Mathematical analysis of non-uniform polyhedra with two regular n-gonal faces and 2n trapezoidal faces inscribed in a sphere

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1. Introduction

In this paper, a class of non-uniform polyhedra is investigated in detail. The polyhedron consists of two congruent regular n-gonal faces, 2n congruent trapezoidal faces, 5n edges & 3n all of which lie on a spherical surface of fixed radius. Each trapezoidal face has three equal side lengths, two equal acute angles α and two equal obtuse angles β, as illustrated in Fig. 1 for the case of a non-uniform tetradecahedron [1]. The constraint that all 3n vertices lie on a common spherical surface imposes strong geometric relations among the defining parameters of the polyhedron. As a consequence, key quantities such as the solid angle subtended by each face at the centre, the normal distance of each face from the centre, the circumscribed (outer) radius, the inscribed (inner) radius, the mean radius, the total surface area, and the enclosed volume can be systematically determined.

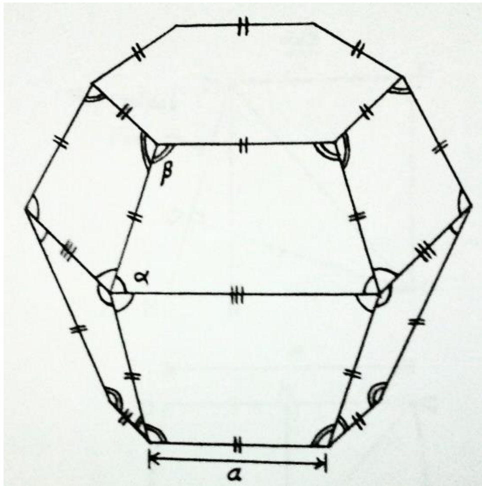

Figure 1: A non-uniform tetradecahedron has 2 congruent regular hexagonal faces each of edge length a & 12 congruent trapezoidal faces. All its 18 vertices eventually & exactly lie on a spherical surface with a certain radius.

It is shown that if the length of one of the two unequal edge types is specified, all remaining dimensions of the non-uniform polyhedron can be uniquely evaluated. In particular, when the edge length a of the regular n-gonal faces is known, the analysis simplifies significantly. A closed-form mathematical relation between the edge length a and the radius R of the circumscribed spherical surface is derived, enabling all geometrical dimensions of the polyhedron to be expressed solely in terms a. Furthermore, the plane angles and the solid angles associated with each face are obtained directly from this formulation [2,3].

$$\text{Total number of faces} = 2n + 2, \ \text{number of regular polygonal faces} = 2,$$
$$\text{noumber of trapezoidal faces} = 2n, \ \text{number of edges} = 5n, \ \text{and number of vertices} = 3n$$

2. Analysis of Non-uniform Polyhedron

For ease of calculations & understanding, let there be a non-uniform polyhedron, with the centre O, having 2 congruent regular n-gonal faces each with edge length a & 2n congruent trapezoidal faces each with three equal sides each a and all its 3n vertices lying on a spherical surface with a radius R_o. Now consider any of 2n congruent trapezoidal faces say ABCD ($AD = BC = CD = a$) & join the vertices A & D to the centre O. (See the Figure 2). Join the centre E of the top regular n-gonal face to the centre O & to the vertex D. Draw a perpendicular DF from the vertex D to the line AO, perpendicular EM from the centre E to the side CD, perpendicular ON from the centre O to the side AB & then join the mid-points M & N of the sides CD & AB respectively in order to obtain trapeziums ADEO & OEMN (See the Figures 3 & 4 below). Now we have,

