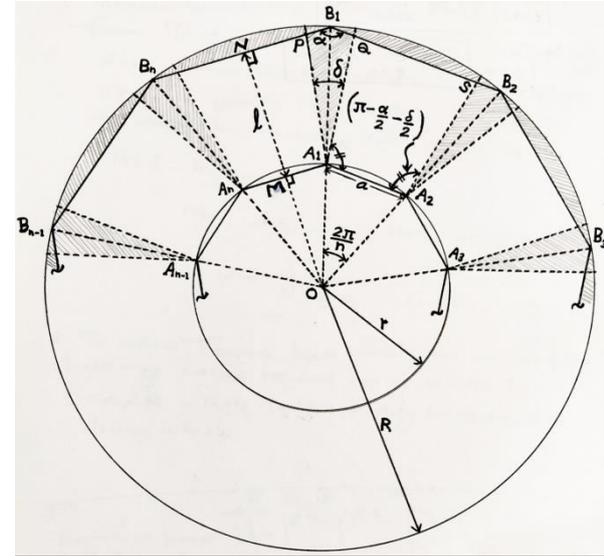Figure-2: δ is V-cut angle of V-parts which is bisected by radial dotted lines joining the vertices of regular n-gons to the centre O. The shaded area is to be cut removed from big circle of radius R. QS is the edge length of regular n-gonal open end of pyramidal flat container.

$$\boldsymbol{\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{\alpha}{2}\right) - \alpha}$$

Setting the value, $\alpha = \frac{(n-2)\pi}{n}$, we get

$$\delta = 2\tan^{-1}\left(\sec\theta\tan\frac{(n-2)\pi}{2n}\right) - \frac{(n-2)\pi}{n}$$

$$= 2\tan^{-1}\left(\sec\theta\tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right)\right) - \pi + \frac{2\pi}{n}$$

$$= 2\tan^{-1}\left(\sec\theta\cot\frac{\pi}{n}\right) - \pi + \frac{2\pi}{n}$$

$$= 2\cot^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right) - \pi + \frac{2\pi}{n} \qquad \left(\text{since}, \quad \tan^{-1}x = \cot^{-1}\frac{1}{x}\right)$$

$$= 2\left(\frac{\pi}{2} - \tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)\right) - \pi + \frac{2\pi}{n} \qquad \left(\text{since}, \quad \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x\right)$$

$$= \pi - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right) - \pi + \frac{2\pi}{n}$$

$$= \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)$$

Hence, **V-cut angle δ,** required to cut remove V-parts & rotate each two co-planar planes about their intersecting straight edges i.e. edges of regular n-gonal base of pyramidal flat container, is given as

$$\boldsymbol{\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)} \qquad\qquad \dots\dots (1)$$

4. Derivation of edge length (a_1) of regular n-gonal open end of pyramidal flat container: QS is the edge length of regular n-gonal open end (see above fig-2).

From fig-2 above,

$$A_1B_1 = OB_1 - OA_1 = R - r$$

$$A_1B_1 = \frac{a}{2}cosec\frac{\pi}{n} + l\,sec\frac{\pi}{n} - \frac{a}{2}cosec\frac{\pi}{n}$$

$$A_1B_1 = l\,sec\frac{\pi}{n}$$

In $\Delta A_1 P B_1$ (See fig-2 above),

$$\boldsymbol{\angle PA_1B_1} = \frac{\angle PA_1Q}{2} = \frac{\delta}{2} \quad \& \quad \angle PB_1A_1 = \frac{\angle PB_1Q}{2} = \frac{\alpha}{2} \quad \forall \quad \alpha = \frac{(n-2)\pi}{n}$$

$$\therefore \ \boldsymbol{\angle A_1PB_1} = \pi - \angle PB_1A_1 - \angle PA_1B_1$$

$$= \pi - \frac{\alpha}{2} - \frac{\delta}{2}$$

Now, using Sine rule in $\Delta A_1 P B_1$ (see fig-2 above),

$$\frac{PB_1}{\sin\angle PA_1B_1} = \frac{A_1B_1}{\sin\angle A_1PB_1}$$

$$\frac{PB_1}{\sin\frac{\delta}{2}} = \frac{l\,sec\frac{\pi}{n}}{\sin\left(\pi - \frac{\alpha}{2} - \frac{\delta}{2}\right)}$$

$$PB_1 = \frac{l\,sec\frac{\pi}{n}\sin\frac{\delta}{2}}{\sin\left(\frac{\alpha+\delta}{2}\right)}$$

$$= \frac{l\,sec\frac{\pi}{n}\sin\left(\frac{\alpha+\delta}{2} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha+\delta}{2}\right)}$$

$$= \frac{l\,sec\frac{\pi}{n}\left(\sin\left(\frac{\alpha+\delta}{2}\right)\cos\frac{\alpha}{2} - \cos\left(\frac{\alpha+\delta}{2}\right)\sin\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha+\delta}{2}\right)}$$

$$= l \sec \frac{\pi}{n} \left(\cos \frac{\alpha}{2} - \frac{\cos \left(\frac{\alpha + \delta}{2} \right) \sin \frac{\alpha}{2}}{\sin \left(\frac{\alpha + \delta}{2} \right)} \right)$$

$$PB_1 = l \sec \frac{\pi}{n} \left(\cos \frac{\alpha}{2} - \frac{\sin \frac{\alpha}{2}}{\tan \left(\frac{\alpha + \delta}{2} \right)} \right) \qquad \ldots \ldots \ldots \ldots \ldots (2)$$

Now, from HCR's Theorem [1], we know that V-cut angle is given as

$$\delta = 2 \tan^{-1} \left(\sec \theta \tan \frac{\alpha}{2} \right) - \alpha$$

$$\alpha + \delta = 2 \tan^{-1} \left(\sec \theta \tan \frac{\alpha}{2} \right)$$

$$\frac{\alpha + \delta}{2} = \tan^{-1} \left(\sec \theta \tan \frac{\alpha}{2} \right)$$

$$\tan \left(\frac{\alpha + \delta}{2} \right) = \sec \theta \tan \frac{\alpha}{2}$$

Setting above value of $\tan \left(\frac{\alpha+\delta}{2} \right)$ in above Eq.(2), we get

$$PB_1 = l \sec \frac{\pi}{n} \left(\cos \frac{\alpha}{2} - \frac{\sin \frac{\alpha}{2}}{\sec \theta \tan \frac{\alpha}{2}} \right)$$

$$= l \sec \frac{\pi}{n} \left(\cos \frac{\alpha}{2} - \frac{\sin \frac{\alpha}{2} \cos \theta \cos \frac{\alpha}{2}}{\sin \frac{\alpha}{2}} \right)$$

$$= l \sec \frac{\pi}{n} \left(\cos \frac{\alpha}{2} - \cos \theta \cos \frac{\alpha}{2} \right)$$

$$= l \sec \frac{\pi}{n} \cos \frac{\alpha}{2} (1 - \cos \theta)$$

$$= l \sec \frac{\pi}{n} \cos \frac{(n-1)\pi}{2n} (1 - \cos \theta)$$

$$= \frac{l}{\cos \frac{\pi}{n}} \cos \left(\frac{\pi}{2} - \frac{\pi}{n} \right) (1 - \cos \theta)$$

$$= \frac{l}{\cos \frac{\pi}{n}} \sin \frac{\pi}{n} (1 - \cos \theta)$$

$$= l \tan \frac{\pi}{n} (1 - \cos \theta)$$

From the symmetry in fig-2 above, we have

$$SB_2 = B_1 Q = PB_1 = l \tan \frac{\pi}{n} (1 - \cos \theta)$$

In right ΔOEB_1 (see figure-3 below)

$$\sin \angle CA_1B_1 = \frac{B_1E}{OB_1} \quad \Rightarrow \quad \sin\frac{\pi}{n} = \frac{B_1E}{R} \quad \Rightarrow B_1E = R\sin\frac{\pi}{n}$$

$$B_1E = \left(\frac{a}{2}\,cosec\,\frac{\pi}{n} + l\,sec\,\frac{\pi}{n}\right)\sin\frac{\pi}{n} \qquad \text{(setting value of } R)$$

$$B_1E = \frac{a}{2} + l\tan\frac{\pi}{n}$$

$$\therefore \; B_1B_2 = 2B_1E = 2\left(\frac{a}{2} + l\tan\frac{\pi}{n}\right) = a + 2l\tan\frac{\pi}{n}$$

Figure-3: This is a part of figure-2 above. A_1QSA_2 is trapezoidal face of pyramidal flat container after cut-removing undesired area.

$$\therefore \; QS = B_1B_2 - B_1Q - SB_2$$

$$= B_1B_2 - 2B_1Q \qquad\qquad (\because \; SB_2 = B_1Q)$$

$$= a + 2l\tan\frac{\pi}{n} - 2\left(l\,\tan\frac{\pi}{n}(1-\cos\theta)\right)$$

$$= a + 2l\tan\frac{\pi}{n} - 2l\tan\frac{\pi}{n} + 2l\cos\theta\,\tan\frac{\pi}{n}$$

$$= a + 2l\cos\theta\,\tan\frac{\pi}{n}$$

Hence, the edge length (a_1) of regular n-gonal open end of pyramidal flat container, is given as

$$\boldsymbol{a_1 = a + 2l\cos\theta\,\tan\frac{\pi}{n}} \qquad \dots\dots\dots\dots\dots \text{ (3)}$$

Now, make the V-cut angle δ at each vertex of small regular n-gon such that δ is bisected by radial dotted line (as shown in fig-2 above) and mark V-parts and undesired area of big circle with radius R (see fig-2 above).

After cutting out the undesired area from big circle we get a blank of paper (as shown in the figure-4). Then, each of trapezoidal faces is rotated about the edges of regular n-gonal base $A_1A_2A_3 \dots\dots A_{n-1}A_n$ until their new edges (generated after cutting V-parts) coincide with one another. Then butt-join the mating lateral edges using suitable adhesive to get a desired pyramidal flat container (as shown in fig-5 below).
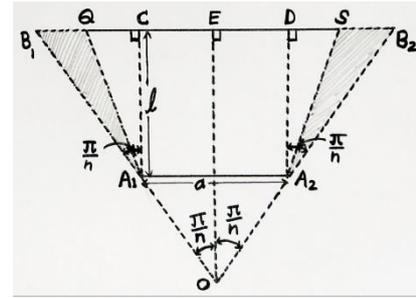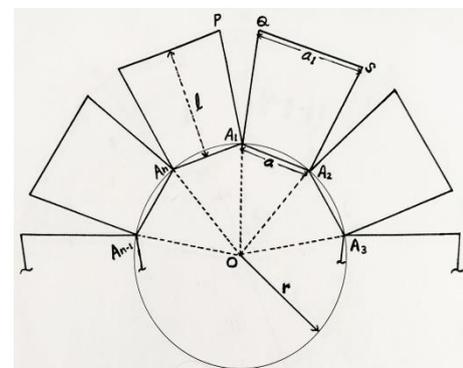

Figure-4: Blank of paper sheet after cutting, removing undesired area from big circle.

5. Derivation of dihedral angle (θ_d) between trapezoidal lateral faces of pyramidal flat container:

The dihedral angle θ_d, between two cut planes (as shown in fig-4) rotated through an angle θ about their straight edges intersecting each other at an angle α so that their new edges coincide, is given by HCR's Corollary [1] as follows

$$\boldsymbol{\cos\frac{\theta_d}{2} = \sin\theta\cos\frac{\alpha}{2}} \qquad \dots\dots\dots\dots \text{ (4)}$$

Now, setting the value $\alpha = \frac{(n-2)\pi}{n}$ in above Eq.(4), we obtain

$$\cos\frac{\theta_d}{2} = \sin\theta\cos\frac{(n-2)\pi}{2n}$$

$$\cos\frac{\theta_d}{2} = \sin\theta\cos\left(\frac{\pi}{2}-\frac{\pi}{n}\right)$$

$$\cos\frac{\theta_d}{2} = \sin\theta\sin\frac{\pi}{n}$$

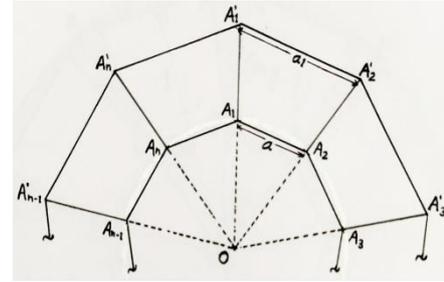$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right)$$

Figure-5: Top view of flat container after rotating lateral faces until their new edges coincide. Eg. new edges PA_1 & QA_1 coincide at A'_1A_1 (i.e. lateral edge).

