Mathematical Analysis of Disphenoid (Isosceles Tetrahedron)

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1. Introduction: A disphenoid is a tetrahedron whose four faces are congruent acute-angled triangles [1,2]. In such a tetrahedron, each pair of opposite edges is equal in length, which is why it is also commonly referred to as an isosceles tetrahedron (see Fig. 1). Owing to this symmetry, the solid angles at all four vertices are equal; equivalently, each face subtends an identical solid angle at its opposite vertex, since all faces have equal areas and are located at equal normal distances from their respective opposite vertices [3-6]. The objective of the present work is to derive general analytical expressions for the fundamental geometric properties of a disphenoid using three-dimensional coordinate geometry. Specifically, closed-form formulae are obtained for the volume and total surface area, as well as for the radii of the inscribed and circumscribed spheres, and solid angle subtended by disphenoid at its vertex. In addition, explicit coordinates of the four vertices are determined, together with the coordinates of the in-center, circum-center, and centroid of the disphenoid. The proof for in-center, circum-center and centroid to be coincident is presented and mathematical equation governing all disphenoids is also derived in a closed form.

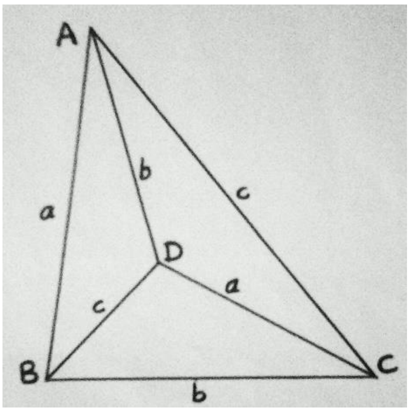

Figure 1: A disphenoid ABCD has each pair of opposite edges equal in length ($AB = CD = a, AD = BC = b, AC = BD = c$.

2. Analysis of disphenoid (isosceles tetrahedron)

Consider a disphenoid ABCD having four congruent triangular faces each of sides a, b & c in 3D space optimally such that its triangular face ABC lies on XY-plane so that vertex A is at origin & vertex B is on the x-axis. (As shown in the Figure 2)

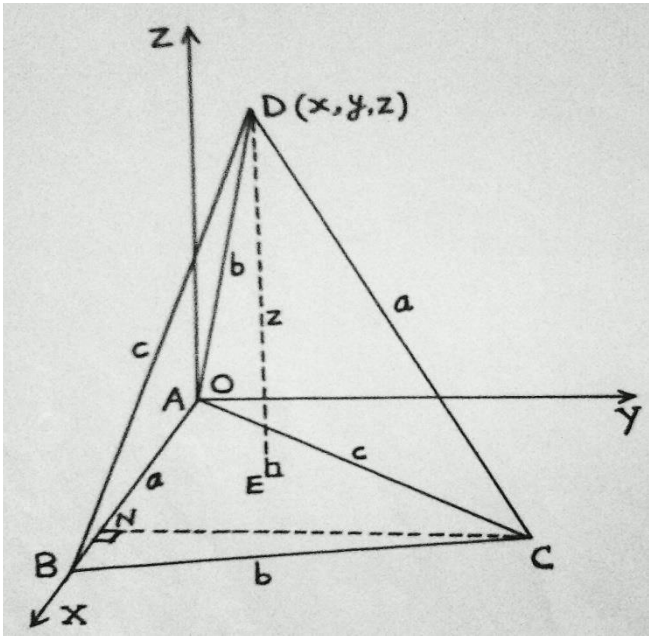

Figure 2: A disphenoid ABCD optimally has its vertex A at origin, vertex B on the x-axis, vertex C on the XY-plane & vertex D at (x, y, z) in the first octant in 3D space.

