

Mathematical derivations of inscribed & circumscribed radii for three externally tangent circles

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1. Introduction:

In this paper, the analytic formulas for the radii of the inscribed and circumscribed circles by three externally touching circles have been derived using geometry and trigonometry. Further analysis has been done for three external touching circles with a common tangent and enclosed in a smallest rectangle. The analytic formulas for the area of intersection and length of common chord for two intersecting circles have also been derived.

Consider three circles having centres A, B & C and radii a, b & c respectively, touching each other externally such that a small circle P is inscribed in the gap & touches them externally & a large circle Q circumscribes them & is touched by them internally. We are to calculate the radii of **inscribed circle P** (touching three circles with centres A, B & C externally) & **circumscribed circle Q** (touched by three circles with centres A, B & C internally) (See Figure 1).

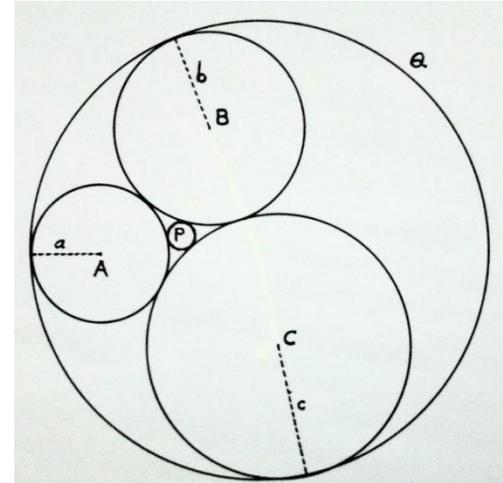


Figure 1: Three circles with centres A, B & C and radii a, b & c respectively are touching each other externally. Inscribed circle P & circumscribed circle Q are touching these circles externally & internally.

2. Derivation of the radius of inscribed circle: Let r be the radius of inscribed circle, with centre O, externally touching the given circles, having centres A, B & C and radii a, b & c , at the points M, N & P respectively. Now join the centre O to the centres A, B & C by dotted straight lines to obtain ΔAOB , ΔBOC & ΔAOC & also join the centres A, B & C by dotted straight lines to obtain ΔABC (as shown in Figure 2 below). Thus we have

$$AM = a, \quad BN = b, \quad CP = c \quad \&$$

$$OM = ON = OP = r \text{ (radius of inscribed circle)}$$

In ΔABC (see Fig. 2)

$$AB = a + b, \quad BC = b + c \quad \& \quad AC = a + c$$

$$\Rightarrow \text{semiperimeter} = \frac{AB + BC + AC}{2}$$

$$s = \frac{a + b + b + c + a + c}{2} = a + b + c$$

$$\sin \frac{\angle ACB}{2} = \sqrt{\frac{(s - AC)(s - BC)}{(AC)(BC)}}$$

$$\Rightarrow \sin \frac{\alpha}{2} = \sqrt{\frac{(a + b + c - a - c)(a + b + c - b - c)}{(a + c)(b + c)}}$$

$$\sin \frac{\alpha}{2} = \sqrt{\frac{ab}{(a + c)(b + c)}} \quad \dots \dots \dots (I)$$

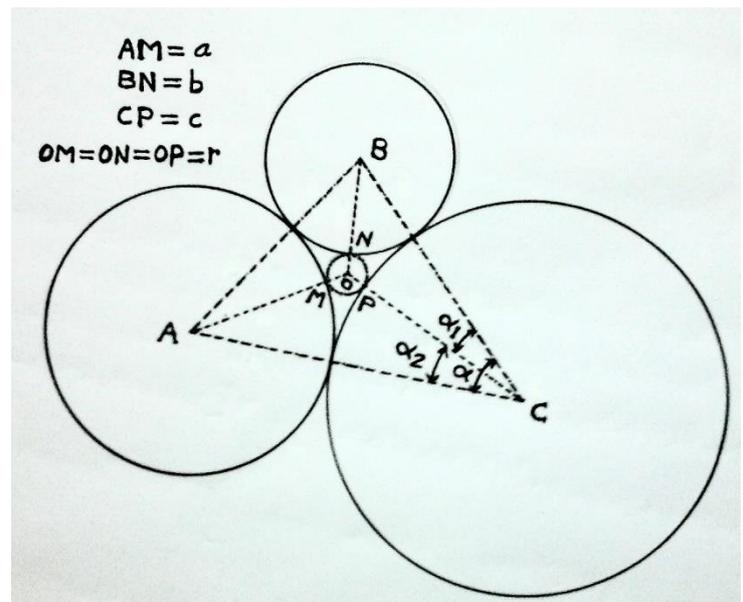


Figure 2: The centres A, B, C & O are joined to each other by dotted straight lines to obtain $\Delta ABC, \Delta AOB, \Delta BOC$ & ΔAOC .

Similarly, in ΔBOC (Fig. 2)

$$\begin{aligned} \text{semiperimeter} &= \frac{OB + BC + OC}{2} \\ \Rightarrow s &= \frac{r + b + b + c + r + c}{2} = b + c + r \\ \cos \frac{\angle BCO}{2} &= \sqrt{\frac{s(s - OB)}{(BC)(OC)}} \\ \Rightarrow \cos \frac{\alpha_1}{2} &= \sqrt{\frac{(b + c + r)(b + c + r - r - b)}{(b + c)(r + c)}} = \sqrt{\frac{c(b + c + r)}{(b + c)(c + r)}} \\ \cos \frac{\alpha_1}{2} &= \sqrt{\frac{c(b + c + r)}{(b + c)(c + r)}} \dots \dots \dots (II) \end{aligned}$$

Similarly, in ΔAOC (see Fig. 2)

$$\begin{aligned} \text{semiperimeter} &= \frac{OA + AC + OC}{2} \Rightarrow s = \frac{r + a + a + c + r + c}{2} = a + c + r \\ \cos \frac{\angle ACO}{2} &= \sqrt{\frac{s(s - OA)}{(AC)(OC)}} \Rightarrow \cos \frac{\alpha_2}{2} = \sqrt{\frac{(a + c + r)(a + c + r - r - a)}{(a + c)(r + c)}} = \sqrt{\frac{c(a + c + r)}{(a + c)(c + r)}} \\ \cos \frac{\alpha_2}{2} &= \sqrt{\frac{c(a + c + r)}{(a + c)(c + r)}} \dots \dots \dots (III) \end{aligned}$$

Now, again in ΔABC , we have

$$\angle ACO + \angle BCO = \angle ACB \Rightarrow \alpha_2 + \alpha_1 = \alpha \text{ or } \frac{\alpha_1}{2} + \frac{\alpha_2}{2} = \frac{\alpha}{2}$$