Hence, the dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces of a pyramidal flat container is given as

$$\boldsymbol{\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right)} \qquad \dots\dots\dots\dots\dots \text{(5)}$$

Special case: If the trapezoidal lateral faces are rotated through right angle i.e. $\theta = 90^o$ then the dihedral angle θ_d between the consecutive lateral faces of pyramidal flat container is given as follows

$$\theta_d = 2\cos^{-1}\left(\sin 90^o\sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\cos\left(\frac{\pi}{2}-\frac{\pi}{n}\right)\right) = 2\left(\frac{\pi}{2}-\frac{\pi}{n}\right) = \frac{(n-2)\pi}{n}$$

6. Derivation of length (L) of each lateral edge of pyramidal flat container: The trapezoidal lateral face A_1QSA_2 with parallel sides a & a_1 at a normal distance l is shown in fig-6. Drop a perpendiculars A_1C & A_2D from vertices A_1 & A_2 to the QS to get two congruent right triangles by symmetry. Let $A_1Q = A_2S =$ edge length = L. Now, we have (see fig-6).
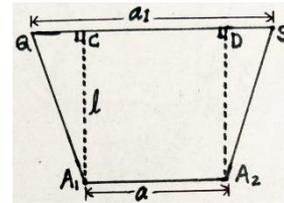

Figure-6: Trapezoidal face of pyramidal flat container. $A_1Q = A_2S =$ edge length = L

$$QC = DS = \frac{a_1 - a}{2}$$

Using Pythagorean Theorem in right ΔA_1CQ as follows

$$(A_1Q)^2 = (QC)^2 + (A_1C)^2$$

$$A_1Q = \sqrt{(QC)^2 + (A_1C)^2}$$

$$= \sqrt{\left(\frac{a_1-a}{2}\right)^2 + (l)^2}$$

$$= \sqrt{\left(\frac{a + 2l\cos\theta\,\tan\frac{\pi}{n} - a}{2}\right)^2 + l^2} \qquad \text{(setting the value of } a_1\text{)}$$

$$= \sqrt{l^2\cos^2\theta\,\tan^2\frac{\pi}{n} + l^2}$$

$$= l\sqrt{1 + \cos^2\theta\,\tan^2\frac{\pi}{n}}$$

Hence, the length (L) of each lateral edge of pyramidal flat container**,** is given as

$$\boldsymbol{L = l\sqrt{1 + \cos^2\theta\,\tan^2\frac{\pi}{n}}} \qquad \dots\dots\dots\dots\dots \text{(6)}$$

7. Derivation of surface area (A_s) of pyramidal flat container: We know that pyramidal flat container has a regular n-gonal base $A_1A_2A_3 \ldots \ldots A_{n-1}A_n$ with each side a (see regular n-gonal base in fig-7) and each of its lateral faces as a trapezoid with parallel sides a & a_1 at a normal distance l (see trapezoidal face in fig-8 below).
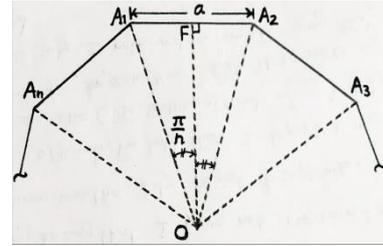

Fogure-7: Regular n-gonal base is divided into n no. of congruent isosceles triangles.

Drop a perpendicular OF from centre O to the side A_1A_2 which bisects angle $\frac{2\pi}{n}$

In right ΔOFA_1 (see fig-7 above)

$$\cot\frac{\pi}{n} = \frac{a/2}{OF} \quad \Rightarrow \quad OF = \frac{a}{2}\cot\frac{\pi}{n}$$

The area of isosceles ΔOA_1A_2 with base $A_1A_2 = a$ & normal height $OF = \frac{a}{2}\cot\frac{\pi}{n}$ is

$$= \frac{1}{2}(a)\left(\frac{a}{2}\cot\frac{\pi}{n}\right) = \frac{a^2}{4}\cot\frac{\pi}{n}$$

Since regular n-gonal base is divided into n no. of congruent isosceles triangle hence **area of regular n-gonal base of pyramidal flat container**

$$= n(\text{area of isosceles } \Delta OA_1A_2) = n\left(\frac{a^2}{4}\cot\frac{\pi}{n}\right) = \frac{1}{4}na^2\cot\frac{\pi}{n}$$

Now, the area of trapezoidal lateral face A_1QSA_2 with parallel sides a & a_1 at a normal distance l (see fig-8)

Figure-8: Trapezoidal face of pyramidal flat container.

$$= \frac{1}{2}(\text{sum of parallel sides})(\text{normal distance})$$

$$= \frac{1}{2}(a + a_1)(l)$$

$$= \frac{1}{2}\left(a + a + 2l\cos\theta\,\tan\frac{\pi}{n}\right)l$$

$$= l\left(a + l\cos\theta\,\tan\frac{\pi}{n}\right)$$

Since the lateral surface of pyramidal flat container consists of n no. of congruent trapezoidal faces, hence **total area of lateral surface of pyramidal flat container**

$$= n(\text{Area of trapezium } A_1QSA_2)$$

$$= nl\left(a + l\cos\theta\,\tan\frac{\pi}{n}\right)$$

Now, total surface area of regular n-gonal base and lateral surface

$$= \frac{1}{4}na^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\,\tan\frac{\pi}{n}\right)$$

Hence, the **total surface area (A_s) of pyramidal flat container (base +lateral surface)**, is given as

$$A_s = \frac{1}{4}na^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\,\tan\frac{\pi}{n}\right) \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

8. Derivation of volume (V) of pyramidal flat container with regular n-gonal base: The pyramidal flat container is actually a frustum of hollow right pyramid with regular n-gonal base. If we extend the lateral edges of pyramidal flat container, they intersect one another at point P (apex) & thus we get an imaginary hollow right pyramid with regular n-gonal base of each side a_1 & normal height H (as shown in the fig-9). Let $O_1O = h$ be the normal height of pyramidal flat container.
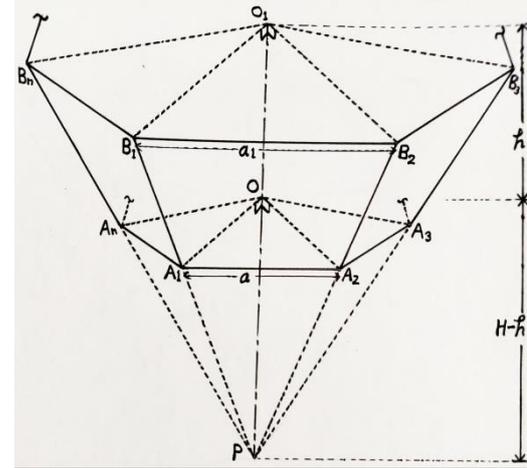
Drop a perpendicular from apex P to the planes of regular n-gonal ends of container which passes through the centres O & O_1 . Join the centres O & O_1 to the vertices of respective regular n-gons by dotted straight lines (as shown in the fig-9).

In similar right triangles ΔPO_1B_1 & ΔPOA_1 (see fig-9)

$$\frac{PB_1}{PA_1} = \frac{O_1P}{OP} \qquad \dots\dots\dots..(I)$$

In similar isosceles triangles ΔPB_1B_2 & ΔPA_1A_2 (See fig-9)

$$\frac{PB_1}{PA_1} = \frac{B_1B_2}{A_1A_2} \qquad \dots\dots\dots..(II)$$

Equating (I) & (II), we get

$$\frac{O_1P}{OP} = \frac{B_1B_2}{A_1A_2}$$

$$\frac{H}{H-h} = \frac{a_1}{a}$$

$$aH = a_1H - a_1h$$

$$H = \frac{a_1h}{a_1 - a}$$

$$H = \frac{\left(a + 2l\cos\theta\ \tan\frac{\pi}{n}\right)h}{a + 2l\cos\theta\ \tan\frac{\pi}{n} - a} \qquad \text{(setting the value of } a_1)$$

$$H = \frac{h\left(a + 2l\cos\theta\ \tan\frac{\pi}{n}\right)}{2l\cos\theta\ \tan\frac{\pi}{n}}$$

$$\boldsymbol{H = h\left(1 + \frac{a}{2l}\sec\theta\ \cot\frac{\pi}{n}\right)} \qquad \dots\dots\dots\dots (8)$$

Figure-9: Lateral edges of pyramidal flat container are extended to intersect at point P to get an imaginary hollow right pyramid of height H.

Above is the general formula to compute the height of imaginary hollow right pyramid complementary of pyramidal flat container.

The volume (V) of pyramidal flat container with regular n-gonal base will be equal to the difference of volumes of big hollow right pyramid of normal height H & small hollow right pyramid of normal height $(H - h)$ (see above fig-9), is given as

$$V = \frac{1}{3}(\text{Area of regular polygon with side } a_1)(H) - \frac{1}{3}(\text{Area of regular polygon with side a})(H - h)$$

$$= \frac{1}{3}\left(\frac{1}{4}na_1{}^2 \cot\frac{\pi}{n}\right)H - \frac{1}{3}\left(\frac{1}{4}na^2 \cot\frac{\pi}{n}\right)(H-h) \qquad \text{(area of regular polygon derived above in fig. 7)}$$

$$= \frac{n}{12}\cot\frac{\pi}{n}(a_1{}^2 H - a^2 H + a^2 h)$$

$$= \frac{n}{12}\cot\frac{\pi}{n}((a_1{}^2 - a^2)H + a^2 h)$$

$$= \frac{n}{12}\cot\frac{\pi}{n}\left(\left(\left(a + 2l\cos\theta\,\tan\frac{\pi}{n}\right)^2 - a^2\right)h\left(1 + \frac{a}{2l}\sec\theta\,\cot\frac{\pi}{n}\right) + a^2 h\right) \qquad \text{(setting values of } a_1 \text{ \& } H)$$

$$= \frac{nh}{12}\cot\frac{\pi}{n}\left(\left(a^2 + 4l^2\cos^2\theta\,\tan^2\frac{\pi}{n} + 4al\cos\theta\,\tan\frac{\pi}{n} - a^2\right)\left(1 + \frac{a}{2l}\sec\theta\,\cot\frac{\pi}{n}\right) + a^2\right)$$

$$= \frac{nh}{12}\cot\frac{\pi}{n}\left(\left(4l^2\cos^2\theta\,\tan^2\frac{\pi}{n} + 4al\cos\theta\,\tan\frac{\pi}{n}\right)\left(1 + \frac{a}{2l}\sec\theta\,\cot\frac{\pi}{n}\right) + a^2\right)$$

$$= \frac{nh}{12}\cot\frac{\pi}{n}\left(4l^2\cos^2\theta\,\tan^2\frac{\pi}{n} + 4al\cos\theta\,\tan\frac{\pi}{n} + 2al\cos\theta\,\tan\frac{\pi}{n} + 2a^2 + a^2\right)$$

$$= \frac{nh}{12}\cot\frac{\pi}{n}\left(3a^2 + 4l^2\cos^2\theta\,\tan^2\frac{\pi}{n} + 6al\cos\theta\,\tan\frac{\pi}{n}\right)$$

$$= \frac{nh}{12}\left(3a^2 \cot\frac{\pi}{n} + 4l^2\cos^2\theta\,\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

The normal height h of pyramidal flat container can be expressed in terms of slant height l & the angle of inclination θ of each trapezoidal face with the plane of base (see fig-10).

In right ΔNTM (see fig-10),

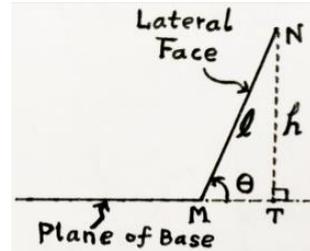$$\sin\theta = h/l \quad \Rightarrow \quad h = l\sin\theta$$

Figure-10: Each lateral trapezoidal face is inclined with the plane of base at an angle θ.