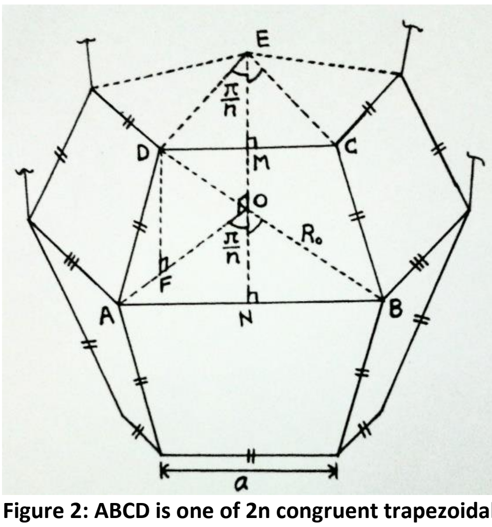

Figure 2: ABCD is one of 2n congruent trapezoidal faces with $AD = BC = CD = a$. ΔCED & ΔAOB are isosceles triangles. ADEO & OEMN are trapeziums.

$$OA = OB = OD = R_o\ , \ AD = \ BC = CD = a, \ \angle CED = \angle AOB = \frac{2\pi}{n}$$

Hence, in isosceles triangles ΔCED & ΔAOB , we have

$$\boldsymbol{EC = ED} \text{ and } \boldsymbol{CD = a}, \text{and } \boldsymbol{OA = OB = R_o}$$

In right ΔEMD (Fig. 2),

$$sin\angle DEM = \frac{DM}{ED} \Rightarrow sin\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{ED} \Rightarrow \boldsymbol{ED = \frac{a}{2}cosec\frac{\pi}{n} = EC = OF}$$

$$tan\angle\, DEM = \frac{DM}{EM} \Rightarrow tan\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{EM} \Rightarrow \boldsymbol{EM = \frac{a}{2}cot\frac{\pi}{n} = OH}$$

In right ΔANO (Fig. 2),

$$sin\angle AON = \frac{AN}{OA} \Rightarrow sin\frac{\pi}{n} = \frac{AN}{R_o} \Rightarrow \boldsymbol{AN = R_o sin\frac{\pi}{n} = NB}$$

$$cos\angle AON = \frac{ON}{OA} \Rightarrow cos\frac{\pi}{n} = \frac{ON}{R_o} \Rightarrow \boldsymbol{ON = R_o cos\frac{\pi}{n}}$$

In right ΔOED (Fig. 3),

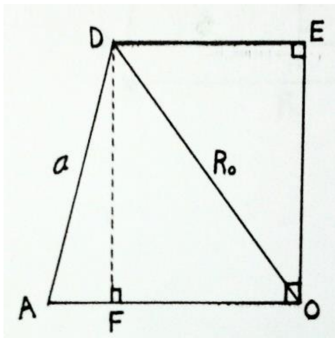

Figure 3: Trapezium ADEO with $AD = a$, $OA = OD = R_o$ & $DF = EO$. The lines DE & AO are parallel.

$$EO = \sqrt{(OD)^2 - (DE)^2} = \sqrt{{R_o}^2 - \left(\frac{a}{2}cosec\frac{\pi}{n}\right)^2}$$

$$\Rightarrow \boldsymbol{EO = DF = MH = \frac{1}{2}\sqrt{4{R_o}^2 - a^2cosec^2\frac{\pi}{n}}}$$

In right ΔAFD (Fig. 3),

$$\Rightarrow (AD)^2 = (AF)^2 + (DF)^2 = (OA - OF)^2 + (DF)^2$$

$$\Rightarrow a^2 = \left(R_o - \frac{a}{2}cosec\frac{\pi}{n}\right)^2 + \left(\frac{1}{2}\sqrt{4{R_o}^2 - a^2cosec^2\frac{\pi}{n}}\right)^2$$

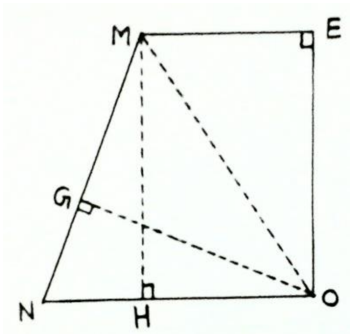

Figure 4: Trapezium OEMN with $EM = OH$ & $EO = MH$. The lines ME & NO are parallel.