Now, taking **cosines** on both the sides of above equation, we have

$$\begin{aligned} \cos \left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2} \right) &= \cos \frac{\alpha}{2} \Rightarrow \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} - \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} = \cos \frac{\alpha}{2} \\ &\Rightarrow \left(\cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} - \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \right)^2 = \left(\cos \frac{\alpha}{2} \right)^2 \\ &\Rightarrow \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} + \sin^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha_2}{2} - 2 \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} = \cos^2 \frac{\alpha}{2} \\ &\Rightarrow \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} + \left(1 - \cos^2 \frac{\alpha_1}{2} \right) \left(1 - \cos^2 \frac{\alpha_2}{2} \right) - \cos^2 \frac{\alpha}{2} = 2 \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \\ &\Rightarrow \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} + 1 - \cos^2 \frac{\alpha_1}{2} - \cos^2 \frac{\alpha_2}{2} + \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha}{2} = 2 \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \\ &\Rightarrow 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} - \cos^2 \frac{\alpha_2}{2} + \sin^2 \frac{\alpha}{2} = 2 \cos \frac{\alpha_1}{2} \cos \frac{\alpha_2}{2} \sqrt{\left(1 - \cos^2 \frac{\alpha_1}{2} \right)} \sqrt{\left(1 - \cos^2 \frac{\alpha_2}{2} \right)} \\ &\Rightarrow \left(2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} - \cos^2 \frac{\alpha_2}{2} + \sin^2 \frac{\alpha}{2} \right)^2 = 4 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \left(1 - \cos^2 \frac{\alpha_1}{2} \right) \left(1 - \cos^2 \frac{\alpha_2}{2} \right) \end{aligned}$$

$$\begin{aligned}
&\Rightarrow 4 \cos^4 \frac{\alpha_1}{2} \cos^4 \frac{\alpha_2}{2} - 2 \cos^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - 2 \cos^2 \frac{\alpha_1}{2} \cos^4 \frac{\alpha_2}{2} + 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha}{2} - 2 \cos^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \\
&\quad + \cos^4 \frac{\alpha_1}{2} + \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha}{2} - 2 \cos^2 \frac{\alpha_1}{2} \cos^4 \frac{\alpha_2}{2} + \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \\
&\quad + \cos^4 \frac{\alpha_2}{2} - \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha}{2} + 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha}{2} - \cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha}{2} - \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha}{2} \\
&\quad + \sin^4 \frac{\alpha}{2} = 4 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - 4 \cos^4 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - 4 \cos^2 \frac{\alpha_1}{2} \cos^4 \frac{\alpha_2}{2} + 4 \cos^4 \frac{\alpha_1}{2} \cos^4 \frac{\alpha_2}{2} \\
&\Rightarrow \cos^4 \frac{\alpha_1}{2} + \cos^4 \frac{\alpha_2}{2} + \sin^4 \frac{\alpha}{2} + 4 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha}{2} - 2 \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} - 2 \cos^2 \frac{\alpha_1}{2} \sin^2 \frac{\alpha}{2} \\
&\quad - 2 \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha}{2} = 0 \\
&\Rightarrow \cos^4 \frac{\alpha_1}{2} + \cos^4 \frac{\alpha_2}{2} + \sin^4 \frac{\alpha}{2} + \cos^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_2}{2} (4 \sin^2 \frac{\alpha}{2} - 2) - 2 \sin^2 \frac{\alpha}{2} (\cos^2 \frac{\alpha_1}{2} + \cos^2 \frac{\alpha_2}{2}) = 0
\end{aligned}$$

Now, substituting all the corresponding values from eq(I), (II) & (III) in above expression, we have

$$\begin{aligned}
&\left(\sqrt{\frac{c(b+c+r)}{(b+c)(c+r)}} \right)^4 + \left(\sqrt{\frac{c(a+c+r)}{(a+c)(c+r)}} \right)^4 + \left(\sqrt{\frac{ab}{(a+c)(b+c)}} \right)^4 \\
&\quad + \left(\sqrt{\frac{c(b+c+r)}{(b+c)(c+r)}} \right)^2 \left(\sqrt{\frac{c(a+c+r)}{(a+c)(c+r)}} \right)^2 \left(4 \left(\sqrt{\frac{ab}{(a+c)(b+c)}} \right)^2 - 2 \right) \\
&\quad - 2 \left(\sqrt{\frac{ab}{(a+c)(b+c)}} \right)^2 \left\{ \left(\sqrt{\frac{c(b+c+r)}{(b+c)(c+r)}} \right)^2 + \left(\sqrt{\frac{c(a+c+r)}{(a+c)(c+r)}} \right)^2 \right\} = 0 \\
&\frac{c^2(b+c+r)^2}{(b+c)^2(c+r)^2} + \frac{c^2(a+c+r)^2}{(a+c)^2(c+r)^2} + \frac{a^2b^2}{(a+c)^2(b+c)^2} + \frac{c^2(a+c+r)(b+c+r)}{(a+c)(b+c)(c+r)^2} \left(\frac{4ab}{(a+c)(b+c)} - 2 \right) \\
&\quad - \frac{2abc}{(a+c)(b+c)(c+r)} \left\{ \frac{(b+c+r)}{(b+c)} + \frac{(a+c+r)}{(a+c)} \right\} = 0
\end{aligned}$$

Now, multiplying the above equation by $(a+c)^2(b+c)^2(c+r)^2$, we get

$$\begin{aligned}
&c^2(a+c)^2(b+c+r)^2 + c^2(b+c)^2(a+c+r)^2 + a^2b^2(c+r)^2 \\
&\quad + c^2(a+c+r)(b+c+r)(2ab - 2bc - 2ac - 2c^2) \\
&\quad - 2abc(c+r)\{(a+c)(b+c+r) + (b+c)(a+c+r)\} = 0 \\
&\Rightarrow c^2(a+c)^2\{r^2 + 2(b+c)r + (b+c)^2\} + c^2(b+c)^2\{r^2 + 2(a+c)r + (a+c)^2\} + a^2b^2\{r^2 + 2cr + c^2\} \\
&\quad + c^2(2ab - 2bc - 2ac - 2c^2)\{r^2 + (a+b+2c)r + (a+c)(b+c)\} \\
&\quad - 2abc(c+r)\{(a+b+2c)r + 2(a+c)(b+c)\} = 0 \\
&\Rightarrow \{c^2(a+c)^2 + c^2(b+c)^2 + a^2b^2 + 2c^2(ab - bc - ac - c^2)\}r^2 \\
&\quad + \{2c^2(a+c)(b+c)(a+b+2c) + 2a^2b^2c + 2c^2(a+b+2c)(ab - bc - ac - c^2)\}r \\
&\quad + 2c^2(a+c)^2(b+c)^2 + a^2b^2c^2 + 2c^2(a+c)(b+c)(ab - bc - ac - c^2) \\
&\quad - 2abc(a+b+2c)r^2 - 2abc\{2(a+c)(b+c) + c(a+b+2c)\}r - 4abc^2(a+c)(b+c) \\
&= 0
\end{aligned}$$