Applying cosine formula in ΔABC as follows

$$\cos A = \frac{(AB)^2 + (AC)^2 - (BC)^2}{2(AB)(AC)} = \frac{a^2 + c^2 - b^2}{2ac}$$

In right ΔANC, $AN = (AC)cosA = c\ cosA$

$$= c\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$AN = \frac{a^2 + c^2 - b^2}{2a} = \textbf{x coordinate of vertex C}$$

Now, drop a perpendicular CN from vertex C to the side AB, the area of ΔABC is as

$$\Delta = \frac{1}{2}(base\ )(normal\ height) = \frac{1}{2}(a)(CN)$$

$$CN = \frac{2\Delta}{a} = \textbf{y coordinate of vertex C}$$

Now, coordinates of vertex C are $(AN, CN, 0) \equiv \left(\frac{a^2+c^2-b^2}{2a}, \frac{2\Delta}{a}, 0\right)$. Let the coordinates of vertex D be (x, y, z)

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Now, using distance formula, the distance between the vertices $D(x, y, z)$ & $A(0, 0, 0)$ is given as

$$DA = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = b$$

$$\boldsymbol{x^2 + y^2 + z^2 = b^2} \qquad \ldots\ldots\ldots\ldots..(1)$$

Similarly, the distance between the vertices $D(x, y, z)$ & $B(a, 0, 0)$ is given as

$$DB = \sqrt{(x-a)^2 + (y-0)^2 + (z-0)^2} = c$$

$$(x-a)^2 + (y-0)^2 + (z-0)^2 = c^2$$

$$x^2 + y^2 + z^2 - 2ax + a^2 = c^2$$

Setting the value of $x^2 + y^2 + z^2$ from (1),

$$b^2 - 2ax + a^2 = c^2$$

$$\boldsymbol{x = \frac{a^2 + b^2 - c^2}{2a}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Similarly, the distance between the vertices $D(x, y, z)$ & $C\left(\frac{a^2+c^2-b^2}{2a}, \frac{2\Delta}{a}, 0\right)$ is given as

$$DC = \sqrt{\left(x - \frac{a^2 + c^2 - b^2}{2a}\right)^2 + \left(y - \frac{2\Delta}{a}\right)^2 + (z-0)^2} = a$$

$$\left(x - \frac{a^2 + c^2 - b^2}{2a}\right)^2 + \left(y - \frac{2\Delta}{a}\right)^2 + (z-0)^2 = a^2$$

$$x^2 + y^2 + z^2 - 2\left(\frac{a^2 + c^2 - b^2}{2a}\right)x + \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 - 2\left(\frac{2\Delta}{a}\right)y + \left(\frac{2\Delta}{a}\right)^2 = a^2$$

Setting the value of $x^2 + y^2 + z^2$ from (1) & the value of x from (2),

$$b^2 - 2\left(\frac{a^2 + c^2 - b^2}{2a}\right)\left(\frac{a^2 + b^2 - c^2}{2a}\right) + \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 - \left(\frac{4\Delta}{a}\right)y + \left(\frac{2\Delta}{a}\right)^2 = a^2$$

$$\left(\frac{4\Delta}{a}\right)y = b^2 - a^2 - \frac{(a^2 + c^2 - b^2)(a^2 + b^2 - c^2)}{2a^2} + \frac{(a^2 + c^2 - b^2)^2}{4a^2} + \frac{4\Delta^2}{a^2}$$

$$\left(\frac{4\Delta}{a}\right)y = \frac{4a^2b^2 - 4a^4 - 2(a^2 + c^2 - b^2)(a^2 + b^2 - c^2) + (a^2 + c^2 - b^2)^2 + 16\Delta^2}{4a^2}$$

But from Heron's formula, $\Delta = \sqrt{s(s-a)(s-b)(s-c)} \Rightarrow 16\Delta^2 = 4a^2b^2 - (a^2 + b^2 - c^2)^2$ Setting the value of $16\Delta^2$ in above equation, we should get

$$\left(\frac{4\Delta}{a}\right)y = \frac{4a^2b^2 - 4a^4 - 2(a^2 + c^2 - b^2)(a^2 + b^2 - c^2) + (a^2 + c^2 - b^2)^2 + 4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2}$$

$$= \frac{8a^2b^2 - 4a^4 + (a^2 + c^2 - b^2)^2 - (a^2 + c^2 - b^2)(a^2 + b^2 - c^2) - (a^2 + c^2 - b^2)(a^2 + b^2 - c^2) - (a^2 + b^2 - c^2)^2}{4a^2}$$

$$= \frac{8a^2b^2 - 4a^4 + (a^2 + c^2 - b^2)(a^2 + c^2 - b^2 - a^2 - b^2 + c^2) - (a^2 + b^2 - c^2)(a^2 + c^2 - b^2 + a^2 + b^2 - c^2)}{4a^2}$$