$$\Rightarrow a^2 = {R_o}^2 + \frac{a^2}{4}cosec^2\frac{\pi}{n} - aR_o cosec\frac{\pi}{n} + {R_o}^2 - \frac{a^2}{4}cosec^2\frac{\pi}{n}$$

$$2{R_o}^2 - aR_o cosec\frac{\pi}{n} - a^2 = 0$$

$$\Rightarrow R_o = \frac{acosec\frac{\pi}{n} \pm \sqrt{\left(-acosec\frac{\pi}{n}\right)^2 + 8a^2}}{4}$$

$$= \frac{acosec\frac{\pi}{n} \pm a\sqrt{8 + cosec^2\frac{\pi}{n}}}{4} = \frac{a}{4}\left(cosec\frac{\pi}{n} \pm \sqrt{8 + cosec^2\frac{\pi}{n}}\right)$$

But, $R_o > a > 0$ by taking positive sign, we get

$$\therefore \quad \boldsymbol{R_o = \frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)} \quad \dots\dots\dots\dots\dots\dots (I)$$

Now, draw a perpendicular OG from the centre O to the trapezoidal face ABCD, perpendicular MH from the mid-point M of the side CD to the line ON. Thus in trapezium OEMN (see the above Figure 4), we have

$$MH = EO = \frac{1}{2}\sqrt{4{R_o}^2 - a^2 cosec^2\frac{\pi}{n}} = \sqrt{\left(\frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\right)^2 - \frac{a^2}{4}cosec^2\frac{\pi}{n}}$$

$$= a\sqrt{\frac{1}{16}cosec^2\frac{\pi}{n} + \frac{1}{16}\left(8 + cosec^2\frac{\pi}{n}\right) + \frac{1}{8}\left(cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}\right) - \frac{1}{4}cosec^2\frac{\pi}{n}}$$

$$= a\sqrt{\frac{1}{16}cosec^2\frac{\pi}{n} + \frac{1}{2} + \frac{1}{16}cosec^2\frac{\pi}{n} + \frac{1}{8}cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - \frac{1}{4}cosec^2\frac{\pi}{n}}$$

$$= a\sqrt{\frac{1}{2} + \frac{1}{8}cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - \frac{1}{8}cosec^2\frac{\pi}{n}} = a\sqrt{\frac{4 + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - cosec^2\frac{\pi}{n}}{8}}$$

$$\therefore \boldsymbol{MH = EO = \frac{a}{2}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}}} \quad \dots\dots\dots\dots\dots\dots (II)$$

$$\boldsymbol{NH} = ON - OH = ON - EM = R_o cos\frac{\pi}{n} - \frac{a}{2}cot\frac{\pi}{n} \quad \text{(see figure 2 \& 4)}$$

$$= \frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)cos\frac{\pi}{n} - \frac{a}{2}cot\frac{\pi}{n} \quad \text{(setting the value of } R_o \text{ from eq(I))}$$

$$= \frac{a}{4}\left(cot\frac{\pi}{n} + cos\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}\right) - \frac{a}{2}cot\frac{\pi}{n} = \frac{a}{4}cos\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - \frac{a}{4}cot\frac{\pi}{n}$$

$$= \frac{a}{4}\left(cos\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - cot\frac{\pi}{n}\right)$$

In right ΔMHN (Fig. 4 above),

$$MN = \sqrt{(MH)^2 + (NH)^2} = \sqrt{(EO)^2 + (NH)^2}$$

$$= \sqrt{\left(\frac{a}{2}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}}\right)^2 + \left(\frac{a}{4}\left(cos\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - cot\frac{\pi}{n}\right)\right)^2}$$

$$= a\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{8} + \frac{1}{16}\left(cos^2\frac{\pi}{n}\left(8 + cosec^2\frac{\pi}{n}\right) + cot^2\frac{\pi}{n} - 2cos\frac{\pi}{n}cot\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}\right)}$$

$$=\frac{a}{4}\sqrt{8-2cosec^2\frac{\pi}{n}+2cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}+8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot^2\frac{\pi}{n}-2cos^2\frac{\pi}{n}cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}$$