$$\begin{aligned} \Rightarrow \{a^2c^2 + c^4 + 2ac^3 + b^2c^2 + c^4 + 2bc^3 + a^2b^2 + 2abc^2 - 2bc^3 - 2ac^3 - 2c^4 - 2a^2bc - 2ab^2c \\ - 4abc^2\}r^2 \\ + \{2a^2bc^2 + 2a^2c^3 + 2bc^4 + 2c^5 + 4abc^3 + 4ac^4 + 2ab^2c^2 + 2b^2c^3 + 2ac^4 + 2c^5 \\ + 4abc^3 + 4bc^4 + 2a^2b^2c + 2a^2bc^2 - 2abc^3 - 2a^2c^3 - 2ac^4 + 2ab^2c^2 - 2b^2c^3 \\ - 2abc^3 - 2bc^4 + 4abc^3 - 4bc^4 - 4ac^4 - 4c^5 - 4a^2b^2c - 6ab^2c^2 - 6a^2bc^2 - 8abc^3\}r \\ + \{a^2b^2c^2 + 4a^2b^2c^2 + 4ab^2c^3 + 4a^2bc^3 + 4abc^4 - 4a^2b^2c^2 - 4ab^2c^3 - 4a^2bc^3 \\ - 4abc^4\} = 0 \end{aligned}$$

$$\Rightarrow \{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)\}r^2 - 2abc(ab + bc + ca)r + a^2b^2c^2 = 0$$

Now, solving the above quadratic equation for the values of r as follows

$$\begin{aligned} r &= \frac{2abc(ab + bc + ca) \pm \sqrt{\{2abc(ab + bc + ca)\}^2 - 4\{a^2b^2c^2\}\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)\}}}{2\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)\}} \\ &= \frac{2abc(ab + bc + ca) \pm 2abc\sqrt{4a^2bc + 4ab^2c + 4abc^2}}{2\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a + b + c)\}} = \frac{abc(ab + bc + ca) \pm abc\sqrt{4abc(a + b + c)}}{(ab + bc + ca)^2 - 4abc(a + b + c)} \\ \Rightarrow r &= abc \left(\frac{(ab + bc + ca) \pm 2\sqrt{abc(a + b + c)}}{(ab + bc + ca)^2 - (2\sqrt{abc(a + b + c)})^2} \right) \end{aligned}$$

Case 1: Taking positive sign, we get

$$r = abc \left(\frac{(ab + bc + ca) + 2\sqrt{abc(a + b + c)}}{(ab + bc + ca)^2 - (2\sqrt{abc(a + b + c)})^2} \right) = abc \left(\frac{1}{(ab + bc + ca) - 2\sqrt{abc(a + b + c)}} \right)$$

$\Rightarrow r < 0 \quad \forall a, b, c > 0$ but $r > 0$ hence this value of radius r is discarded

Case 2: Taking negative sign, we get

$$r = abc \left(\frac{(ab + bc + ca) - 2\sqrt{abc(a + b + c)}}{(ab + bc + ca)^2 - (2\sqrt{abc(a + b + c)})^2} \right) = abc \left(\frac{1}{(ab + bc + ca) + 2\sqrt{abc(a + b + c)}} \right)$$

$\Rightarrow r > 0 \quad \forall a, b, c > 0$ hence this value of radius r is accepted

Hence, the radius (r) of inscribed circle is given as

$$r = \frac{abc}{2\sqrt{abc(a + b + c)} + (ab + bc + ca)} \quad (r > 0 \quad \forall a, b, c > 0)$$

Above is the required expression to compute the radius (r) of the inscribed circle which externally touches three given circles with radii a, b & c touching each other externally.

3. Derivation of the radius of circumscribed circle: Let R be the radius of circumscribed circle, with centre O , is internally touched by the given circles, having centres A, B & C and radii a, b & c , at the points M, N & P respectively. Now join the centre O to the centres A, B & C by dotted straight lines to obtain $\triangle AOB, \triangle BOC$ & $\triangle AOC$ & also join the centres A, B & C by dotted straight lines to obtain $\triangle ABC$ (As shown in the Figure 3) Thus we have

$$AM = a, \quad BN = b, \quad CP = c \quad \&$$

$$OM = ON = OP = R \text{ (radius of circumscribed circle)}$$

$$OA = OM - AM = R - a, \quad OB = R - b \quad \& \quad OC = R - c$$

In $\triangle AOB$ (Fig. 3)

$$OA = R - a, \quad AB = a + b \quad \& \quad OB = R - b$$

$$\text{semiperimeter} = \frac{OA + AB + OB}{2}$$

$$s = \frac{R - a + a + b + R - b}{2} = R$$

$$\sin \frac{\angle AOB}{2} = \sqrt{\frac{(s - OA)(s - OB)}{(OA)(OB)}} = \sqrt{\frac{(R - R + a)(R - R + b)}{(R - a)(R - b)}} = \sqrt{\frac{ab}{(R - a)(R - b)}} \quad (\text{let } \angle AOB = \alpha)$$

$$\Rightarrow \sin \frac{\alpha}{2} = \sqrt{\frac{ab}{(R - a)(R - b)}} \quad \dots \dots \dots (I)$$

Similarly, in $\triangle BOC$ (Fig. 3)

$$\text{semiperimeter} = \frac{OB + BC + OC}{2}$$

$$s = \frac{R - b + b + c + R - c}{2} = R$$

$$\cos \frac{\angle BOC}{2} = \sqrt{\frac{s(s - BC)}{(OB)(OC)}} \Rightarrow \cos \frac{\alpha_1}{2} = \sqrt{\frac{R(R - b - c)}{(R - b)(R - c)}} \quad \dots \dots \dots (II)$$

Similarly, in $\triangle AOC$ (Fig. 3)

$$\text{semiperimeter} = \frac{OA + AC + OC}{2} \Rightarrow s = \frac{R - a + a + c + R - c}{2} = R$$

$$\cos \frac{\angle AOC}{2} = \sqrt{\frac{s(s - AC)}{(OA)(OC)}} \Rightarrow \cos \frac{\alpha_2}{2} = \sqrt{\frac{R(R - a - c)}{(R - a)(R - c)}} \quad \dots \dots \dots (III)$$