Setting the value of normal height h of container in terms of slant height l in expression of volume, we get

$$V = \frac{nl\sin\theta}{12}\left(3a^2 \cot\frac{\pi}{n} + 4l^2\cos^2\theta\,\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

Hence, the volume (V) of pyramidal flat container with regular n-gonal base of each side a, slant height l & angle of inclination θ of each trapezoidal face with the plane of base, is given as

$$\boldsymbol{V = \frac{nl\sin\theta}{12}\left(3a^2 \cot\frac{\pi}{n} + 4l^2\cos^2\theta\,\tan\frac{\pi}{n} + 6al\cos\theta\right)} \quad \text{............ (9)}$$

Special case: If the angle of inclination of each trapezoidal face with the plane of base becomes 90° i.e. $\theta = 90^o$, then the volume of pyramidal flat container is obtained by setting $\theta = 90^o$ in above formula as follows

$$V = \frac{nl\sin 90^o}{12}\left(3a^2 \cot\frac{\pi}{n} + 4l^2\cos^2 90^o\,\tan\frac{\pi}{n} + 6al\cos 90^o\right)$$

$$= \frac{nl}{12}\left(3a^2 \cot\frac{\pi}{n}\right)$$

$$= \left(\frac{1}{4}na^2 \cot\frac{\pi}{n}\right)l$$

$$= \left(\frac{1}{4}na^2 \cot\frac{\pi}{n}\right)h \qquad \text{(since, } h = l\sin 90^o = l)$$

$$= (\text{Area of polygon with } n \text{ no. of sides each of length } a)(\text{Normal height})$$

$$= \text{Volume of right prism with regular polygonal cross section of side } a \text{ \& height } h$$

The above result is true because a pyramidal flat container with lateral faces inclined at right angle with the plane of regular n-gonal base of side a becomes a right prism with regular n-gonal cross-section of side a & height h. This result verifies the above formula of volume of pyramidal flat container.

9. Construction of pyramidal flat container: For making a pyramidal flat container, having regular n-gonal base of each side a, slant height l & angle of inclination θ of each trapezoidal face with the plane of base, using a thin sheet of paper, polymer, metal or alloy of desired thickness which can easily be cut, bent and butt-joined at mating edges, following steps are followed

Step 1: Draw a circle with radius $r = \frac{a}{2}cosec\frac{\pi}{n}$ & centre 'O' on the sheet of desired material and thickness.

Step 2: Draw another circle of radius $R = \frac{a}{2}cosec\frac{\pi}{n} + l\sec\frac{\pi}{n}$ concentric with small circle of radius r on the sheet

Step 3: Take aperture angle $\frac{2\pi}{n}$ & draw a regular polygon $A_1A_2A_3 \ldots \ldots A_{n-1}A_n$ with n no. of sides each of length a circumscribed by the small circle of radius r (as shown in fig-1 above).

Step 4: Join the vertices $A_1, A_2, A_3, \ldots \ldots A_{n-1}$ & A_n to the centre O by dotted straight lines which when extended intersect big circle at the points $B_1, B_2, B_3 \ldots \ldots B_{n-1}$ & B_n. Joint these points by straight lines to get a big regular n-gon $B_1B_2B_3 \ldots \ldots B_{n-1}B_n$ (as shown in fig-1 above)

Step 5: Make V-cut angle $\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)$ at each vertex of small regular n-gon such that δ is bisected by the radial line passing through that vertex (as shown in fig-2 above).

Step 6: Mark V-cut parts & undesired area (as shaded in the fig-2 above) & cut remove undesired area from big circular sheet of radius R to get a blank of sheet (as shown in the fig-4 above).

Step 7: Rotate or bend each of trapezoidal faces about the edges of regular n-gonal base $A_1A_2A_3 \ldots \ldots A_{n-1}A_n$ until their new edges (generated after cutting V-parts) coincide with one another. Then butt-join the mating lateral edges using a suitable adhesive or welding process to get a desired pyramidal flat container (top view of flat container is shown in fig-5 above).

10. Geometric parameters of pyramidal flat container: Thus, a desired pyramidal flat container (as obtained from the above steps), with regular n-gonal base of each side a, slant height l & angle of inclination θ of each trapezoidal face with the plane of base, has the following important parameters.

1. Edge length (a_1) of regular n-gonal open end of pyramidal flat container, is given as

$$\boldsymbol{a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n}}$$

2. Length (L) of each lateral edge of pyramidal flat container, is given as

$$\boldsymbol{L = l\sqrt{1 + \cos^2\theta\tan^2\frac{\pi}{n}}}$$

3. Normal height (h) of pyramidal flat container, is given as

$$h = l\sin\theta$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces of pyramidal flat container, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right)$$

5. Total surface area (A_s) of pyramidal flat container (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\,\tan\frac{\pi}{n}\right)$$

6. Volume (V) of pyramidal flat container, is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

We will now make some typical paper models of a pyramidal flat container using general steps as mentioned above & the generalized formula as derived above.

10.1. Pyramidal flat container with square base: Here are the steps to make a paper model of pyramidal flat container with square base of each side $5.4\ cm$, slant height $4.5\ cm$ and each trapezoidal lateral face is inclined with the plane of base at an angle 65°.

Given data: $n = $ number of sides of square base $= 4$

$\qquad a = $ edge length of square base $= 5.4$ cm

$\qquad l = $ slant height of pyramidal flat container $= 4.5$ cm

$\qquad \theta = $ angle of inclination of each trapezoidal lateral face with the plane of base $= 65^o$

The following steps (as shown in figure-11 below) are used to make a pyramidal flat container (of desired dimensions) with square base using sheet of paper

Step 1. Calculate radius (r) of small circle

$$r = \frac{a}{2}cosec\frac{\pi}{n} = \frac{5.4}{2}cosec\frac{\pi}{4} = 3.82\ cm$$

Draw a circle with radius $r = 3.82\ cm$ on the sheet of paper (see fig-11 (i)).

Step 2. Calculate radius (R) of big circle

$$R = \frac{a}{2}cosec\frac{\pi}{n} + l\sec\frac{\pi}{n} = \frac{5.4}{2}cosec\frac{\pi}{4} + 4.5\sec\frac{\pi}{4} = 10.18\ cm$$

Draw another circle of radius $R = 10.18\ cm$ concentric with small circle (see fig-11(ii)).

Step 3. Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{4} = 90^o$ & draw a square of each side $a = 5.4\ cm$ circumscribed by the small circle of radius $r = 3.82\ cm$ (as shown in fig-11(iii)). Join the vertices of square to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 10.18 cm (as shown in fig-11(iv) below).
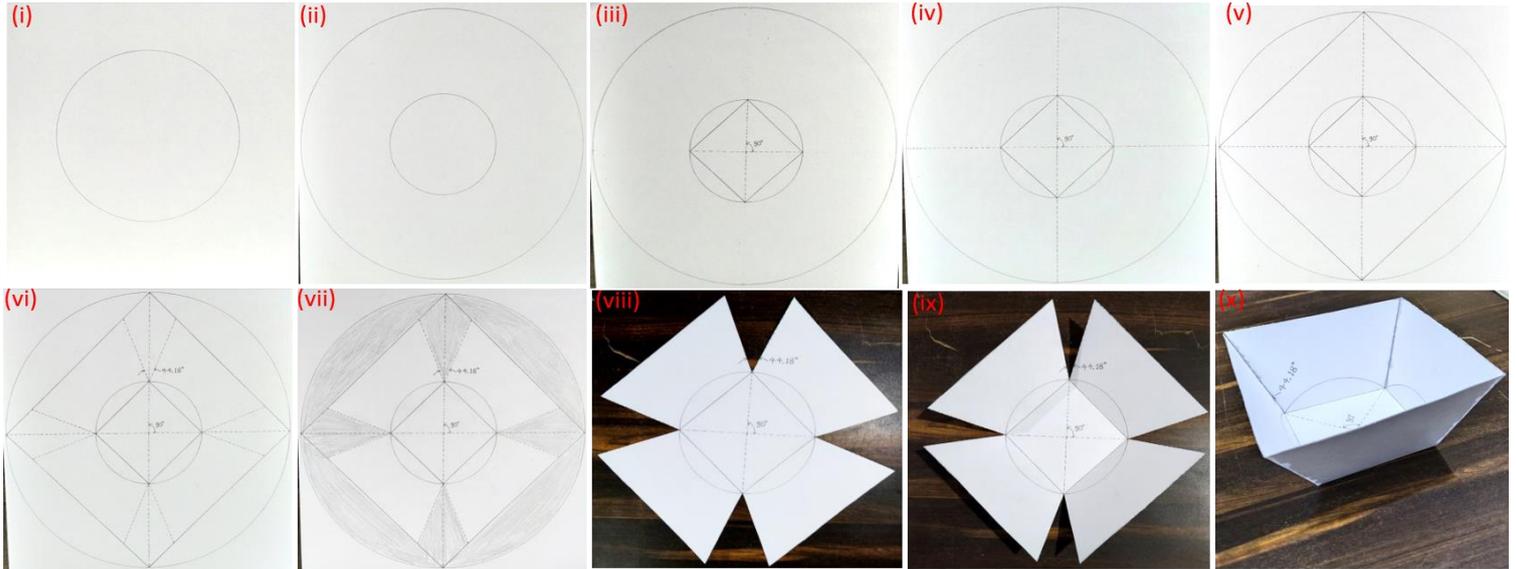
Figure-11: (i) A circle of radius 3.82 cm is drawn on the sheet of paper **(ii)** a larger circle of radius 10.18 cm drawn on the sheet of paper is concentric with small circle of radius 3.82 cm **(iii)** a square of side 5.4 cm drawn on the sheet of paper is circumscribed by small circle of radius 3.82 cm **(iv)** dotted straight lines joining the vertices of square to the centre are extended to intersect big circle of radius 10.18 cm **(v)** a big square is obtained by joining points of intersection of radial lines and the big circle by straight lines **(vi)** V-cut angle of 44.18^o is made at each vertex of small square **(vii)** undesired area (including V-parts) to be cut-removed from big circle is shaded **(viii)** blank of paper after cutting & removing undesired area (including V-parts) from big circle **(ix)** rotating trapezoidal faces about the edges of square (base) which become lateral faces after meeting at the cut edges **(x)** desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive.

Step 4. Join the points of intersection of radial lines & big circle by the straight lines to get a big square (See fig-11(v) above).

Step 5. Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\,\tan\frac{\pi}{n}\right) = \frac{2\pi}{4} - 2\tan^{-1}\left(\cos 65^o\,\tan\frac{\pi}{4}\right) = 44.18^o$$

Draw V-cut angle $\delta = 44.18^o$ at each vertex of small square such that δ is bisected by the radial line passing through that vertex (as shown in fig-11(vi) above) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 10.18 cm (as shown in fig-11(vii) above).

Step 6. Cut & remove undesired area (as shaded in fig-11(vii) above) from big circle (sheet) to get a blank of sheet (as shown in fig-11(viii) above). Rotate the trapezoidal faces about the edges of small square (base) (as shown in fig-11(ix) above)**.**

Step 7. Rotate the trapezoidal faces about the edges of square (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with square base (as shown in fig-11(x) above).

The desired pyramidal flat container with square base has some important parameters which are analytically calculated by using generalized formula (as derived in Sec. 10 above) as follows

1. Edge length (a_1) of square open end, is given as

$$a_1 = a + 2l\cos\theta\,\tan\frac{\pi}{n} = 5.4 + 2(4.5)\cos 65^o\,\tan\frac{\pi}{4} = 9.203\ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1 + \cos^2\theta \, \tan^2\frac{\pi}{n}} = 4.5\sqrt{1 + \cos^2 65^o \, \tan^2\frac{\pi}{4}} = 4.885 \; cm$$

3. Normal height (h), is given as

$$h = l\sin\theta = 4.5 \sin 65^o = 4.078 \; cm$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta \, \sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 65^o \, \sin\frac{\pi}{4}\right) = 100.29^o$$

5. Total surface area (A_s) (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta \, \tan\frac{\pi}{n}\right) = \frac{1}{4}(4)\,(5.4)^2\cot\frac{\pi}{4} + 4(4.5)\left(5.4 + 4.5\cos 65^o \, \tan\frac{\pi}{4}\right)$$

$$= 160.592 cm^2$$

6. Volume (V), is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta \, \tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{4(4.5)\sin 65^o}{12}\left(3(5.4)^2\cot\frac{\pi}{4} + 4(4.5)^2\cos^2 65^o \, \tan\frac{\pi}{4} + 6(5.4)(4.5)\cos 65^o\right)$$

$$= 222.36 \; cm^3$$

All the above values of important parameters of the pyramidal flat container with a square base can be verified by accurate and precise measurements.