$$=\frac{a}{4}\sqrt{8-2-2cot^2\frac{\pi}{n}+8cos^2\frac{\pi}{n}+2cot^2\frac{\pi}{n}+2\left(1-cos^2\frac{\pi}{n}\right)cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}$$

$$=\frac{a}{4}\sqrt{6+8cos^2\frac{\pi}{n}+2sin^2\frac{\pi}{n}cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}\quad=\frac{a}{4}\sqrt{6+8cos^2\frac{\pi}{n}+2\sqrt{8sin^2\frac{\pi}{n}+1}}$$

$$=\frac{a}{4}\sqrt{6+8cos^2\frac{\pi}{n}+2\sqrt{9-8cos^2\frac{\pi}{n}}}=\frac{a}{2}\sqrt{\frac{6+8cos^2\frac{\pi}{n}+2\sqrt{9-8cos^2\frac{\pi}{n}}}{4}}$$

$$\therefore\quad \boldsymbol{MN=\frac{a}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}}\qquad \dots\dots\dots\dots\dots.(III)$$

Now, area of ΔOMN can be calculated as follows (from the above Figure 4)

$$\text{Area of }\Delta OMN=\frac{1}{2}[(MN)\times(OG)]=\frac{1}{2}[(ON)\times(MH)]\ \Rightarrow\ (MN)\times(OG)=(ON)\times(MH)$$

$$\Rightarrow\ OG=\frac{(ON)\times(MH)}{MN}=\frac{\left(R_o cos\frac{\pi}{n}\right)\times\left(\frac{a}{2}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}\right)}{\left(\frac{a}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)}$$

$$=\frac{\frac{a}{4}\left(cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)cos\frac{\pi}{n}\sqrt{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}{\sqrt{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$$

$$=\frac{\frac{a}{4}cos\frac{\pi}{n}\sqrt{\left(4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}\right)\left(cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)^2}}{\sqrt{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$$

$$=\frac{\frac{a}{4}cos\frac{\pi}{n}\sqrt{32+16cosec^2\frac{\pi}{n}+16cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}{\sqrt{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$$

$$= acos\frac{\pi}{n}\sqrt{\frac{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}} = a\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$$

$$\therefore\ \boldsymbol{OG = a\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots.(IV)$$

2.1. Normal distance (H_{n-gon}) of regular n-gonal faces from the centre of non-uniform polyhedron

The normal distance (H_{n-gon}) of each of 2 congruent regular n-gonal faces from the centre O of a non-uniform polyhedron is given as

$$H_{n-gon} = EO = \frac{a}{2}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}} \qquad \text{(from the eq(II) above)}$$

$$\therefore\ \boldsymbol{H_{n-gon} = \frac{a}{2}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}} \qquad \forall\ n \in N\ \&\ n \geq 3$$

It's clear that both the congruent regular n-gonal faces are at an equal normal distance H_{n-gon} from the centre of a non-uniform polyhedron.

2.2. Solid angle (ω_{n-gon}) subtended by each of 2 congruent regular n-gonal faces at the centre of non-uniform polyhedron

We know that the solid angle (ω) subtended by any regular polygon with each side of length a at any point lying at a distance H on the vertical axis passing through the centre of plane is given by following equation [2,3],

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\frac{2Hsin\frac{\pi}{n}}{\sqrt{4H^2+a^2cot^2\frac{\pi}{n}}}\right)}$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle subtended by each regular n-gonal face at the centre of the non-uniform polyhedron as follows

$$\omega_{n-gon} = 2\pi - 2n\sin^{-1}\left(\frac{2(EO)sin\frac{\pi}{n}}{\sqrt{4(EO)^2+a^2cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{2\left(\frac{a}{2}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}}\right)sin\frac{\pi}{n}}{\sqrt{4\left(\frac{a}{2}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}}\right)^2 + a^2cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{sin\frac{\pi}{n}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}}}{\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2} + cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{sin\frac{\pi}{n}\sqrt{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}}{\sqrt{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} + 2cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sqrt{4sin^2\frac{\pi}{n} + sin\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - 1}}{\sqrt{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} + 2cosec^2\frac{\pi}{n} - 2}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4sin^2\frac{\pi}{n} - 1 + \sqrt{8sin^2\frac{\pi}{n} + 1}}{2 + cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4 - 4cos^2\frac{\pi}{n} - 1 + \sqrt{8 - 8cos^2\frac{\pi}{n} + 1}}{2 + cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}}\right)$$

$$\therefore\ \boldsymbol{\omega_{n-gon} = 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{3 - 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2 + cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}}\right)} \qquad \forall\ n \in N\ \&\ n \geq 3$$