Now, again in $\triangle AOB$ (Fig. 3), we have

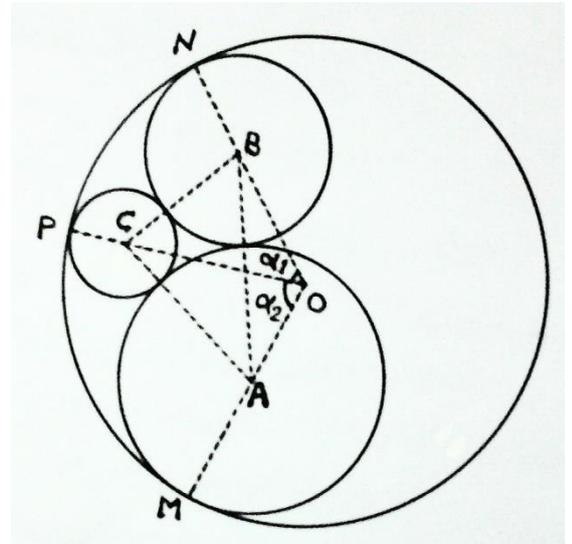


Figure 3: The centres A, B, C & O are joined to each other by dotted straight lines to obtain $\triangle ABC, \triangle AOB, \triangle BOC$ & $\triangle AOC$.

$$\angle BOC + \angle AOC = \angle AOB \Rightarrow \alpha_2 + \alpha_1 = \alpha \text{ or } \frac{\alpha_1}{2} + \frac{\alpha_2}{2} = \frac{\alpha}{2}$$

Now, taking **cosines** on both the sides, we have

$$\begin{aligned} \cos\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) &= \cos\frac{\alpha}{2} \Rightarrow \cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2} = \cos\frac{\alpha}{2} \\ &\Rightarrow \left(\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\right)^2 = \left(\cos\frac{\alpha}{2}\right)^2 \\ &\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} + \sin^2\frac{\alpha_1}{2}\sin^2\frac{\alpha_2}{2} - 2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} = \cos^2\frac{\alpha}{2} \\ &\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} + \left(1 - \cos^2\frac{\alpha_1}{2}\right)\left(1 - \cos^2\frac{\alpha_2}{2}\right) - \cos^2\frac{\alpha}{2} = 2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} \\ &\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} + 1 - \cos^2\frac{\alpha_1}{2} - \cos^2\frac{\alpha_2}{2} + \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - \cos^2\frac{\alpha}{2} = 2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} \\ &\Rightarrow 2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - \cos^2\frac{\alpha_1}{2} - \cos^2\frac{\alpha_2}{2} + \sin^2\frac{\alpha}{2} = 2\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\sqrt{\left(1 - \cos^2\frac{\alpha_1}{2}\right)}\sqrt{\left(1 - \cos^2\frac{\alpha_2}{2}\right)} \\ &\Rightarrow \left(2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - \cos^2\frac{\alpha_1}{2} - \cos^2\frac{\alpha_2}{2} + \sin^2\frac{\alpha}{2}\right)^2 = 4\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\left(1 - \cos^2\frac{\alpha_1}{2}\right)\left(1 - \cos^2\frac{\alpha_2}{2}\right) \\ &\Rightarrow 4\cos^4\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2} - 2\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - 2\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2} + 2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2} - 2\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} \\ &\quad + \cos^4\frac{\alpha_1}{2} + \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - \cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha}{2} - 2\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2} + \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} \\ &\quad + \cos^4\frac{\alpha_2}{2} - \cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2} + 2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2} - \cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha}{2} - \cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2} \\ &\quad + \sin^4\frac{\alpha}{2} = 4\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - 4\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - 4\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2} + 4\cos^4\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2} \\ &\Rightarrow \cos^4\frac{\alpha_1}{2} + \cos^4\frac{\alpha_2}{2} + \sin^4\frac{\alpha}{2} + 4\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2} - 2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - 2\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha}{2} \\ &\quad - 2\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2} = 0 \\ &\Rightarrow \cos^4\frac{\alpha_1}{2} + \cos^4\frac{\alpha_2}{2} + \sin^4\frac{\alpha}{2} + \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\left(4\sin^2\frac{\alpha}{2} - 2\right) - 2\sin^2\frac{\alpha}{2}\left(\cos^2\frac{\alpha_1}{2} + \cos^2\frac{\alpha_2}{2}\right) = 0 \end{aligned}$$

Now, substituting all the corresponding values from Eq.(I), (II) & (III) in above expression, we have

$$\begin{aligned} &\left(\sqrt{\frac{R(R-b-c)}{(R-b)(R-c)}}\right)^4 + \left(\sqrt{\frac{R(R-a-c)}{(R-a)(R-c)}}\right)^4 + \left(\sqrt{\frac{ab}{(R-a)(R-b)}}\right)^4 \\ &\quad + \left(\sqrt{\frac{R(R-b-c)}{(R-b)(R-c)}}\right)^2 \left(\sqrt{\frac{R(R-a-c)}{(R-a)(R-c)}}\right)^2 \left(4\left(\sqrt{\frac{ab}{(R-a)(R-b)}}\right)^2 - 2\right) \\ &\quad - 2\left(\sqrt{\frac{ab}{(R-a)(R-b)}}\right)^2 \left\{\left(\sqrt{\frac{R(R-b-c)}{(R-b)(R-c)}}\right)^2 + \left(\sqrt{\frac{R(R-a-c)}{(R-a)(R-c)}}\right)^2\right\} = 0 \end{aligned}$$

$$\begin{aligned} & \frac{R^2(R-b-c)^2}{(R-b)^2(R-c)^2} + \frac{R^2(R-a-c)^2}{(R-a)^2(R-c)^2} + \frac{a^2b^2}{(R-a)^2(R-b)^2} \\ & + \frac{R^2(R-b-c)(R-a-c)}{(R-a)(R-b)(R-c)^2} \left(\frac{4ab}{(R-a)(R-b)} - 2 \right) \\ & - \frac{2abR}{(R-a)(R-b)(R-c)} \left\{ \frac{(R-b-c)}{(R-b)} + \frac{(R-a-c)}{(R-a)} \right\} = 0 \end{aligned}$$