10.2. Pyramidal flat container with regular hexagonal base: Here are the steps to make a paper model of pyramidal flat container with regular hexagonal base of each side 6 cm , slant height 3.5 cm and each trapezoidal lateral face is inclined with the plane of base at an angle 70°.

Given data: $n =$ number of sides of regular hexagonal base $= 6$

$a =$ edge length of regular hexagonal base $= 6$ cm

$l =$ slant height of pyramidal flat container $= 3.5$ cm

$\theta =$ angle of inclination of each trapezoidal lateral face with plane of base $= 70^o$

The following steps (as shown in figure-12 below) are used to make a pyramidal flat container (of desired dimensions) with regular hexagonal base using a sheet of paper.

Step 1. Calculate radius (r) of small circle

$$r = \frac{a}{2}cosec\frac{\pi}{n} = \frac{6}{2}cosec\frac{\pi}{6} = 6 \; cm$$

Draw a circle with radius $r = 6 \; cm$ on the sheet of paper (See fig-12(i) below).

Step 2. Calculate radius (R) of big circle

$$R = \frac{a}{2}cosec\frac{\pi}{n} + l\sec\frac{\pi}{n} = \frac{6}{2}cosec\frac{\pi}{6} + 3.5\sec\frac{\pi}{6} = 10.04 \; cm$$
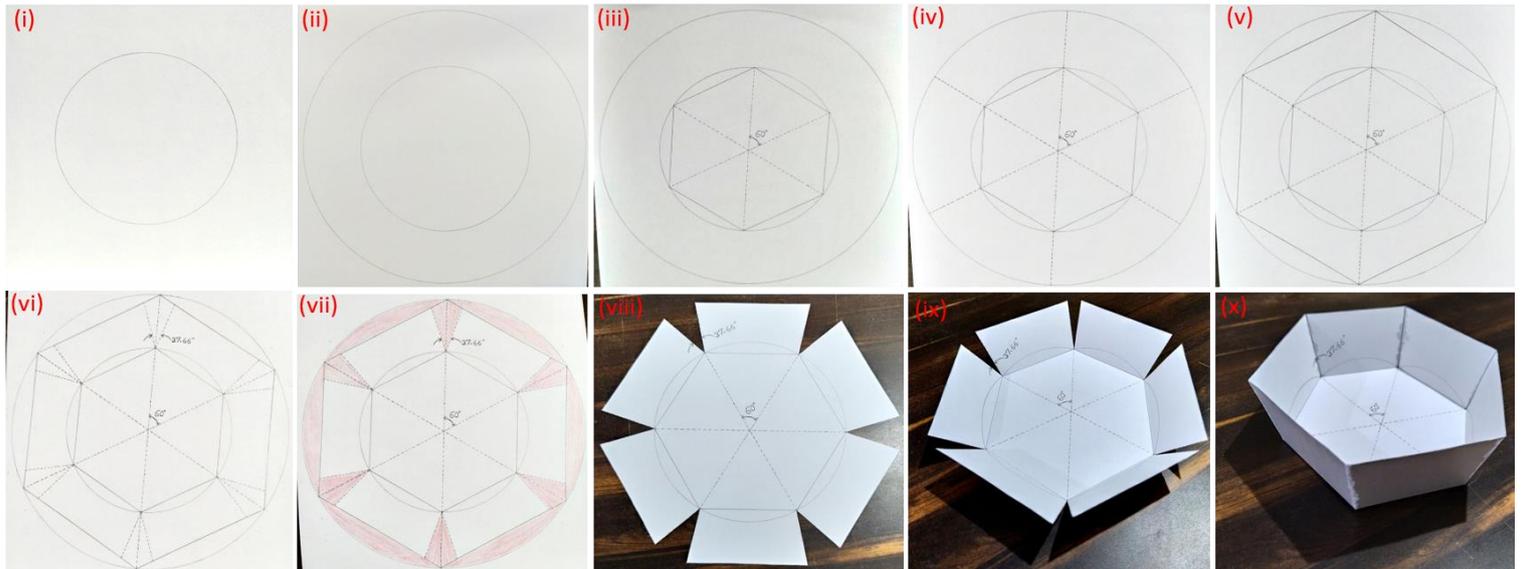
Figure-12: (i) A circle of radius 6 cm is drawn on the sheet of paper **(ii)** a larger circle of radius 10.04 cm drawn on the sheet of paper is concentric with small circle of radius 6 cm **(iii)** a regular hexagon of side6 cm drawn on the sheet of paper is circumscribed by small circle of radius 6 cm **(iv)** dotted straight lines joining the vertices of regular hexagon to the centre are extended so as to intersect big circle of radius 10.04 cm **(v)** a big regular hexagon is obtained by joining points of intersection of radial lines and the big circle by straight line **(vi)** V-cut angle of 37.66^o is made at each vertex of small regular hexagon **(vii)** undesired area (including V-parts) to be removed from big circle is shaded **(viii)** blank of paper after cutting & removing undesired area (including V-parts) from big circle **(ix)** rotating trapezoidal faces about the edges of regular hexagon (base) which become lateral faces after meeting at the edges **(x)** desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive.

Draw another circle of radius $R = 10.04\ cm$ concentric with small circle (see above fig-12(ii)).

Step 3. Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{6} = 60^o$ & draw a regular hexagon of each side $a = 6\ cm$ circumscribed by the small circle of radius $r = 6\ cm$ (as shown in fig-12(iii) above). Join the vertices of regular hexagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 10.04 cm (as shown in fig-12(iv) above).

Step 4. Join the points of intersection of radial lines & big circle by the straight lines to get a big regular hexagon (See fig-12(v) above)

Step 5. Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right) = \frac{2\pi}{6} - 2\tan^{-1}\left(\cos 70^o \tan\frac{\pi}{6}\right) = 37.66^o$$

Draw V-cut angle $\delta = 37.66^o$ at each vertex of small regular hexagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-12(vi) above) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 10.04 cm (as shown in fig-12(vii) above).

Step 6. Cut & remove undesired area (as shaded in fig-12(vii) above) from big circle (sheet) to get a blank of paper sheet (as shown in fig-12(xiii) above). Rotate the trapezoidal faces about the edges of small regular hexagon (base) (as shown in fig-12(ix) above).

Step 7. Rotate the trapezoidal faces about the edges of regular hexagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with regular hexagonal base (as shown in fig-12(x) above).

The desired pyramidal flat container with regular hexagonal base has some important parameters which are analytically calculated by using generalized formula (as derived in Sec. 10 above) as follows

1. Edge length (a_1) of regular hexagonal open end, is given as

$$a_1 = a + 2l \cos\theta \ \tan\frac{\pi}{n} = 6 + 2(3.5)\cos 70^o \ \tan\frac{\pi}{6} = 7.382 \ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1 + \cos^2\theta \tan^2\frac{\pi}{n}} = 3.5\sqrt{1 + \cos^2 70^o \tan^2\frac{\pi}{6}} = 3.567 \ cm$$

3. Normal height (h), is given as

$$h = l \sin\theta = 3.5 \sin 70^o = 3.289 \ cm$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta \sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 70^o \sin\frac{\pi}{6}\right) = 123.95^o$$

5. Total surface area (A_s) (base +lateral surface), is given as

$$A_s = \frac{1}{4}n \, a^2 \cot\frac{\pi}{n} + nl\left(a + l \cos\theta \ \tan\frac{\pi}{n}\right) = \frac{1}{4}(6)\,(6)^2\cot\frac{\pi}{6} + 6(3.5)\left(6 + 3.5 \cos 70^o \ \tan\frac{\pi}{6}\right)$$

$$= 234.044 \ cm^2$$

6. Volume (V), is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2 \cot\frac{\pi}{n} + 4l^2\cos^2\theta \tan\frac{\pi}{n} + 6al \cos\theta \ \right)$$

$$= \frac{6(3.5)\sin 70^o}{12}\left(3(6)^2 \cot\frac{\pi}{6} + 4(3.5)^2\cos^2 70^o \tan\frac{\pi}{6} + 6(6)(3.5) \cos 70^o \ \right)$$

$$= 383.925 \ cm^3$$

All the values of parameters computed above can be verified by the accurate and precise measurements.

10.3. Pyramidal flat container with regular pentagonal base: Let's make a paper model of pyramidal flat container with regular pentagonal base of each side $5 \ cm$, slant height $4 \ cm$ and each trapezoidal lateral face is inclined with the plane of base at an angle 80°.

Given data: $n = $ number of sides of regular pentagonal base $= 5$

$\qquad a = $ edge length of regular pentagonal base $= 5$ cm

$\qquad l = $ slant height of pyramidal flat container $= 4$ cm

$\qquad \theta = $ angle of inclination of each trapezoidal lateral face with plane of base $= 80^o$

The following steps (as shown in figure-13 below) are used to make a pyramidal flat container (of desired dimensions) with regular pentagonal base using a sheet of paper
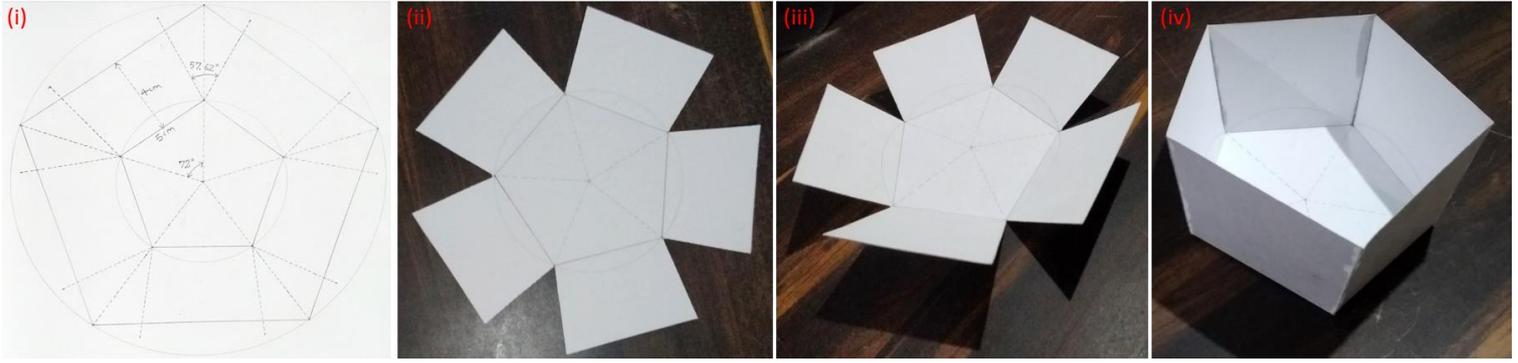
Figure-13: (i) Complete drawing on the sheet of paper to make desired pyramidal flat container with regular pentagonal base inside a large circular sheet of radius 9.2 cm **(ii)** blank of paper after cutting & removing undesired area (including V-parts) from big circle **(iii)** rotating trapezoidal faces about the edges of regular pentagon (base) which become lateral faces after meeting at the edges **(iv)** desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive.

Step 1. Calculate radius (r) of small circle

$$r = \frac{a}{2} cosec\frac{\pi}{n} = \frac{5}{2} cosec\frac{\pi}{5} = 4.25\ cm$$

Draw a circle with radius $r = 4.25\ cm$ on the sheet of paper (see fig-13(i) above).

Step 2. Calculate radius (R) of big circle

$$R = \frac{a}{2} cosec\frac{\pi}{n} + l\sec\frac{\pi}{n} = \frac{5}{2} cosec\frac{\pi}{5} + 4\sec\frac{\pi}{5} = 9.2\ cm$$

Draw another circle of radius $R = 9.2\ cm$ concentric with small circle (See fig-13(i) above).