Now, the area of each of two regular n-gonal faces is given as,

$$Area\ of\ each\ n-gonal\ face,\ \ \boldsymbol{A_{n-gon} = \frac{1}{4}na^2cot\frac{\pi}{n}}$$

2.3. Normal distance (H_t) of trapezoidal faces from the centre of non-uniform polyhedron

The normal distance (H_t) of each of 2n congruent trapezoidal faces from the centre of non-uniform polyhedron is given as

$$H_t = OG = a\sqrt{\frac{2cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n} + cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n}}}{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}} \quad \text{(from the eq(IV) above)}$$

$$\therefore \ \boldsymbol{H_t = a\sqrt{\frac{2cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n} + cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n}}}{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}}} \quad \forall \ n \in N \ \& \ n \geq 3$$

It's clear that all 2n congruent trapezoidal faces are at an equal normal distance H_t from the centre of any non-uniform polyhedron.

2.4. Solid angle (ω_t) subtended by each of 2n congruent trapezoidal faces at the centre of non-uniform polyhedron

Since a non-uniform polyhedron is a closed surface & we know that the total solid angle, subtended by any closed surface at any point lying inside it, is $4\pi \ sr$ (Ste-radian) [4] hence the sum of solid angles subtended by 2 congruent regular n-gonal & 2n congruent trapezoidal faces at the centre of the non-uniform polyhedron must be $4\pi \ sr$. Thus we have

$$2[\omega_{n-gon}] + 2n[\omega_{trapezium}] = 4\pi \ \ or \ \ 2n[\omega_{trapezium}] = 4\pi - 2[\omega_{n-gon}]$$

$$\omega_{trapezium} = \frac{2\pi - \omega_{n-gon}}{n} = \frac{2\pi - \left[2\pi - 2n\sin^{-1}\left(\sqrt{\frac{3 - 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2 + cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}}\right)\right]}{n}$$

$$= 2\sin^{-1}\left(\sqrt{\frac{3 - 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2 + cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}}\right)$$

$$\therefore \ \boldsymbol{\omega_t = 2\sin^{-1}\left(\sqrt{\frac{3 - 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2 + cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}}\right)} \quad \forall \ n \in N \ \& \ n \geq 3$$

2.5. Interior angles $(\alpha \ \& \ \beta)$ of the trapezoidal faces of non-uniform polyhedron

From the above Figures 1 & 2, let α be acute angle & β be obtuse angle. Acute angle α is determined as follows

$$\sin\angle BAD = \frac{MN}{AD} \Rightarrow sin\alpha = \frac{\frac{a}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}}{a} \quad \text{(from eq(III) above)}$$

$$= \frac{1}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}} \quad or \ \ \alpha = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)$$

$$\therefore \text{Acute angle,} \ \ \boldsymbol{\alpha = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)} \quad \forall \ n \in N \ \& \ n \geq 3$$

In trapezoidal face ABCD, we know that the sum of all interior angles (of a quadrilateral) is 360^o.

$$\therefore 2\alpha + 2\beta = 360^o \ \ or \ \ \beta = 180^o - \alpha$$

$$\therefore \text{Obtuse angle,} \ \ \boldsymbol{\beta = 180^o - \alpha}$$

2.6. Sides of the trapezoidal face of non-uniform polyhedron

All the sides of each trapezoidal face can be determined as follows (see figure 2 above),

$$\boldsymbol{AD = BC = CD = a} \ \text{and} \ \boldsymbol{AB = 2R_o sin\frac{\pi}{n} = \frac{a}{2}\left(1+\sqrt{1+8sin^2\frac{\pi}{n}}\right)} \quad \text{(from eq(I) above)}$$