Now, on multiplying the above equation by $(R-a)^2(R-b)^2(R-c)^2$, we get

$$\begin{aligned} & R^2(R-a)^2(R-b-c)^2 + R^2(R-b)^2(R-a-c)^2 + a^2b^2(R-c)^2 \\ & + R^2(R-b-c)(R-a-c)\{2ab - 2R^2 + 2(a+b)R\} \\ & - 2abR(R-c)\{(R-a)(R-b-c) + (R-b)(R-a-c)\} = 0 \\ \Rightarrow & R^2(R-a)^2\{R^2 - 2(b+c)R + (b+c)^2\} + R^2(R-b)^2\{R^2 - 2(a+c)R + (a+c)^2\} \\ & + a^2b^2\{R^2 - 2cR + c^2\} \\ & + R^2\{R^2 - (a+b+2c)R + (a+c)(b+c)\}\{2ab - 2R^2 + 2(a+b)R\} \\ & - 2abR(R-c)\{2R^2 - 2(a+b+c)R + a(b+c) + b(a+c)\} = 0 \\ \Rightarrow & \{R^4 - 2(b+c)R^3 + (b+c)^2R^2\}(R^2 + a^2 - 2aR) + \{R^4 - 2(a+c)R^3 + (a+c)^2R^2\}(R^2 + b^2 - 2bR) \\ & + a^2b^2R^2 - 2a^2b^2cR + a^2b^2c^2 \\ & + \{R^4 - (a+b+2c)R^3 + (a+c)(b+c)R^2\}\{2ab - 2R^2 + 2(a+b)R\} \\ & + (2abcR - 2abR^2)\{2R^2 - 2(a+b+c)R + a(b+c) + b(a+c)\} = 0 \\ \Rightarrow & R^6 - 2(b+c)R^5 + (b+c)^2R^4 + a^2R^4 - 2a^2(b+c)R^3 + a^2(b+c)^2R^2 - 2aR^5 + 4a(b+c)R^4 \\ & - 2a(b+c)^2R^3 + R^6 - 2(a+c)R^5 + (a+c)^2R^4 + b^2R^4 - 2b^2(a+c)R^3 \\ & + b^2(a+c)^2R^2 - 2bR^5 + 4b(a+c)R^4 - 2b(a+c)^2R^3 + a^2b^2R^2 - 2a^2b^2cR + a^2b^2c^2 \\ & + 2abR^4 - 2ab(a+b+2c)R^3 + 2ab(a+c)(b+c)R^2 - 2R^6 + 2(a+b+2c)R^5 \\ & - 2(a+c)(b+c)R^4 - 4abc(a+b+c)R^2 + 4ab(a+b+c)R^3 + 2abc(2ab+ac+bc)R \\ & - 2ab(2ab+ac+bc)R^2 = 0 \\ \Rightarrow & \{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a+b+c)\}R^2 + 2abc(ab+bc+ca)R + a^2b^2c^2 = 0 \end{aligned}$$

Now, solving the above quadratic equation for the values of R as follows

$$\begin{aligned} R &= \frac{-2abc(ab+bc+ca) \pm \sqrt{\{-2abc(ab+bc+ca)\}^2 - 4\{a^2b^2c^2\}\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a+b+c)\}}}{2\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a+b+c)\}} \\ &= \frac{-2abc(ab+bc+ca) \pm 2abc\sqrt{4a^2bc + 4ab^2c + 4abc^2}}{2\{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a+b+c)\}} = \frac{-abc(ab+bc+ca) \pm abc\sqrt{4abc(a+b+c)}}{(ab+bc+ca)^2 - 4abc(a+b+c)} \\ \Rightarrow R &= abc \left(\frac{-(ab+bc+ca) \pm 2\sqrt{abc(a+b+c)}}{(ab+bc+ca)^2 - (2\sqrt{abc(a+b+c)})^2} \right) \end{aligned}$$

Case 1: Taking positive sign, we get

$$R = abc \left(\frac{-(ab+bc+ca) + 2\sqrt{abc(a+b+c)}}{(ab+bc+ca)^2 - (2\sqrt{abc(a+b+c)})^2} \right) = -abc \left(\frac{1}{(ab+bc+ca) + 2\sqrt{abc(a+b+c)}} \right)$$

$\Rightarrow R < 0 \quad \forall a, b, c > 0$ but $R > 0$ hence this value of radius R is discarded

Case 2: Taking negative sign, we get

$$R = abc \left(\frac{-(ab + bc + ca) - 2\sqrt{abc(a+b+c)}}{(ab + bc + ca)^2 - (2\sqrt{abc(a+b+c)})^2} \right) = -abc \left(\frac{1}{(ab + bc + ca) - 2\sqrt{abc(a+b+c)}} \right)$$

$$= abc \left(\frac{1}{2\sqrt{abc(a+b+c)} - (ab + bc + ca)} \right)$$

$\Rightarrow R > 0 \quad \forall a, b, c > 0$ hence this value of radius R is accepted

Hence, the radius (R) of circumscribed circle is given as

$$R = \frac{abc}{2\sqrt{abc(a+b+c)} - (ab + bc + ca)} \quad (R > 0 \quad \forall a, b, c > 0)$$

Above is the required expression to compute the radius (R) of the circumscribed circle which is internally touched by three given circles with radii a, b & c touching each other externally.

NOTE: The circumscribed circle will exist for three given radii a, b & c ($a \geq b \geq c > 0$) if & only if the following inequality is satisfied

$$c > \frac{ab}{(\sqrt{a} + \sqrt{b})^2}$$

For any other value of radius c (of smallest circle) not satisfying the above inequality, the circumscribed circle will not exist i.e. there will be no circle which circumscribes & internally touches three externally touching circles if the above inequality fails to hold good.

Special case: If three circles of equal radius a are touching each other externally then the radii r & R of inscribed & circumscribed circles respectively are obtained by setting $a = b = c = a$ in the above expressions as follows

$$\Rightarrow r = \frac{abc}{2\sqrt{abc(a+b+c)} + (ab + bc + ca)} = \frac{a^3}{2\sqrt{a^3(a+a+a)} + (a^2 + a^2 + a^2)} = \frac{a^3}{2\sqrt{3}a^2 + 3a^2}$$

$$= \frac{a}{2\sqrt{3} + 3} = \frac{a(2\sqrt{3} - 3)}{(2\sqrt{3} + 3)(2\sqrt{3} - 3)} = \frac{a(2\sqrt{3} - 3)}{3} = a \left(\frac{2}{\sqrt{3}} - 1 \right) \approx 0.154700538a$$

$$\Rightarrow R = \frac{abc}{2\sqrt{abc(a+b+c)} - (ab + bc + ca)} = \frac{a^3}{2\sqrt{a^3(a+a+a)} - (a^2 + a^2 + a^2)} = \frac{a^3}{2\sqrt{3}a^2 - 3a^2}$$

$$= \frac{a}{2\sqrt{3} - 3} = \frac{a(2\sqrt{3} + 3)}{(2\sqrt{3} - 3)(2\sqrt{3} + 3)} = \frac{a(2\sqrt{3} + 3)}{3} = a \left(\frac{2}{\sqrt{3}} + 1 \right) \approx 2.154700538a$$

4. Derivation of the radius of inscribed circle: Let r be the radius of inscribed circle, with centre C , externally touching two given externally touching circles, having centres A & B and radii a & b respectively, and their common tangent MN . Now join the centres A , B & C to each other as well as to the points of tangency M , N & P respectively by dotted straight lines. Draw the perpendicular AT from the centre A to the line BN . Also draw a line passing through the centre C & parallel to the tangent MN which intersects the lines AM & BN at the points Q & S respectively. (As shown in the Figure 4) Thus we have

$$AM = a, \quad BN = b, \quad CP = r = ?$$

In right $\triangle ATB$

$$AB = a + b \text{ \& } BT = BN - TN = BN - AM = b - a$$

$$\Rightarrow AT = \sqrt{(AB)^2 - (BT)^2}$$

$$= \sqrt{(a + b)^2 - (b - a)^2} = \sqrt{4ab} = 2\sqrt{ab}$$

$$\therefore AT = QS = MN = 2\sqrt{ab} \dots \dots \dots (I)$$

In right $\triangle AQC$ (Fig. 4)

$$AC = a + r \text{ \& } AQ = AM - QM = AM - CP = a - r$$

$$\Rightarrow QC = \sqrt{(AC)^2 - (AQ)^2}$$

$$= \sqrt{(a + r)^2 - (a - r)^2} = \sqrt{4ar} = 2\sqrt{ar} \therefore QC = MP = 2\sqrt{ar} \dots \dots \dots (II)$$