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$$=\frac{8a^2b^2-4a^4+(a^2+c^2-b^2)(2c^2-2b^2)-(a^2+b^2-c^2)(2a^2)}{4a^2}$$

$$=\frac{4a^2b^2-2a^4+(a^2+c^2-b^2)(c^2-b^2)-a^2(a^2+b^2-c^2)}{2a^2}$$

$$\left(\frac{4\Delta}{a}\right)y=\frac{4a^2b^2-2a^4+a^2c^2+c^4-b^2c^2-a^2b^2-b^2c^2+b^4-a^4-a^2b^2+a^2c^2}{2a^2}$$

$$\boldsymbol{y}=\frac{\mathbf{2}\boldsymbol{a}^{\mathbf{2}}\boldsymbol{b}^{\mathbf{2}}-\mathbf{2}\boldsymbol{b}^{\mathbf{2}}\boldsymbol{c}^{\mathbf{2}}+\mathbf{2}\boldsymbol{a}^{\mathbf{2}}\boldsymbol{c}^{\mathbf{2}}-\mathbf{3}\boldsymbol{a}^{\mathbf{4}}+\boldsymbol{b}^{\mathbf{4}}+\boldsymbol{c}^{\mathbf{4}}}{\mathbf{8}\boldsymbol{a}\Delta}\quad\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Now, setting the values of x & y from (2) & (3) respectively into (1), we should get

$$\left(\frac{a^2+b^2-c^2}{2a}\right)^2+\left(\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}\right)^2+z^2=b^2$$

$$z=\sqrt{b^2-\left(\frac{a^2+b^2-c^2}{2a}\right)^2-\left(\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}\right)^2}$$

$$=\sqrt{\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2}-\frac{(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}{64a^2\Delta^2}}$$

$$=\sqrt{\frac{2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4}{4a^2}-\frac{(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}{64a^2\Delta^2}}$$

$$=\sqrt{\frac{16\Delta^2(2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4)-(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}{64a^2\Delta^2}}$$

$$=\frac{\sqrt{16\Delta^2(2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4)-(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}}{8a\Delta}$$

But from Heron's formula, $\Delta=\sqrt{s(s-a)(s-b)(s-c)}\ \Rightarrow\ 16\Delta^2=2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4$
Setting the value of $16\Delta^2$ in above equation, we should get

$$z=\frac{\sqrt{(2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4)^2-(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}}{8a\Delta}$$

$$=\frac{\sqrt{(2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4+2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)(2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4-2a^2b^2+2b^2c^2-2a^2c^2+3a^4-b^4-c^4)}}{8a\Delta}$$

$$=\frac{\sqrt{(4a^2b^2+4a^2c^2-4a^4)(4b^2c^2+2a^4-2b^4-2c^4)}}{8a\Delta}$$

$$=\frac{\sqrt{8a^2(b^2+c^2-a^2)(2b^2c^2+a^4-b^4-c^4)}}{8a\Delta}$$

$$=\frac{2a\sqrt{2}\sqrt{(b^2+c^2-a^2)(a^4-(b^2-c^2)^2)}}{8a\Delta}$$

$$=\frac{\sqrt{(b^2+c^2-a^2)(a^2+b^2-c^2)(a^2-b^2+c^2)}}{2\Delta\sqrt{2}}$$

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$$z=\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

Thus, setting the values of x, y & z from (2), (3) & (4), the coordinates of vertex D are given as

$$D\equiv\left(\frac{a^2+b^2-c^2}{2a},\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta},\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)$$