Distance between parallel sides AB & CD of trapezoidal face ABCD

$$\therefore \ \boldsymbol{MN = \frac{a}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}} \quad \text{(from eq(III) above)}$$

Hence, the area of each of 2n congruent trapezoidal faces of a non-uniform polyhedron is given as follows

$$\text{Area of trapezium } ABCD = \frac{1}{2}(\text{sum of parallel sides}) \times (\text{normal distance between parallel sides})$$

$$\Rightarrow A_t = \frac{1}{2}(AB+CD)(MN) = \frac{1}{2}(R_o + a)\left(\frac{a}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)$$

$$= \frac{a}{4}\left(\frac{a}{4}\left(cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)+a\right)\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}$$

$$= \frac{a^2}{16}\left(4+cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}$$

$$\therefore\ \boldsymbol{A_t = \frac{a^2}{16}\left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2}}}$$

3. Important parameters of a non-uniform polyhedron

1. **Inner (inscribed) radius ($\boldsymbol{R_i}$):** It is the radius of the largest sphere inscribed (trapped inside) by a non-uniform polyhedron. The largest inscribed sphere either touches both the congruent regular n-gonal faces or touches all 2n congruent trapezoidal faces depending on the value of no. of sides n of the regular polygonal face & is equal to the minimum value out of H_{n-gon} & H_t & is given as follows

$$\boldsymbol{R_i = Min\left(H_{n-gon},\ H_t\right)}$$

$$Where, \qquad H_{n-gon} = \frac{a}{2}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}} \qquad \&$$

$$H_t = a\sqrt{\frac{2cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n} + cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n}}}{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}}$$

2. **Outer (circumscribed) radius ($\boldsymbol{R_o}$):** It is the radius of the smallest sphere circumscribing a non-uniform polyhedron or it's the radius of a spherical surface passing through all 3n vertices of a non-uniform polyhedron. It is given from eq(I) as follows

$$\boldsymbol{R_o = \frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)}$$

3. **Surface area ($\boldsymbol{A_s}$):** We know that a non-uniform polyhedron has 2 congruent regular n-gonal faces & 2n congruent trapezoidal faces. Hence, its surface area is given as follows

$$A_s = 2(\text{area of regular polygon}) + 2\text{n}(\text{area of trapezium } ABCD) \qquad (\text{see Figure 2 above})$$

The area of any regular n-polygon with each side of length a is given from the following formula [5],

$$\boldsymbol{A = \frac{1}{4}na^2cot\frac{\pi}{n}}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 2 \times \left(\frac{1}{4}na^2cot\frac{\pi}{n}\right) + 2n \times \left(\frac{1}{2}(AB + CD)(MN)\right)$$

$$= 2 \times \left(\frac{1}{4}na^2cot\frac{\pi}{n}\right) + 2n \times \left(\frac{a^2}{16}\left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2}}\right)$$

$$=\frac{na^2}{8}\left(4cot\frac{\pi}{n}+\left(4+cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)$$

$$\therefore\ \boldsymbol{A_s=\frac{na^2}{8}\left(4cot\frac{\pi}{n}+\left(4+cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)}$$

4. **Volume $(\boldsymbol{V})$**: We know that a non-uniform polyhedron has 2 congruent regular n-gonal & 2n congruent trapezoidal faces. Hence, the volume (V) of the non-uniform polyhedron is the sum of volumes of all its $(2n+2)$ elementary right pyramids with regular n-gonal & trapezoidal bases (faces) given as follows

$$V=2(\text{volume of right pyramid with regular polygonal base})\\ +2n(\text{volume of right pyramid with trapezoidal base } ABCD)$$

$$=2\left(\frac{1}{3}(\text{area of regular polygon})\times H_{n-gon}\right)+2n\left(\frac{1}{3}(\text{area of trapezium } ABCD)\times H_t\right)$$