In right $\triangle BSC$ (Fig. 4)

$$BC = b + r \text{ \& } BS = BN - SN = BN - CP = b - r$$

$$\Rightarrow CS = \sqrt{(BC)^2 - (BS)^2} = \sqrt{(b + r)^2 - (b - r)^2} = \sqrt{4br} = 2\sqrt{br} \therefore CS = PN = 2\sqrt{br} \dots \dots \dots (III)$$

From the above figure 4, it is obvious that $MP + PN = MN$ now, substituting the corresponding values, we get

$$2\sqrt{ar} + 2\sqrt{br} = 2\sqrt{ab} \Rightarrow \sqrt{r}(\sqrt{a} + \sqrt{b}) = \sqrt{ab} \Rightarrow \sqrt{r} = \frac{\sqrt{ab}}{(\sqrt{a} + \sqrt{b})}$$

$$\Rightarrow r = \left(\frac{\sqrt{ab}}{(\sqrt{a} + \sqrt{b})} \right)^2 = \frac{ab}{a + b + 2\sqrt{ab}}$$

$$\therefore r = \frac{ab}{a + b + 2\sqrt{ab}} = \frac{ab}{(\sqrt{a} + \sqrt{b})^2} \quad (r > 0 \ \forall \ a, b > 0)$$

Above is the required expression to compute the radius (r) of the inscribed circle which externally touches two given circles with radii a & b & their common tangent.

Special case: If two circles of equal radius a are touching each other externally then the radius r of inscribed circle externally touching them as well as their common tangent, is obtained by setting $a = b = a$ in the above expressions as follows

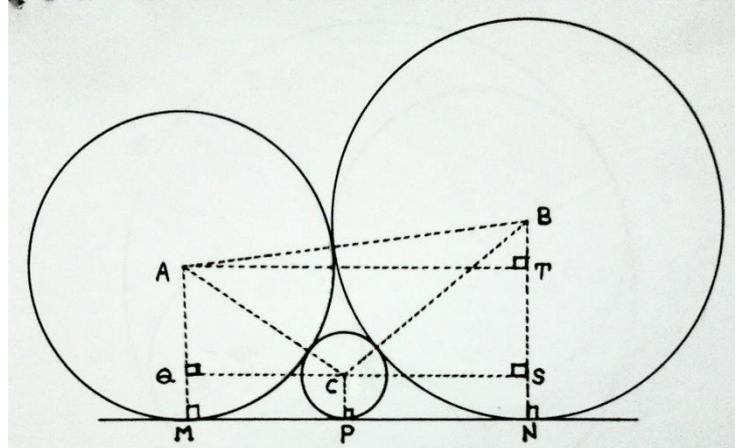


Figure 4: A small circle with centre C is externally touching two given externally touching circles with centres A & B and their common tangent MN .

$$r = \frac{ab}{a + b + 2\sqrt{ab}} = \frac{a^2}{a + a + 2\sqrt{a^2}} = \frac{a^2}{4a} = \frac{a}{4} \Rightarrow r = \frac{a}{4}$$

5. Relationship of the radii of three externally touching circles enclosed in a smallest rectangle:

Consider any three externally touching circles with the centres A, B & C and their radii $a, b \text{ \& } c$ ($\forall a > b \geq c$) respectively enclosed in a smallest rectangle PQR (as shown in Figure 5).

Now, draw the perpendiculars AD, AF & AH from the centre A of the biggest circle to the sides PQ, RS & QR respectively. Also draw the perpendiculars CE & CM from the centre C to the straight lines PQ & DF respectively and the perpendiculars BG & BN from the centre B to the straight lines RS & DF respectively. Then join the centres A, B & C to each other by the (dotted) straight lines to obtain ΔABC . Now, we have

$$AD = AF = a, \quad BG = b \text{ \& } CE = c \quad (\forall b, c < a)$$

$$AB = a + b, \quad BC = b + c \text{ \& } AC = a + c$$

Now, applying cosine rule in right ΔABC

$$\cos \angle BAC = \frac{(AB)^2 + (AC)^2 - (BC)^2}{2(AB)(AC)} \Rightarrow \cos \alpha = \frac{(a + b)^2 + (a + c)^2 - (b + c)^2}{2(a + b)(a + c)}$$

$$\cos \alpha = \frac{a^2 + b^2 + 2ab + a^2 + c^2 + 2ac - b^2 - c^2 - 2bc}{2(a + b)(a + c)} = \frac{a^2 + ab + ac - bc}{(a + b)(a + c)} = \frac{a(a + b) + c(a - b)}{(a + b)(a + c)}$$

$$\cos \alpha = \frac{a(a + b) + c(a - b)}{(a + b)(a + c)} \dots \dots \dots (I)$$

$$\Rightarrow \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left(\frac{a(a + b) + c(a - b)}{(a + b)(a + c)} \right)^2} \quad (\forall \alpha \in [0, \pi])$$

$$= \frac{\sqrt{a^2(a + b)^2 + c^2(a + b)^2 + 2ac(a + b)^2 - a^2(a + b)^2 - c^2(a - b)^2 - 2ac(a^2 - b^2)}}{(a + b)(a + c)}$$

$$\sin \alpha = \frac{\sqrt{4abc^2 + 4abc(a + b)}}{(a + b)(a + c)} \dots \dots \dots (II)$$

In right ΔANB (Fig. 5)

$$\sin \angle ABN = \frac{AN}{AB} = \frac{AF - NF}{AB} = \frac{AF - BG}{AB}$$

$$\Rightarrow \sin \theta = \frac{a - b}{a + b} \dots \dots \dots (III)$$

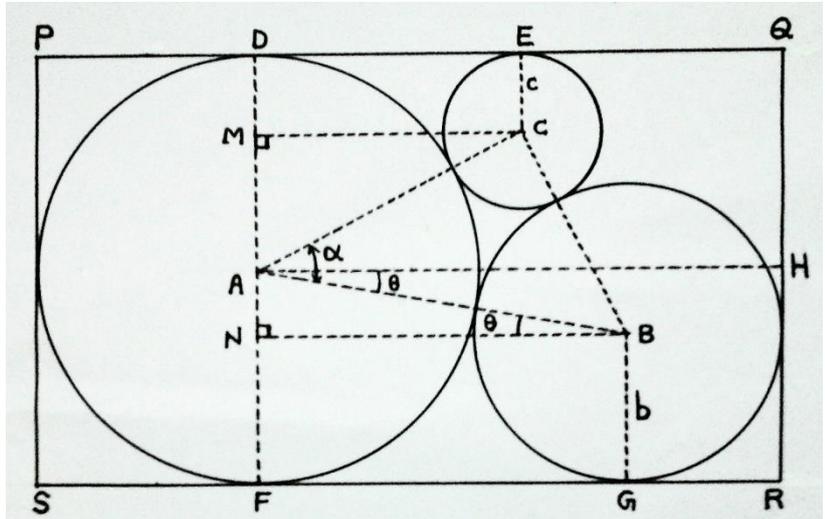


Figure 5: Three externally touching circles with their centres A, B & C and radii $a, b \text{ \& } c$ ($\forall a > b \geq c$) respectively are enclosed in a smallest rectangle PQRS.