Step 3. Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{5} = 72^o$ & draw a regular pentagon of each side $a = 5\ cm$ circumscribed by the small circle of radius $4.25\ cm$. Join the vertices of regular pentagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 9.2 cm (as shown in fig-13(i) above).

Step 4. Join the points of intersection of radial lines & big circle by the straight lines to get a big regular pentagon (see fig-13(i) above).

Step 5. Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right) = \frac{2\pi}{5} - 2\tan^{-1}\left(\cos 80^o\tan\frac{\pi}{5}\right) = 57.62^o$$

Draw V-cut angle $\delta = 57.62^o$ at each vertex of small regular pentagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-13(i) above) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 9.2 cm (Follow the same steps as mentioned in Sec. 10.1 and Sec. 10.2 above).

Step 6. Cut & remove undesired area (see fig-13(i) above) from big circle (sheet) to get a blank of paper sheet (as shown in fig-13(ii) above). Rotate the trapezoidal faces about the edges of small regular pentagon (base) (as shown in fig-13(iii) above).

Step 7. Rotate the trapezoidal faces about the edges of regular pentagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with regular pentagonal base (as shown in fig-13(iv) above).

The desired pyramidal flat container with regular pentagonal base has some important parameters which are analytically calculated by using generalized formula (as derived in Sec. 10 above) as follows

1. Edge length (a_1) of regular pentagonal open end, is given as

$$a_1 = a + 2l \cos \theta \ \tan \frac{\pi}{n} = 5 + 2(4) \cos 80^o \ \tan \frac{\pi}{5} = 6 \ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l \sqrt{1 + \cos^2 \theta \tan^2 \frac{\pi}{n}} = 4 \sqrt{1 + \cos^2 80^o \tan^2 \frac{\pi}{5}} = 4.032 \ cm$$

3. Normal height (h), is given as

$$h = l \sin\theta = 4 \sin 80^o = 3.939 \ cm$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2 \cos^{-1} \left(\sin \theta \sin \frac{\pi}{n} \right) = 2 \cos^{-1} \left(\sin 80^o \sin \frac{\pi}{5} \right) = 109.26^o$$

5. Total surface area (A_s) (base +lateral surface), is given as

$$A_s = \frac{1}{4} n \, a^2 \cot \frac{\pi}{n} + nl \left(a + l \cos \theta \ \tan \frac{\pi}{n} \right) = \frac{1}{4} (5) \, (5)^2 \cot \frac{\pi}{5} + 5(4) \left(5 + 4 \cos 80^o \ \tan \frac{\pi}{5} \right)$$

$$= 153.105 \ cm^2$$

6. Volume (V), is given as

$$V = \frac{nl \sin \theta}{12} \left(3a^2 \cot \frac{\pi}{n} + 4l^2 \cos^2 \theta \tan \frac{\pi}{n} + 6al \cos \theta \ \right)$$

$$= \frac{5(4) \sin 80^o}{12} \left(3(5)^2 \cot \frac{\pi}{5} + 4(4)^2 \cos^2 80^o \tan \frac{\pi}{5} + 6(5)(4) \cos 80^o \ \right)$$

$$= 205.937 \ cm^3$$

All the above values of important parameters of pyramidal flat container with regular pentagonal base can be verified by accurate and precise measurements.

10.4. Pyramidal flat container with regular heptagonal base: Let's make a paper model of pyramidal flat container with regular heptagonal base of each side $4 \ cm$, slant height $4 \ cm$ and each trapezoidal lateral face is inclined with the plane of base at an angle 110°.

Given data: $n = $ number of sides of regular heptagonal base $= 7$

$\qquad a = $ edge length of regular heptagonal base $= 4$ cm

$\qquad l = $ slant height of pyramidal flat container $= 4$ cm

$\qquad \theta = $ angle of inclination of each trapezoidal lateral face with plane of base $= 110^o$

The following steps (as shown in figure-14 below) are used to make a pyramidal flat container (of desired dimensions) with regular heptagonal base using a sheet of paper.
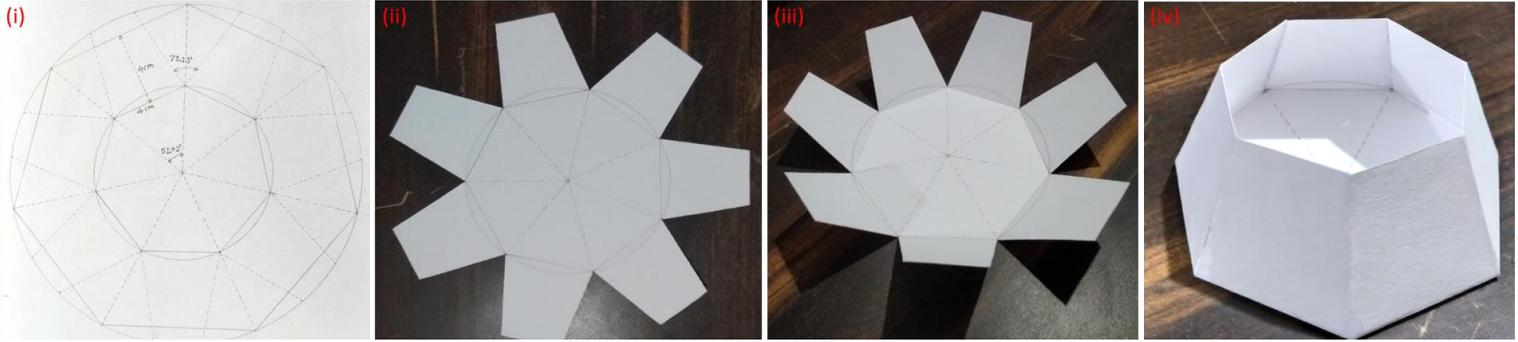
Figure-14: (i) Complete drawing on the sheet of paper to make desired pyramidal flat container with regular heptagonal base inside a large circular sheet of radius 9.05 cm (ii) blank of paper after cutting & removing undesired area (including V-parts) from big circle (iii) rotating trapezoidal faces about the edges of regular heptagon (base) which become lateral faces after meeting at the edges (iv) desired pyramidal flat container after butt-joining the mating lateral edges using suitable adhesive.

Step 1. Calculate radius (r) of small circle

$$r = \frac{a}{2} cosec \frac{\pi}{n} = \frac{4}{2} cosec \frac{\pi}{7} = 4.61 \ cm$$

Draw a circle with radius $r = 4.61 \ cm$ on the sheet of paper (see fig-14(i) above).

Step 2. Calculate radius (R) of big circle

$$R = \frac{a}{2} cosec \frac{\pi}{n} + l \sec \frac{\pi}{n} = \frac{4}{2} cosec \frac{\pi}{7} + 4 \sec \frac{\pi}{7} = 9.05 \ cm$$

Draw another circle of radius $R = 9.05 \ cm$ concentric with small circle (see fig-14(i) above).

Step 3. Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{7} = 51.43^o$ & draw a regular heptagon of each side $a = 4 \ cm$ circumscribed by the small circle of radius $4.61 \ cm$. Join the vertices of regular heptagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 9.05 cm (as shown in fig-14(i) above).

Step 4. Join the points of intersection of radial lines & big circle by the straight lines to get a big regular heptagon (see fig-14(i) above).

Step 5. Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2 \tan^{-1} \left(\cos \theta \tan \frac{\pi}{n} \right) = \frac{2\pi}{7} - 2 \tan^{-1} \left(\cos 110^o \tan \frac{\pi}{7} \right) = 70.13^o$$

Make V-cut angle $\delta = 70.13^o$ at each vertex of small regular heptagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-14(i) above) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 9.05 cm (Follow the same steps as mentioned in Sec. 10.1 and Sec. 10.2 above).

Step 6. Cut & remove undesired area (see fig-14(i) above) from big circle (sheet) to get a blank of paper sheet (as shown in fig-14(ii) above). Rotate the trapezoidal faces about the edges of small regular heptagon (base) (as shown in fig-14(iii) above).

Step 7. Rotate the trapezoidal faces about the edges of regular heptagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired pyramidal flat container with regular heptagonal base (as shown in fig-14(iv) above).

The desired pyramidal flat container with regular heptagonal base has some important parameters which are analytically calculated by using generalized formula (as derived in Sec. 10 above) as follows

1. Edge length (a_1) of regular heptagonal open end, is given as

$$a_1 = a + 2l\cos\theta\ \tan\frac{\pi}{n} = 4 + 2(4)\cos 110^o\ \tan\frac{\pi}{7} = 2.682\ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1 + \cos^2\theta\tan^2\frac{\pi}{n}} = 4\sqrt{1 + \cos^2 110^o\tan^2\frac{\pi}{7}} = 4.054\ cm$$

3. Normal height (h), is given as

$$h = l\sin\theta = 4\sin 110^o = 3.759\ cm$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta\sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 110^o\sin\frac{\pi}{7}\right) = 131.88^o$$

5. Total surface area (A_s) (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\ \tan\frac{\pi}{n}\right) = \frac{1}{4}(7)\,(4)^2\cot\frac{\pi}{7} + 7(4)\left(4 + 4\cos 110^o\ \tan\frac{\pi}{7}\right)$$

$$= 151.695\ cm^2$$

6. Volume (V), is given as

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{7(4)\sin 110^o}{12}\left(3(4)^2\cot\frac{\pi}{7} + 4(4)^2\cos^2 110^o\tan\frac{\pi}{7} + 6(4)(4)\cos 110^o\right)$$

$$= 154.458\ cm^3$$

All the above values of important parameters of the pyramidal flat container with a regular heptagonal base can be verified by accurate and precise measurements.

10.5. Pyramidal flat container with regular octagonal base: Let's make a paper model of pyramidal flat container with regular octagonal base of each side $4\ cm$, slant height $4\ cm$ and each trapezoidal lateral face is inclined with the plane of base at an angle 60°.

Given data: n = number of sides of regular octagonal base = 8

$\qquad\qquad a$ = edge length of regular octagonal base = 4 cm

$\qquad\qquad l$ = slant height of pyramidal flat container = 4 cm

$\qquad\qquad \theta$ = angle of inclination of each trapezoidal lateral face with plane of base = 60°

The following steps (as shown in figure-15 below) are used to make a pyramidal flat container (of desired dimensions) with regular octagonal base using a sheet of paper

Step 1. Calculate radius (r) of small circle

$$r = \frac{a}{2}\csc\frac{\pi}{n} = \frac{4}{2}\csc\frac{\pi}{8} = 5.23 \; cm$$

Draw a circle with radius $r = 5.23 \; cm$ on the sheet of paper (see fig-15(i) below).

Step 2. Calculate radius (R) of big circle

$$R = \frac{a}{2}\csc\frac{\pi}{n} + l\sec\frac{\pi}{n} = \frac{4}{2}\csc\frac{\pi}{8} + 4\sec\frac{\pi}{8} = 9.55 \; cm$$

Draw another circle of radius $R = 9.55 \; cm$ concentric with small circle (see fig-15(i) below).

Step 3. Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{8} = 45^o$ & draw a regular octagon of each side $a = 4 \; cm$ circumscribed by the small circle of radius $5.23 \; cm$. Join the vertices of regular octagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 9.55 cm (as shown in fig-15(i) below).