2.1. Coordinates of four vertices of disphenoid (isosceles tetrahedron): The coordinates of all four vertices A, B, C & D of disphenoid ABCD with three unequal edges a, b & c for the optimal configuration in 3D space are given as $\boldsymbol{A(0,0,0)}, \boldsymbol{B(a,0,0)}$, $\boldsymbol{C\left(\frac{a^2+c^2-b^2}{2a},\frac{2\Delta}{a},0\right)}$ &

$$\boldsymbol{D\left(\frac{a^2+b^2-c^2}{2a},\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta},\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)}$$

Where, Δ is the area of each acute triangular face with sides a, b & c of the disphenoid.

2.2. Vertical height of disphenoid (isosceles tetrahedron): The disphenoid has four congruent acute-triangular faces thus by symmetry, the vertical height of each vertex from its opposite face (base) is equal & it is equal to the vertical height DE of vertex D from triangular face ABC lying on the XY-plane

$$DE=\text{z coordinate of vertex D}=\boldsymbol{\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}}$$

2.3. Volume of disphenoid (isosceles tetrahedron): The volume (V) of disphenoid ABCD (see figure 2 above) is given as

$$V=\frac{1}{3}(\text{Area of triangular base ABC})(\text{normal height of vertex D from base ABC})=\frac{1}{3}(\Delta)(DE)$$

$$=\frac{1}{3}(\Delta)\left(\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)$$

$$=\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{6\sqrt{2}}$$

$$\textbf{Volume of disphenoid},\ \boldsymbol{V=\sqrt{\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{72}}}$$

Above is the general formula to compute the volume of a disphenoid having four congruent faces each as an acute-angled triangle with sides a, b & c

Now, the vertical height (H) of disphenoid in terms of volume (V) & area of face (Δ) is given as

$$\boldsymbol{H}=\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}=\frac{3\sqrt{\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{72}}}{\Delta}=\boldsymbol{\frac{3V}{\Delta}}$$

2.4. Surface area of disphenoid (isosceles tetrahedron): The disphenoid consists of four congruent triangular faces hence the surface area of disphenoid

$$A_S=4(\text{Area of acute triangular face ABC})=4\Delta=\boldsymbol{4\sqrt{s(s-a)(s-b)(s-c)}}$$

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Where, $s=\frac{a+b+c}{2}$ is the semi-perimeter of any of four congruent acute-triangular faces of a disphenoid

2.5. Radius of sphere inscribed by the disphenoid (isosceles tetrahedron): Let r be the radius of sphere inscribed by the disphenoid ABCD, the centre O of inscribed sphere is at an equal normal distance r from all four congruent triangular faces (As shown in the figure 3) Drop the perpendiculars from in-centre O to four faces to get four congruent elementary tetrahedrons each of base area Δ & normal height r. If we add the volumes of these four congruent elementary tetrahedrons then we get the volume of original disphenoid thus the volume of disphenoid

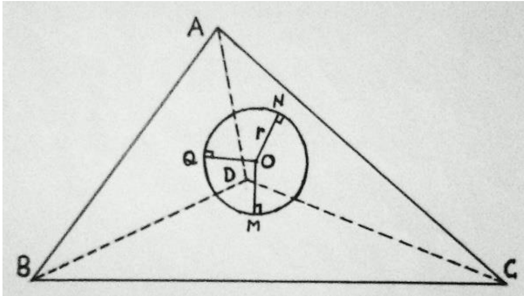

Figure 3: The centre O of inscribed sphere is at an equal normal distance r from all four congruent acute-triangular faces of disphenoid ABCD.

$$V=4(\text{volume of elementary tetrahedron})$$

$$V=4\left(\frac{1}{3}(\Delta)(r)\right)\ \Rightarrow r=\frac{3V}{4\Delta}$$

$$\textbf{Inradius of disphenoid},\ \boldsymbol{r}=\frac{\boldsymbol{3V}}{\boldsymbol{4\Delta}}$$

2.6. Radius of sphere circumscribing the disphenoid (isosceles tetrahedron): Let R be the radius of sphere circumscribing the disphenoid ABCD, the centre P of the circumscribed sphere is at an equal distance R from all four vertices A, B, C & D (As shown in the figure 4). Let the coordinates of circum-centre P be (h,k,l) in 3D space.