$$\begin{aligned} \cos \angle ABN &= \frac{BN}{AB} = \frac{\sqrt{(AB)^2 - (AN)^2}}{AB} = \frac{\sqrt{(a+b)^2 - (a-b)^2}}{a+b} = \frac{\sqrt{4ab}}{a+b} \\ \Rightarrow \cos \theta &= \frac{2\sqrt{ab}}{a+b} \quad \dots \dots \dots (IV) \end{aligned}$$

In right $\triangle AMC$ (Fig. 5)

$$\sin \angle ACM = \frac{AM}{AC} = \frac{AD - MD}{AC} = \frac{AD - CE}{AC} \Rightarrow \sin(\alpha - \theta) = \frac{a - c}{a + c}$$

$$\Rightarrow (a + c)\sin(\alpha - \theta) = a - c \text{ or } (a + c)(\sin\alpha\cos\theta - \cos\alpha\sin\theta) = a - c$$

Now, by substituting the corresponding values from the eq(I), (II), (III) & (IV) in the above expression, we get

$$\begin{aligned} (a + c) \left(\frac{\sqrt{4abc^2 + 4abc(a+b)}}{(a+b)(a+c)} \times \frac{2\sqrt{ab}}{a+b} - \frac{a(a+b) + c(a-b)}{(a+b)(a+c)} \times \frac{a-b}{a+b} \right) &= a - c \\ \Rightarrow (a + c) \left(\frac{4ab\sqrt{c^2 + c(a+b)}}{(a+b)^2(a+c)} - \frac{a(a+b)(a-b) + c(a-b)^2}{(a+b)^2(a+c)} \right) &= a - c \\ \Rightarrow 4ab\sqrt{c^2 + c(a+b)} - a(a^2 - b^2) - c(a-b)^2 &= (a-c)(a+b)^2 \\ \Rightarrow 4ab\sqrt{c^2 + c(a+b)} &= a(a+b)^2 - c(a+b)^2 + a(a^2 - b^2) + c(a-b)^2 \\ \Rightarrow 4ab\sqrt{c^2 + c(a+b)} &= a\{(a+b)^2 + a^2 - b^2\} - c\{(a+b)^2 - (a-b)^2\} \\ \Rightarrow 4ab\sqrt{c^2 + c(a+b)} &= a(2a^2 + 2ab) - c(4ab) \\ \Rightarrow 2b\sqrt{c^2 + c(a+b)} &= a(a+b) - 2bc \end{aligned}$$

Now, taking the square on both the sides, we get

$$\begin{aligned} \left(2b\sqrt{c^2 + c(a+b)} \right)^2 &= (a(a+b) - 2bc)^2 \\ \Rightarrow 4b^2(c^2 + c(a+b)) &= a^2(a+b)^2 + 4b^2c^2 - 4abc(a+b) \\ \Rightarrow 4b^2c^2 + 4b^2c(a+b) &= a^2(a+b)^2 + 4b^2c^2 - 4abc(a+b) \\ \Rightarrow 4b^2c(a+b) + 4abc(a+b) &= a^2(a+b)^2 \\ \Rightarrow 4bc(a+b)(b+a) &= a^2(a+b)^2 \text{ or } 4bc(a+b)^2 = a^2(a+b)^2 \Rightarrow 4bc = a^2 \end{aligned}$$

$$\Rightarrow a^2 = 4bc \text{ or } a = 2\sqrt{bc} \quad \forall a > b \geq c$$

Above relation is very important for computing any of the radii a, b & c if other two are known for three externally touching circles enclosed in a smallest rectangle.

Dimensions of the smallest enclosing rectangle: The length L & width B of the smallest rectangle PQRS enclosing three externally circles touching circles are calculated as follows (see the figure 5 above)

$$\text{Length, } L = PQ = RS = SF + FG + GR = a + 2\sqrt{ab} + b = a + b + 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2$$

$$\text{Width, } B = PS = QR = DM = 2a$$

$$\therefore \text{Length, } L = (\sqrt{a} + \sqrt{b})^2 \quad \& \quad \text{Width, } B = 2a \quad \forall \quad a^2 = 4bc \quad \& \quad a > b \geq c$$

Thus, above expressions can be used to compute the dimensions of the smallest rectangle enclosing three externally touching circles having radii a, b & c ($a > b \geq c$).

6. Length of common chord of two intersecting circles: Consider two circles with centres O_1 & O_2 and radii r_1 & r_2 respectively, at a distance d between their centres, intersecting each other at the points A & B (As shown in the Figure 6). Join the centres O_1 & O_2 to the point A. The line O_1O_2 bisects the common chord AB perpendicularly at the point M. Let $AM = x$ then the length of common chord $AB = 2x$. Now

In right triangle ΔAMO_1 ,

$$O_1M = \sqrt{(O_1A)^2 - (AM)^2} = \sqrt{r_1^2 - x^2}$$

Similarly, In right triangle ΔAMO_2 ,

$$MO_2 = \sqrt{(O_2A)^2 - (AM)^2} = \sqrt{r_2^2 - x^2}$$

Now,

$$O_1O_2 = O_1M + MO_2$$

Substituting the corresponding values, we get

$$d = \sqrt{r_1^2 - x^2} + \sqrt{r_2^2 - x^2}$$

Taking squares on both the sides,

$$d^2 = \left(\sqrt{r_1^2 - x^2} + \sqrt{r_2^2 - x^2} \right)^2$$

$$r_1^2 - x^2 + r_2^2 - x^2 + 2\sqrt{(r_1^2 - x^2)(r_2^2 - x^2)} = d^2$$

$$2\sqrt{(r_1^2 - x^2)(r_2^2 - x^2)} = 2x^2 + d^2 - r_1^2 - r_2^2$$

$$4(r_1^2 - x^2)(r_2^2 - x^2) = (2x^2 + d^2 - r_1^2 - r_2^2)^2$$

$$4r_1^2r_2^2 - 4(r_1^2 + r_2^2)x^2 + 4x^4 = 4x^4 + (d^2 - r_1^2 - r_2^2)^2 + 4(d^2 - r_1^2 - r_2^2)x^2$$