Step 4. Join the points of intersection of radial lines & big circle by the straight lines to get a big regular octagon (see fig-15(i) below)

Step 5. Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right) = \frac{2\pi}{8} - 2\tan^{-1}\left(\cos 60^o\tan\frac{\pi}{8}\right) = 21.6^o$$

Draw V-cut angle $\delta = 21.6^o$ at each vertex of small regular octagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-15(i) below) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 9.55 cm (Follow the same steps as mentioned in Sec. 10.1 and Sec. 10.2 above).
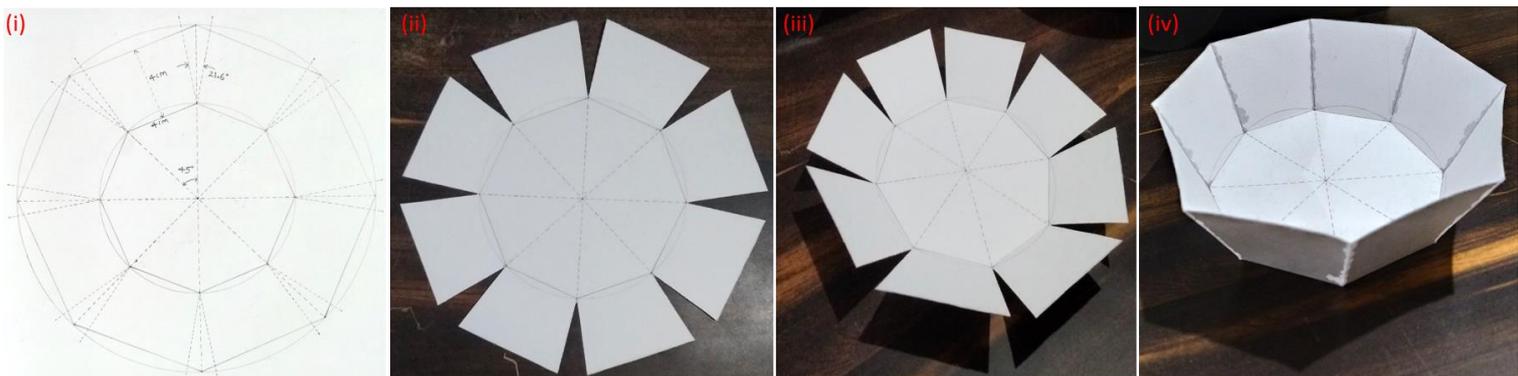

Figure-15: (i) Complete drawing on the sheet of paper to make desired pyramidal flat container with regular octagonal base inside a large circular sheet of radius 9.55 cm (ii) blank of paper after cutting & removing undesired area (including V-parts) from big circle (iii) rotating trapezoidal faces about the edges of regular octagon (base) which become lateral faces after meeting at the edges (iv) desired pyramidal flat container after butt-joining the mating lateral edges using a suitable adhesive.

Step 6. Cut & remove undesired area (see fig-15(i) above) from big circle (sheet) to get a blank of paper sheet (as shown in fig-15(ii) above). Rotate the trapezoidal faces about the edges of small regular octagon (base) (as shown in fig-15(iii) above).

Step 7. Rotate the trapezoidal faces about the edges of regular octagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using a suitable adhesive to get the desired pyramidal flat container with regular octagonal base (as shown in fig-15(iv)).

The desired pyramidal flat container with regular octagonal base has some important parameters which are analytically calculated by using generalized formula (as derived in Sec. 10 above) as follows

 1. Edge length (a_1) of regular octagonal open end, is given as

$$a_1 = a + 2l \cos\theta \ \tan\frac{\pi}{n} = 4 + 2(4) \cos 60^o \ \tan\frac{\pi}{8} = 5.657 \ cm$$

2. Length (L) of each lateral edge, is given as

$$L = l\sqrt{1 + \cos^2\theta \tan^2\frac{\pi}{n}} = 4\sqrt{1 + \cos^2 60^o \tan^2\frac{\pi}{8}} = 4.085 \ cm$$

3. Normal height (h), is given as

$$h = l \sin\theta = 4 \sin 60^o = 3.464 \ cm$$

4. Dihedral angle (θ_d) between any two consecutive trapezoidal lateral faces, is given as

$$\theta_d = 2\cos^{-1}\left(\sin\theta \sin\frac{\pi}{n}\right) = 2\cos^{-1}\left(\sin 60^o \sin\frac{\pi}{8}\right) = 141.29^o$$

5. Total surface area (A_s) (base +lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta \ \tan\frac{\pi}{n}\right) = \frac{1}{4}(8)\,(4)^2\cot\frac{\pi}{8} + 8(4)\left(4 + 4\cos 60^o \ \tan\frac{\pi}{8}\right)$$

$$= 231.764 \ cm^2$$

6. Volume (V), is given as

$$V = \frac{nl \sin\theta}{12}\left(3a^2 \cot\frac{\pi}{n} + 4l^2\cos^2\theta \tan\frac{\pi}{n} + 6al \cos\theta\right)$$

$$= \frac{8(4) \sin 60^o}{12}\left(3(4)^2 \cot\frac{\pi}{8} + 4(4)^2\cos^2 60^o \tan\frac{\pi}{8} + 6(4)(4) \cos 60^o\right)$$

$$= 393.775 \ cm^3$$

The area of circular sheet of radius 9.55 cm removed as scrap, is equal to the difference of area of circular sheet and the total surface area of container which is given as

$$= \pi(9.55)^2 - 231.764$$

$$= 54.757 \ cm^2$$

The percentage (%) of total area of circular sheet used to make pyramidal flat container with regular octagonal base is

$$= \frac{\text{Total surface area of container}}{\text{Area of circular sheet of radius 9.55 cm}} \times 100 \ \%$$

$$= \frac{231.764}{\pi (9.55)^2} \times 100 \ \%$$

$$= 80.89 \ \%$$

All the above values of important parameters of the pyramidal flat container with regular octagonal base can be verified by accurate and precise measurements.

So far, we have derived generalized formulas & discussed in detail the method to make a pyramidal flat container with a regular n-gonal base. Applying the same approach & method as in the case of the pyramidal flat container, we will derive a generalized formula & discuss in detail the method to make right pyramid with regular n-gonal base.

11. Right pyramid (shell) with regular n-gonal base: We know that all the triangular lateral faces of a right pyramid intersect each other at the apex. If we rotate all the trapezoidal lateral faces of a pyramidal flat container about the edges of its regular n-gonal base through an equal angle in such a way that all the lateral faces meet at a single point (i.e. apex) then a pyramidal flat container (having slant height l & edge length of regular n-gonal base a and an angle θ (measured externally) of inclination of each lateral face with the plane of base) becomes a right pyramid (having slant height l, regular n-gonal base of edge length a & an angle θ (measured externally) of inclination of each lateral face with the plane of base). In this case, each trapezoidal lateral face of the pyramidal flat container becomes an isosceles triangular lateral face of a regular n-gonal right pyramid [2,3,4] (see fig-16 below).

11.1. Angle of inclination (θ) of each isosceles triangular lateral face with the plane of regular n-gonal base of right pyramid

We know that edge length (a_1) of regular n-gonal open end of pyramidal flat container is given as

$$a_1 = a + 2l \cos \theta \ \tan \frac{\pi}{n}$$

Now, in order to make all the trapezoidal lateral faces of flat container meet at a point, rotate all the lateral faces about the edges of its base through an angle θ in such a way that each trapezoidal lateral face of pyramidal flat container becomes an isosceles triangular lateral face of right pyramid i.e. the edge length a_1 of trapezoidal lateral face must be zero i.e. $a_1 = 0$ (as shown in fig-16).
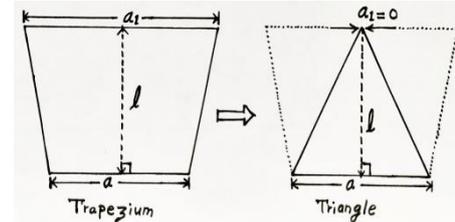

Figure-16: A trapezoidal lateral face of flat container becomes a triangular lateral face of right pyramid when edge length $a_1 = 0$.

$$\therefore \ a + 2l \cos \theta \ \tan \frac{\pi}{n} = 0$$

$$2l \cos \theta \ \tan \frac{\pi}{n} = -a$$

$$\cos \theta \ = \frac{-a}{2l \tan \dfrac{\pi}{n}}$$

$$\cos \theta \ = \frac{-a}{2l} \ \cot \frac{\pi}{n} \qquad \qquad \left(- \ ve \ \text{sign indicates that } \frac{\pi}{2} < \theta < \pi \right)$$

$$\theta \ = \pi - \cos^{-1} \left(\frac{a}{2l} \cot \frac{\pi}{n} \right)$$

Hence, the angle of inclination of each isosceles triangular lateral face (θ), measured externally with the plane of regular n-gonal base with each side a in a right pyramid of slant height l, is given as

$$\boldsymbol{\theta} = \boldsymbol{\pi} - \mathbf{cos}^{-1}\left(\frac{\boldsymbol{a}}{\boldsymbol{2l}}\mathbf{cot}\frac{\boldsymbol{\pi}}{\boldsymbol{n}}\right) \qquad \ldots\ldots\ldots\ldots(10)$$

The above formula is essential for computing the angle of inclination of isosceles triangular lateral face with the plane of regular n-gonal base of right pyramid when number of sides n & edge length a of regular n-gonal base and slant height l are known.

11.2. V-cut angle (δ) required to make right pyramid with regular n-gonal base from a circular sheet

Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula of V-cut angle for pyramidal flat container with regular n-gonal base, V-cut angle δ required to cut remove V-parts & rotate each two co-planar planes about their intersecting straight edges i.e. edges of regular n-gonal base of right pyramid is obtained as follows

$$\delta = \frac{2\pi}{n} - 2\tan^{-1}\left(\cos\theta\tan\frac{\pi}{n}\right)$$

$$= \frac{2\pi}{n} - 2\tan^{-1}\left(\frac{-a}{2l}\cot\frac{\pi}{n}\tan\frac{\pi}{n}\right) \qquad (\text{setting the value of }\cos\theta)$$

$$= \frac{2\pi}{n} - 2\tan^{-1}\left(\frac{-a}{2l}\right)$$

$$= \frac{2\pi}{n} + 2\tan^{-1}\left(\frac{a}{2l}\right)$$

Hence, V-cut angle δ required to cut remove V-parts & rotate each two co-planar planes about their intersecting straight edges i.e. edges of regular n-gonal base of right pyramid, is given as

$$\boldsymbol{\delta} = \frac{\boldsymbol{2\pi}}{\boldsymbol{n}} + \boldsymbol{2}\tan^{-1}\left(\frac{\boldsymbol{a}}{\boldsymbol{2l}}\right) \qquad \ldots\ldots\ldots\ldots(11)$$

11.3. Length (L) of each lateral edge of right pyramid

Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in the generalized formula for length of lateral edge of pyramidal flat container, the length of each lateral edge of a right pyramid with regular n-gonal base, is obtained as follows

$$L = l\sqrt{1 + \cos^2\theta\tan^2\frac{\pi}{n}}$$

$$= l\sqrt{1 + \left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2\tan^2\frac{\pi}{n}} \qquad (\text{setting the value of }\cos\theta)$$

$$= l\sqrt{1 + \frac{a^2}{4l^2}\cot^2\frac{\pi}{n}\tan^2\frac{\pi}{n}}$$

$$= l\sqrt{1 + \frac{a^2}{4l^2}}$$

$$= \sqrt{l^2 + \left(\frac{a}{2}\right)^2}$$

Above result can also be derived by Pythagorean Theorem.