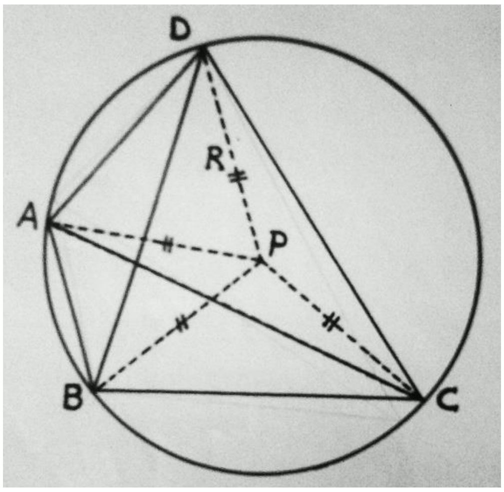

Figure 4: The centre P of circumscribed sphere is at an equal distance R from all four vertices A, B, C & D of disphenoid ABCD.

Now, using distance formula, the distance between the circum-centre $P(h,k,l)$ & the vertex $A(0,0,0)$ of disphenoid ABCD is given as

$$PA=\sqrt{(h-0)^2+(k-0)^2+(l-0)^2}=R$$

$$\boldsymbol{h^2+k^2+l^2=R^2} \quad \ldots\ldots\ldots\ldots..(5)$$

Similarly, the distance between $P(h,k,l)$ & $B(a,0,0)$ is given as

$$PB=\sqrt{(h-a)^2+(k-0)^2+(l-0)^2}=R$$

$$(h-a)^2+(k-0)^2+(l-0)^2=R^2$$

$$h^2+k^2+l^2-2ah+a^2=R^2$$

Setting the value of $h^2+k^2+l^2$ from (5),

$$R^2-2ah+a^2=R^2$$

$$\boldsymbol{h=\frac{a}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots.(6)$$

Similarly, the distance between $P(h,k,l)$ & $C\left(\frac{a^2+c^2-b^2}{2a},\frac{2\Delta}{a},0\right)$ is given as

$$PC=\sqrt{\left(h-\frac{a^2+c^2-b^2}{2a}\right)^2+\left(k-\frac{2\Delta}{a}\right)^2+(l-0)^2}=R$$

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$$\left(h-\frac{a^2+c^2-b^2}{2a}\right)^2+\left(k-\frac{2\Delta}{a}\right)^2+(l-0)^2=R^2$$

$$h^2+k^2+l^2-2\left(\frac{a^2+c^2-b^2}{2a}\right)h+\left(\frac{a^2+c^2-b^2}{2a}\right)^2-2\left(\frac{2\Delta}{a}\right)k+\left(\frac{2\Delta}{a}\right)^2=R^2$$

Setting the value of $h^2+k^2+l^2$ from (5) & the value of h from (6),

$$R^2-2\left(\frac{a^2+c^2-b^2}{2a}\right)\left(\frac{a}{2}\right)+\left(\frac{a^2+c^2-b^2}{2a}\right)^2-\left(\frac{4\Delta}{a}\right)k+\left(\frac{2\Delta}{a}\right)^2=R^2$$

$$\left(\frac{4\Delta}{a}\right)k=-\frac{(a^2+c^2-b^2)}{2}+\frac{(a^2+c^2-b^2)^2}{4a^2}+\frac{4\Delta^2}{a^2}$$

$$=\frac{-2a^2(a^2+c^2-b^2)+(a^2+c^2-b^2)^2+16\Delta^2}{4a^2}$$

$$=\frac{(a^2+c^2-b^2)(a^2+c^2-b^2-2a^2)+16\Delta^2}{4a^2}$$

$$=\frac{b^4+c^4-a^4-2b^2c^2+16\Delta^2}{4a^2}$$

But from Heron's formula, $\Delta=\sqrt{s(s-a)(s-b)(s-c)}\ \Rightarrow\ 16\Delta^2=2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4$
Setting the value of $16\Delta^2$ in above equation, we should get