$$4d^2x^2 = 4r_1^2r_2^2 - (d^2 - r_1^2 - r_2^2)^2$$

$$4x^2 = \frac{(2r_1r_2)^2 - (d^2 - r_1^2 - r_2^2)^2}{d^2}$$

$$4x^2 = \frac{(2r_1r_2 + d^2 - r_1^2 - r_2^2)(2r_1r_2 - d^2 + r_1^2 + r_2^2)}{d^2}$$

$$4x^2 = \frac{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}{d^2}$$

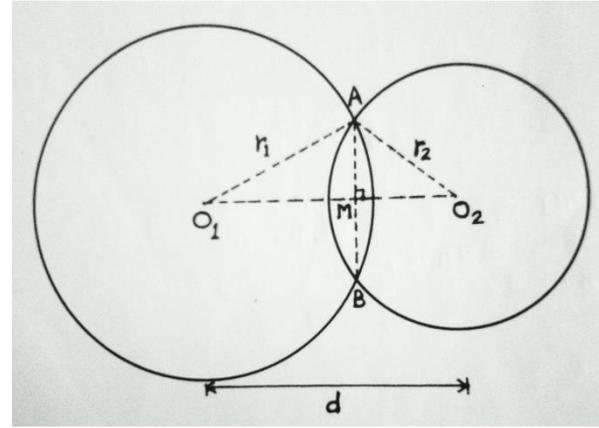


Figure 6: Two circles with the centres O_1 & O_2 and radii r_1 & r_2 respectively at a distance d between their centres, intersecting each other at the points A & B.

$$2x = \sqrt{\frac{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}{d^2}}$$

$$\Rightarrow AB = \frac{\sqrt{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}}{d}$$

Hence, the length of the common chord of two intersecting circles with radii r_1 & r_2 at a distance d between their centres is

$$\text{Length of common chord, } L = \frac{\sqrt{\{d^2 - (r_1 - r_2)^2\}\{(r_1 + r_2)^2 - d^2\}}}{d} \quad \forall |r_1 - r_2| \leq d \leq r_1 + r_2$$

Special case: If $r_1 \neq r_2$ then the **maximum length of common chord** of two intersecting circles

$$= 2 \times \min(r_1, r_2) = \text{diameter of smaller circle at a central distance } d = \sqrt{|r_1^2 - r_2^2|}$$

Angles of intersection of two intersecting circles: Let $\angle AO_1M = \theta_1$ & $\angle AO_2M = \theta_2$ (See above Fig. 6).

In right $\triangle AMO_1$ (Fig. 6), we have

$$\sin \angle AO_1M = \frac{AM}{AO_1} \Rightarrow \sin \theta_1 = \frac{L/2}{r_1} = \frac{L}{2r_1}$$

$$\Rightarrow \theta_1 = \sin^{-1}\left(\frac{L}{2r_1}\right) = \text{semi aperture angle subtended by common chord AB at centre } O_1$$

Similarly, in right $\triangle AMO_2$,

$$\angle AO_2M = \theta_2 = \sin^{-1}\left(\frac{L}{2r_2}\right) = \text{semi aperture angle subtended by common chord AB at centre } O_2$$

Now, it can be easily proved that one of the two supplementary angles of intersection (θ) is given as the sum of above two semi-aperture angles θ_1 and θ_2 subtended by common chord AB at the centres O_1 & O_2 of two intersecting circles (see above Fig. 6)

$$\theta = \theta_1 + \theta_2 = \sin^{-1}\left(\frac{L}{2r_1}\right) + \sin^{-1}\left(\frac{L}{2r_2}\right)$$

Hence, both the **supplementary angles of intersection of two intersecting circles** are given as follows

$$\theta = \sin^{-1}\left(\frac{L}{2r_1}\right) + \sin^{-1}\left(\frac{L}{2r_2}\right) \quad \& \quad \pi - \theta$$

$$\text{Where, } L = \frac{\sqrt{\{d^2 - (r_1 - r_2)^2\}\{(r_1 + r_2)^2 - d^2\}}}{d} \quad \forall |r_1 - r_2| \leq d \leq r_1 + r_2$$

Area of intersection (A) of two intersecting circles: As we have computed above, $2\theta_1$ is the angle of aperture subtended by common chord AB at the centre O_1 of circle with a radius r_1 hence the area (A_1) of segment of corresponding circle is give as (Refer to fig.6 above)

$$A_1 = \text{Area of sector } O_1AB - \text{area of isosceles } \triangle O_1AB$$

$$= \frac{1}{2}(2\theta_1)r_1^2 - \frac{1}{2}(r_1 \times r_1) \sin 2\theta_1$$

$$\begin{aligned}
&= \frac{1}{2}(2\theta_1 - \sin 2\theta_1)r_1^2 \\
&= \frac{1}{2}(2\theta_1 - 2 \sin \theta_1 \cos \theta_1)r_1^2 \\
&= (\theta_1 - \sin \theta_1 \cos \theta_1)r_1^2
\end{aligned}$$

Similarly, the area (A_2) of segment of circle with a radius r_2 & aperture angle $2\theta_2$ subtended by common chord AB at the centre O_2 , is given as follows

$$A_2 = (\theta_2 - \sin \theta_2 \cos \theta_2)r_2^2$$

Now, the area of intersection (A) of two intersecting circles will be equal to the sum of areas A_1 & A_2 of segments as computed above

$$A = A_1 + A_2 = (\theta_1 - \sin \theta_1 \cos \theta_1)r_1^2 + (\theta_2 - \sin \theta_2 \cos \theta_2)r_2^2$$

Hence, the **area (A) of intersection of any two intersecting circles of radii r_1 & r_2 separated by a central distance d** is given as

$$A = (\theta_1 - \sin \theta_1 \cos \theta_1)r_1^2 + (\theta_2 - \sin \theta_2 \cos \theta_2)r_2^2$$

$$\text{Where, } \theta_1 = \sin^{-1}\left(\frac{L}{2r_1}\right), \quad \theta_2 = \sin^{-1}\left(\frac{L}{2r_2}\right) \quad \& \quad L = \frac{\sqrt{\{d^2 - (r_1 - r_2)^2\}\{(r_1 + r_2)^2 - d^2\}}}{d}$$

$$\forall \quad |r_1 - r_2| \leq d \leq r_1 + r_2$$

Conclusion: All the results presented above are derived using elementary geometric and trigonometric principles. The resulting analytical formulas are simple, explicit, and convenient to apply in case studies and practical problems in two-dimensional geometry. Moreover, these results remain valid for the corresponding three-dimensional configuration of three spheres touching one another externally, thereby extending their applicability beyond planar geometry.

Note: Above articles had been derived & illustrated by **Mr H.C. Rajpoot (B Tech, Mechanical Engineering)**

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