$$=2\left(\frac{1}{3}\left(\frac{1}{4}na^2cot\frac{\pi}{n}\right)\times\frac{a}{2}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}\right)$$

$$+2n\left(\frac{1}{3}\left(\frac{a^2}{16}\left(4+cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)\right.$$

$$\left.\times a\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}\right)$$

$$=\frac{1}{12}na^3cot\frac{\pi}{n}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}$$

$$+\frac{1}{24}na^3\left(4+cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{2}}$$

$$\therefore\ V=\frac{1}{24}na^3\left(2\cot\frac{\pi}{n}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}\right.$$

$$\left.+\left(4+cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{2\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}+\cot\frac{\pi}{n}\sqrt{8\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}}}{2}}\right)$$

$$\forall\ \boldsymbol{n}\in\boldsymbol{N}\ \&\ \boldsymbol{n}\geq\boldsymbol{3}$$

5. **Mean radius (R_m):** It is the radius of the sphere having a volume equal to that of a non-uniform polyhedron. It is calculated as follows

$$\text{Volume of sphere with mean radius } R_m=\text{volume of the non uniform polyhedron}$$

$$\frac{4}{3}\pi(R_m)^3=V\ \Rightarrow (R_m)^3=\frac{3V}{4\pi}\quad\Rightarrow\quad \boldsymbol{R_m}=\left(\frac{\boldsymbol{3V}}{\boldsymbol{4\pi}}\right)^{\frac{1}{3}}$$

For finite value of edge length a of regular n-gonal face $\Rightarrow$ $\boldsymbol{R_i}<\boldsymbol{R_m}<\boldsymbol{R_o}$

Hence, by setting different values of no. of sides $n=3,4,5,6,7\ldots\ldots\ldots\ldots\ldots$, we can find out all the important parameters of any non-uniform polyhedron with known value of side a of regular n-gonal face.

Conclusions: All the formula above are generalised which are applicable to calculate the important parameters, of any non-uniform polyhedron having 2 congruent regular n-gonal faces, 2n congruent trapezoidal faces with three equal sides, 5n edges & 3n vertices lying on a spherical surface, such as solid angle subtended by each face at the centre, normal distance of each face from the centre, inner radius, outer radius, mean radius, surface area & volume.

Let there be any non-uniform polyhedron having 2 congruent regular n-gonal faces each with edge length a, $2n$ congruent trapezoidal faces each with three sides equal to a & forth equal *to* $2R_o\sin{\pi}/{n}$, $5n$ edges and $3n$ vertices lying on a spherical surface then all its important parameters are calculated as tabulated below.

Congruent polygonal faces	No. of faces	Normal distance of each face from the centre of the non-uniform polyhedron	Solid angle subtended by each face at the centre of the non-uniform polyhedron (in sr)
Regular polygon	2	$\frac{a}{2}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}$	$2\pi-2n\sin^{-1}\sqrt{\frac{3-4\cos^2\frac{\pi}{n}+\sqrt{9-8\cos^2\frac{\pi}{n}}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}$
Trapezium	$2n$	$a\sqrt{\frac{2\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}+\cot\frac{\pi}{n}\sqrt{8\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}}}{3+4\cos^2\frac{\pi}{n}+\sqrt{9-8\cos^2\frac{\pi}{n}}}}$	$2\sin^{-1}\left(\sqrt{\frac{3-4\cos^2\frac{\pi}{n}+\sqrt{9-8\cos^2\frac{\pi}{n}}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}\right)$

Inner (inscribed) radius (R_i)	$R_i = Minimum\ normal\ distance\ of\ any\ face\ from\ the\ centre$
Outer (circumscribed) radius (R_o)	$R_o = \frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)$
Mean radius (R_m)	$R_m = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$
Surface area (A_s)	$A_s = \frac{na^2}{8}\left(4cot\frac{\pi}{n} + \left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2}}\right)$
Volume (V)	$V = \frac{1}{24}na^3\left(2cot\frac{\pi}{n}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}} + \left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{2cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n} + cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n}}}{2}}\right)$

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