Hence, the length (L) of each lateral edge of a right pyramid with slant height l & regular n-gonal base of each side a, is given as

$$L = \sqrt{l^2 + \left(\frac{a}{2}\right)^2} \qquad \ldots\ldots\ldots\ldots (12)$$

11.4. Normal height (h) of right pyramid

Substituting $\cos\theta = \frac{-a}{2l} \cot\frac{\pi}{n}$ in generalized formula for normal height of pyramidal flat container with regular n-gonal base, the normal height of a right pyramid with regular n-gonal base, is obtained as follows

$$h = l \sin\theta = l\sqrt{1 - \cos^2\theta}$$

$$= l\sqrt{1 - \left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2} \qquad (\text{setting the value of } \cos\theta)$$

$$= l\sqrt{1 - \frac{a^2}{4l^2}\cot^2\frac{\pi}{n}}$$

$$= \sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}}$$

Hence, the normal height (h) of a right pyramid with slant height l & regular n-gonal base of each side a, is given as

$$h = \sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}} \qquad \ldots\ldots\ldots\ldots (13)$$

11.5. Dihedral angle (θ_d) between any two consecutive lateral faces of right pyramid

Substituting $\cos\theta = \frac{-a}{2l} \cot\frac{\pi}{n}$ in generalized formula of dihedral angle for pyramidal flat container with regular n-gonal base, the dihedral angle between any two consecutive lateral faces of a right pyramid with regular n-gonal base, is obtained as follows

$$\theta_d = 2\cos^{-1}\left(\sin\theta \sin\frac{\pi}{n}\right)$$

$$= 2\cos^{-1}\left(\sin\frac{\pi}{n}\sqrt{1 - \cos^2\theta}\right)$$

$$= 2\cos^{-1}\left(\sin\frac{\pi}{n}\sqrt{1 - \left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2}\right) \qquad (\text{setting the value of } \cos\theta)$$

$$= 2\cos^{-1}\left(\sin\frac{\pi}{n}\sqrt{1 - \frac{a^2}{4l^2}\cot^2\frac{\pi}{n}}\right)$$

$$= 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{n} - \frac{a^2}{4l^2}\cos^2\frac{\pi}{n}}\right)$$

Hence, the dihedral angle (θ_d) between any two adjacent lateral faces of a right pyramid with slant height l & regular n-gonal base of each side a, is given as

$$\boldsymbol{\theta_d = 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{n} - \frac{a^2}{4l^2}\cos^2\frac{\pi}{n}}\right)} \qquad \ldots\ldots\ldots\ldots (14)$$

11.6. Total surface area (A_s) of right pyramid (base + lateral surface)

Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula for total surface area of pyramidal flat container with regular n-gonal base, the total surface area of a right pyramid with regular n-gonal base, is obtained as follows

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\cos\theta\,\tan\frac{\pi}{n}\right)$$

$$= \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a + l\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)\tan\frac{\pi}{n}\right) \qquad \text{(setting the value of } \cos\theta)$$

$$= \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + nl\left(a - \frac{a}{2}\right)$$

$$= \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + n\left(\frac{1}{2}al\right)$$

$$= \text{Area of regular polygonal base} + \text{Area of lateral surface}$$

Hence, the total surface area (A_s) (base + lateral surface) of a right pyramid with slant height l & regular n-gonal base of each side a, is given as

$$\boldsymbol{A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + n\left(\frac{1}{2}al\right)} \qquad \ldots\ldots\ldots\ldots (15)$$

11.7. Volume (V) of right pyramid

Substituting $\cos\theta = \frac{-a}{2l}\cot\frac{\pi}{n}$ in generalized formula for volume of pyramidal flat container with regular n-gonal base, the volume of a right pyramid with regular n-gonal base, is obtained as follows

$$V = \frac{nl\sin\theta}{12}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{nl}{12}\sqrt{1 - \cos^2\theta}\left(3a^2\cot\frac{\pi}{n} + 4l^2\cos^2\theta\tan\frac{\pi}{n} + 6al\cos\theta\right)$$

$$= \frac{nl}{12}\sqrt{1 - \left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2}\left(3a^2\cot\frac{\pi}{n} + 4l^2\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)^2\tan\frac{\pi}{n} + 6al\left(\frac{-a}{2l}\cot\frac{\pi}{n}\right)\right)$$

$$= \frac{nl}{12}\sqrt{1 - \frac{a^2}{4l^2}\cot^2\frac{\pi}{n}}\left(3a^2\cot\frac{\pi}{n} + a^2\cot\frac{\pi}{n} - 3a^2\cot\frac{\pi}{n}\right)$$

$$= \frac{n}{12}\sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}}\left(a^2\cot\frac{\pi}{n}\right)$$

$$= \frac{1}{3} \times \frac{1}{4}na^2\cot\frac{\pi}{n}\sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}}$$

$$= \frac{1}{3} \times \text{(Area of regular polygonal base)} \times \text{(Normal height)}$$

Hence, the volume (V) of a right pyramid with slant height l & regular n-gonal base of each side a, is given as

$$\boldsymbol{V = \frac{1}{12}na^2\cot\frac{\pi}{n}\sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}}} \qquad \dots\dots\dots\dots(16)$$

11.8. Construction of right pyamid with regular n-gonal base

The general steps, for making a right pyramid having regular n-gonal base of each side a, slant height l [2,3,4], using a thin sheet of paper, polymer, metal or alloy of desired thickness which can easily be cut, bent and butt-joined at mating edges, are the same as that of making a pyramidal flat container (as discussed above in details) but there is also an alternative & easier way to make a right pyramid with desired dimensions as follows

Step 1: Draw a circle with radius $r = \frac{a}{2}cosec\frac{\pi}{n}$ & centre 'O' on the sheet of desired material and thickness (see fig. 17 below).

Step 2: Draw another circle of radius $R = \frac{a}{2}cosec\frac{\pi}{n} + l\sec\frac{\pi}{n}$ concentric with small circle of radius r on the sheet (see fig. 17 below).

Step 3: Take aperture angle $\frac{2\pi}{n}$ & draw a regular polygon $A_1A_2A_3 \dots\dots A_{n-1}A_n$ with n no. of sides each of length a circumscribed by the small circle of radius r (as shown in fig-17 below).

Step 4: Join the vertices $A_1, A_2, A_3, \dots\dots A_{n-1}$ & A_n to the centre O by dotted straight lines which when extended intersect big circle at the points $B_1, B_2, B_3 \dots\dots B_{n-1}$ & B_n. Joint these points by straight lines to get a big regular n-gon $B_1B_2B_3 \dots\dots B_{n-1}B_n$ (as shown in fig-17 below below).

Step 5: Draw V-cut angle $\delta = \frac{2\pi}{n} + 2\tan^{-1}\left(\frac{a}{2l}\right)$ at each vertex of small regular n-gon such that δ is bisected by the radial line passing through that vertex or alternatively join the mid-point of each side of big regular n-gon to two closest vertices of small regular n-gon to get the same drawing (as shown in fig-18 below).

Step 6: Mark V-cut parts & undesired area (as shaded in the fig-19 below) & cut remove undesired area from big circular sheet of radius R to get a blank of sheet (as shown in the fig-20 below).

Step 7: Rotate or bend each of triangular faces about the edges of regular n-gonal base $A_1A_2A_3 \dots\dots A_{n-1}A_n$ until their new edges (generated after cutting V-parts) coincide with one another & meet at apex. Then butt-join the mating lateral edges using a suitable adhesive or welding process to get a desired right pyramid.
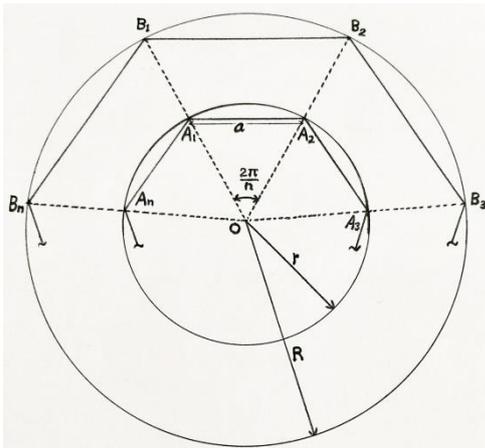
Figure-17: Small regular n-gon is circumscribed by circle of radius r & big regular n-gon is obtained by joining points of intersection of extended radial lines and the big circle.
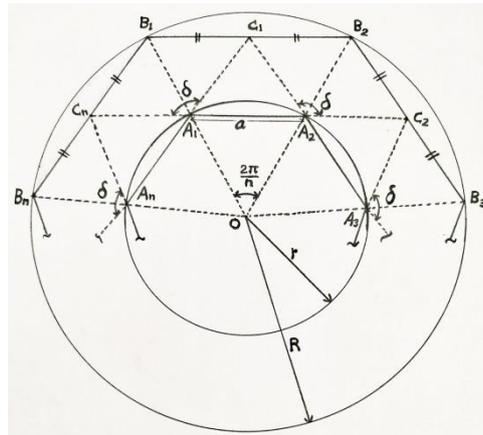

Figure-18: V-cut angle is made at each vertex of small regular n-gon to cut remove undesired area from big circle of radius R.
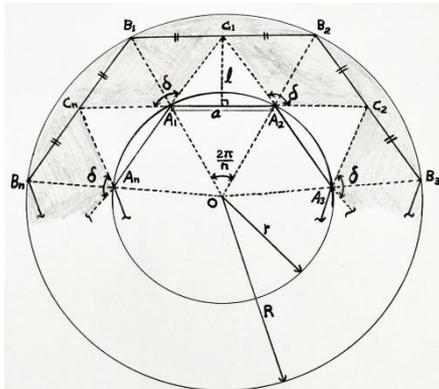

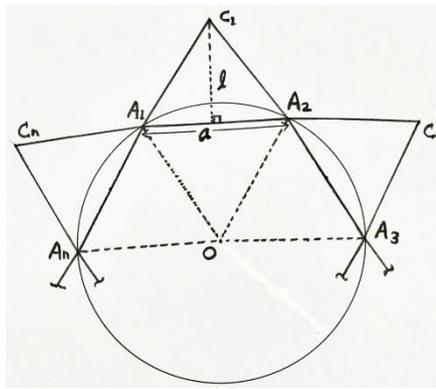**Figure-19: Undesired area to be cut removed is shaded.**

Figure-20: Blank of sheet after cut-removing undesired area.

Thus, a desired right pyramid (as obtained above) with a regular n-gonal base of each side a & slant height l, has the following important parameters

1. Length (L) of each lateral edge of right pyramid, is given as

$$L = \sqrt{l^2 + \left(\frac{a}{2}\right)^2}$$

2. Normal height (h) of right pyramid, is given as

$$h = \sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}}$$

3. Angle of inclination (θ) of each isosceles triangular lateral face with the plane of regular n-gonal base, is given as

$$\theta = \pi - \cos^{-1}\left(\frac{a}{2l}\cot\frac{\pi}{n}\right)$$

4. Dihedral angle (θ_d) between any two consecutive lateral faces of right pyramid, is given as

$$\theta_d = 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{n} - \frac{a^2}{4l^2}\cos^2\frac{\pi}{n}}\right)$$

5. Total surface area (A_s) of right pyramid (base + lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + n\left(\frac{1}{2}al\right)$$

6. Volume (V) of right pyramid, is given as

$$V = \frac{1}{12}na^2\cot\frac{\pi}{n}\sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}}$$

We will now make a paper model of a regular n-gonal right pyramid using the general steps as mentioned above & the generalized formula as derived in Sec. 11.8 above.

11.9. Right pyramid (shell) with regular octagonal base: Let's make a paper model of right pyramid with regular octagonal base of each side $3\ cm$ & slant height $5\ cm$.

Given data: $n = $ number of sides of regular octagonal base $= 8$

$a = $ edge length of regular octagonal base $= 3$ cm

$l = $ slant height of right pyramid $= 5$ cm

The following steps (as shown in figure-21 below) are used to make a right pyramid (of desired dimensions) with regular octagonal base using a sheet of paper.

Step 1. Calculate radius (r) of small circle

$$r = \frac{a}{2}cosec\frac{\pi}{n} = \frac{3}{2}cosec\frac{\pi}{8} = 3.92\ cm$$

Draw a circle with radius $r = 3.92\ cm$ on the sheet of paper (see fig-21(i)).

Step 2. Calculate radius (R) of big circle

$$R = \frac{a}{2}cosec\frac{\pi}{n} + l\sec\frac{\pi}{n} = \frac{3}{2}cosec\frac{\pi}{8} + 5\sec\frac{\pi}{8} = 9.33\ cm$$

Draw another circle of radius $R = 9.33\ cm$ concentric with small circle (see fig-21(i)).

Step 3. Take aperture angle $\frac{2\pi}{n} = \frac{2\pi}{8} = 45^o$ & draw a regular octagon of each side $a = 3\ cm$ circumscribed by the small circle of radius $3.92\ cm$ (as shown in fig-21(ii) below). Join the vertices of regular octagon to the centre by dotted straight lines and extend these lines so as to intersect big circle of radius 9.33 cm (as shown in fig-21(iii) below).

Step 4. Join the points of intersection of radial lines & big circle by the straight lines to get a big regular octagon (see fig-21(iv) below)**.**
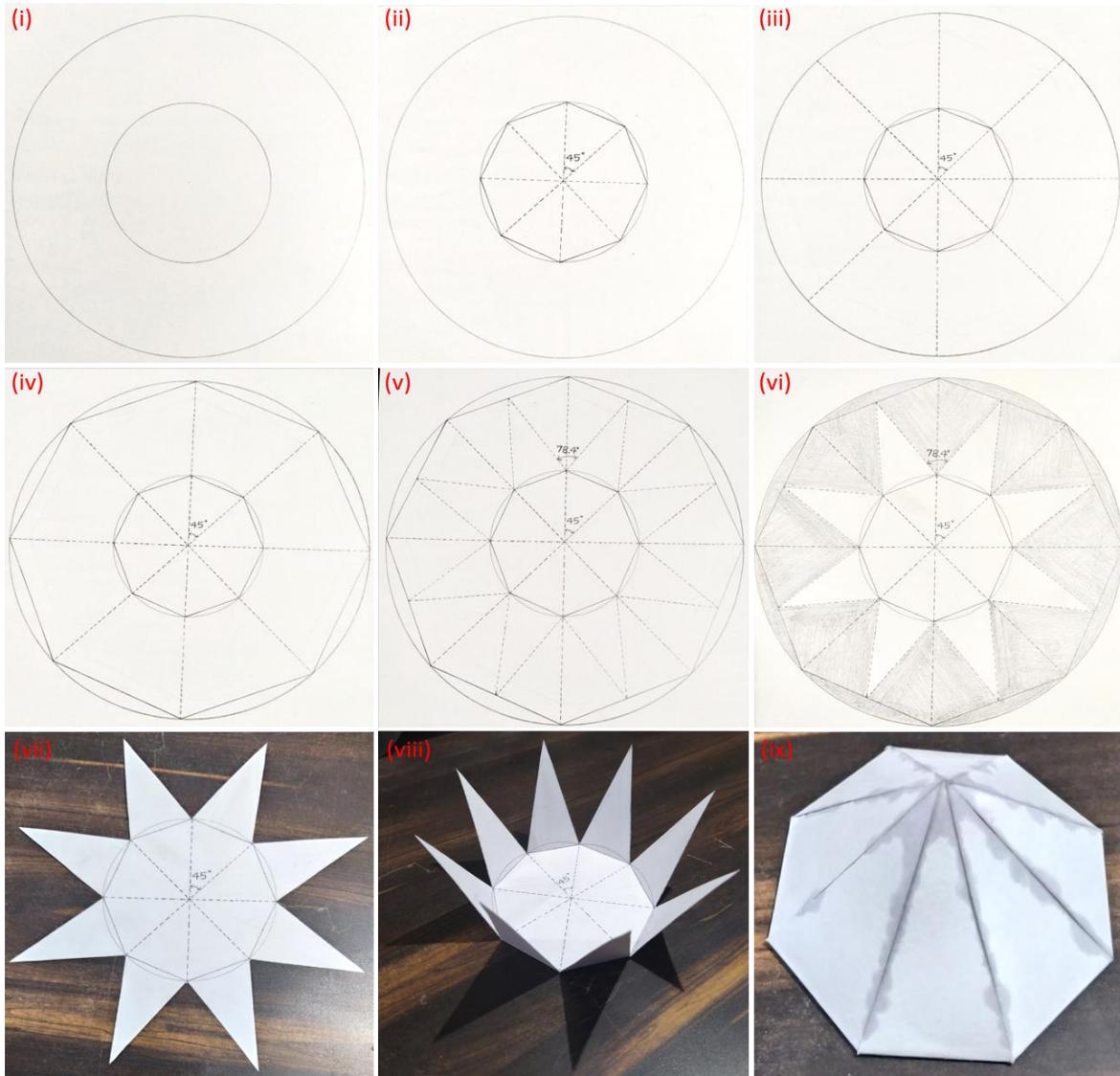
Figure-21: (i) Two concentric circles of radii 3.92 cm & 9.33 cm are drawn on the sheet of paper **(ii)** a regular octagon of side 3 cm drawn on the sheet of paper is circumscribed by small circle of radius 3.92 cm **(iii)** dotted straight lines joining the vertices of regular octagon to the centre are extended so as to intersect big circle of radius 9.33 cm **(iv)** a big regular octagon is obtained by joining points of intersection of radial lines and the big circle by straight lines **(v)** V-cut angle of 78.4^o is made at each vertex of small regular octagon **(vi)** undesired area (including V-parts) to be cut-removed from big circle is shaded **(vii)** blank of paper after cutting & removing undesired area (including V-parts) from big circle **(viii)** rotating isosceles triangular faces about the edges of regular octagon (base) which become lateral faces after meeting at the edges **(ix)** desired right pyramid after butt-joining the mating lateral edges using suitable adhesive.

Step 5. Calculate V-cut angle (δ)

$$\delta = \frac{2\pi}{n} + 2\tan^{-1}\left(\frac{a}{2l}\right) = \frac{2\pi}{8} + 2\tan^{-1}\left(\frac{3}{2 \times 5}\right) = 78.4^o$$

Draw V-cut angle $\delta = 78.4^o$ at each vertex of small regular octagon such that δ is bisected by the radial line passing through that vertex (as shown in fig-21(v) above) & then shade the undesired area (including V-parts) which is to be cut-removed from big circle of radius 9.33 cm (as shown in fig-21(vi) above).

Step 6. Cut & remove undesired area (as shaded in fig21(vi) above) from big circle (sheet) to get a blank of sheet (as shown in fig-21(vii) above). Rotate all the isosceles triangular faces about the edges of small regular octagon (base) (as shown in fig-21(viii) above).

Step 7. Rotate the isosceles triangular faces about the edges of regular octagon (base) until their new edges (generated after removing V-parts) coincide & then butt-join the mating lateral edges using suitable adhesive to get the desired right pyramid with regular octagonal base (as shown in fig-21(ix) above).

The desired right pyramid with a regular octagonal base has some important parameters which are analytically calculated by using generalized formula (as derived in Sec. 11.8 above) as follows

1. Length (L) of each lateral edge of right pyramid, is given as

$$L = \sqrt{l^2 + \left(\frac{a}{2}\right)^2} = \sqrt{(5)^2 + \left(\frac{3}{2}\right)^2} = 5.22 \ cm$$

2. Normal height (h) of right pyramid, is given as

$$h = \sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}} = \sqrt{(5)^2 - \frac{(3)^2}{4}\cot^2\frac{\pi}{8}} = 3.448 \ cm$$

3. Angle of inclination (θ) of each isosceles triangular lateral face with the plane of base , is given as

$$\theta = \pi - \cos^{-1}\left(\frac{a}{2l}\cot\frac{\pi}{n}\right) = \pi - \cos^{-1}\left(\frac{3}{2\times 5}\cot\frac{\pi}{8}\right) = 136.41^o$$

4. Dihedral angle (θ_d) between any two consecutive lateral faces of right pyramid, is given as

$$\theta_d = 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{n} - \frac{a^2}{4l^2}\cos^2\frac{\pi}{n}}\right) = 2\cos^{-1}\left(\sqrt{\sin^2\frac{\pi}{8} - \frac{3^2}{4(5)^2}\cos^2\frac{\pi}{8}}\right) = 149.4^o$$

5. Total surface area (A_s) of right pyramid (base + lateral surface), is given as

$$A_s = \frac{1}{4}n\,a^2\cot\frac{\pi}{n} + n\left(\frac{1}{2}al\right) = \frac{1}{4}(8)(3)^2\cot\frac{\pi}{8} + 8\left(\frac{1}{2}(3)(5)\right) = 103.456 \ cm^2$$

6. Volume (V) of right pyramid, is given as

$$V = \frac{1}{12}na^2\cot\frac{\pi}{n}\sqrt{l^2 - \frac{a^2}{4}\cot^2\frac{\pi}{n}} = \frac{1}{12}(8)(3)^2\cot\frac{\pi}{8}\sqrt{5^2 - \frac{3^2}{4}\cot^2\frac{\pi}{8}} = 49.94 \ cm^3$$

All the above values of important parameters of the right pyramid with a regular octagonal base can be verified by accurate and precise measurements.

12. Polyhedron with two regular n-gonal and 2n trapezoidal faces: So far, we have analysed in detail the pyramidal flat container with a regular n-gonal base & trapezoidal lateral faces. If we butt-join two identical pyramidal flat containers (having a regular n-gonal base), i.e., one pyramidal flat container inverted and put over the other, such that the edges of their regular n-gonal open ends coincide, then we get a closed surface i.e. a polyhedron with two regular n-gonal and 2n trapezoidal faces (i.e. a polyhedron as shown in fig-22 below) [5,6,7]. All the important parameters, like edge length, dihedral angle, surface area, and the volume of a

polyhedron with two regular n-gonal and 2n trapezoidal faces, can be determined by using a generalized formula for a pyramidal flat container with a regular n-gonal base as derived in Sec. 10 above.

Thus, we can make a desired polyhedron with 2 regular n-gonal and 2n trapezoidal faces by joining two identical pyramidal flat containers at the edges of their open ends i.e. one pyramidal flat container is inverted (upside down) over the other identical one (upright) such that the edges of their regular n-gonal open-ends coincide then the mating edges are butt-joined to get e desired polyhedron (as shown in figure-22 below).
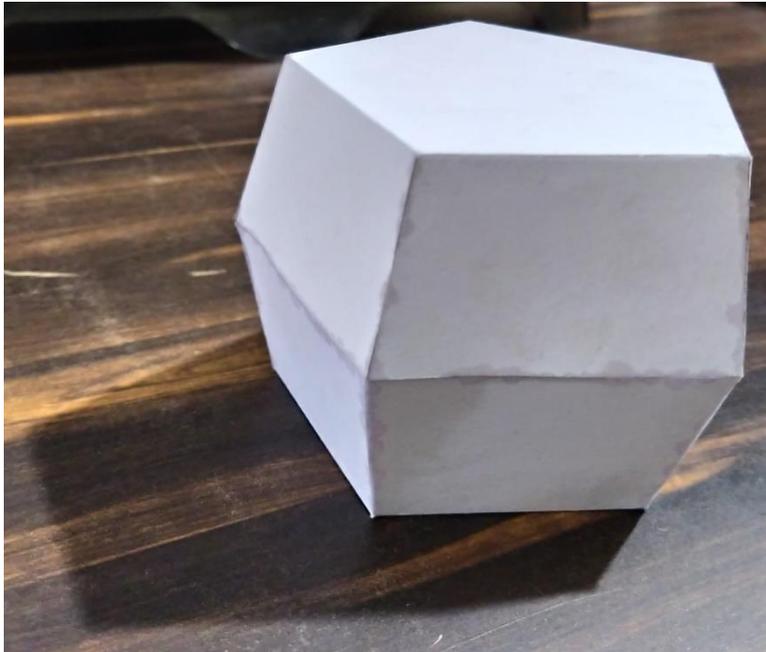

Figure-22: A paper model of a polyhedron with two regular pentagonal and ten trapezoidal faces is made by butt-joining two identical pyramidal flat containers, with a regular pentagonal base, at the edges of their regular pentagonal open-ends.

Note: Above articles had been *derived & illustrated* by **Mr H.C. Rajpoot** (**M Tech, Production Engineering**)

Indian Institute of Technology Delhi   *10 September 2019*

Email:hcrajpoot.iitd@gmail.com

Author's Home Page: https://notionpress.com/author/HarishChandraRajpoot

References

[1] Rajpoot HC. HCR's theorem and corollary on the rotation of two coplanar planes about their intersecting edges. ResearchGate; 2019 Aug. doi:10.13140/RG.2.2.35034.08643

[2] Rajpoot HC. Mathematical analysis of spheres at polyhedral vertices, face filleting, and sphere packing in right pyramids and Platonic solids. ResearchGate; 2015 Jun. doi:10.13140/RG.2.2.12157.99048

[3] Rajpoot HC. HCR's theorem for dihedral angles in a regular n-gonal right pyramid. 2015 Jul. doi:10.13140/RG.2.2.16488.74242.

[4] Rajpoot HC. Solid angle subtended by a regular n-gonal right pyramid (solid or hollow) at its apex. ResearchGate; 2019 Jul. doi:10.13140/RG.2.2.16771.72483

[5] Rajpoot HC. Mathematical analysis of non-uniform polyhedra with two regular n-gonal faces and 2n trapezoidal faces inscribed in a sphere. ResearchGate; 2015 Mar. doi:10.13140/RG.2.2.10899.69923

[6] Altes GK. Paper models of polyhedra. Acesso em: dez. 2013.

[7] Cromwell PR. Polyhedra. Cambridge University Press; 1997.