$$\left(\frac{4\Delta}{a}\right)k=\frac{b^4+c^4-a^4-2b^2c^2+2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4}{4a^2}$$

$$k=\frac{b^4+c^4-a^4-2b^2c^2+2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4}{16a\Delta}$$

$$k=\frac{2a^2b^2+2a^2c^2-2a^4}{16a\Delta}$$

$$\boldsymbol{k=\frac{ab^2+ac^2-a^3}{8\Delta}} \qquad \dots\dots\dots\dots\dots\dots\dots(7)$$

Similarly, the distance between $P(h,k,l)$ & $D\left(\frac{a^2+b^2-c^2}{2a},\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta},\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)$ is given as

$$PD=\sqrt{\left(h-\frac{a^2+b^2-c^2}{2a}\right)^2+\left(k-\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}\right)^2+\left(l-\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)^2}=R$$

$$\left(h-\frac{a^2+b^2-c^2}{2a}\right)^2+\left(k-\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}\right)^2+\left(l-\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)^2=R^2$$

$$h^2+k^2+l^2+\left(\frac{a^2+b^2-c^2}{2a}\right)^2-2\left(\frac{a^2+b^2-c^2}{2a}\right)h+\left(\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}\right)^2-2\left(\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}\right)k$$
$$+\left(\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)^2-2\left(\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)l=R^2$$

Setting the value of $h^2+k^2+l^2$ from (5), the value of h from (6) & the value of k from (7),

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$$R^2+\frac{(a^2+b^2-c^2)^2}{4a^2}-2\left(\frac{a^2+b^2-c^2}{2a}\right)\left(\frac{a}{2}\right)+\frac{(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}{64a^2\Delta^2}$$
$$-2\left(\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}\right)\left(\frac{ab^2+ac^2-a^3}{8\Delta}\right)+\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{8\Delta^2}$$
$$-2\left(\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)l=R^2$$

$$\left(\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{\Delta\sqrt{2}}\right)l$$
$$=\frac{(a^2+b^2-c^2)^2}{4a^2}-\frac{a^2+b^2-c^2}{2}+\frac{(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}{64a^2\Delta^2}$$
$$-\frac{2(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)(ab^2+ac^2-a^3)}{64a\Delta^2}+\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{8\Delta^2}$$

$$=\frac{(a^2+b^2-c^2)(b^2-c^2-a^2)}{4a^2}+\frac{(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}{64a^2\Delta^2}-\frac{2(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)(ab^2+ac^2-a^3)}{64a\Delta^2}$$
$$+\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{8\Delta^2}$$

$$=\frac{16\Delta^2(a^2+b^2-c^2)(b^2-c^2-a^2)}{64a^2\Delta^2}+\frac{(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2}{64a^2\Delta^2}-\frac{2a(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)(ab^2+ac^2-a^3)}{64a^2\Delta^2}$$
$$+\frac{8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{64a^2\Delta^2}$$

$$64a^2\Delta^2\left(\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{\Delta\sqrt{2}}\right)l$$
$$=16\Delta^2(a^2+b^2-c^2)(b^2-c^2-a^2)+(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2$$
$$-2a(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)(ab^2+ac^2-a^3)+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$\left(32\sqrt{2}a^2\Delta\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}\right)l$$
$$=16\Delta^2(a^2+b^2-c^2)(b^2-c^2-a^2)+(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)^2$$
$$-2a(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)(ab^2+ac^2-a^3)+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$=16\Delta^2(b^4+c^4-2b^2c^2-a^4)+(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4-2a^2b^2-2a^2c^2+2a^4)$$
$$+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$=16\Delta^2(b^4+c^4-2b^2c^2-a^4)+(2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)(b^4+c^4-2b^2c^2-a^4)+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$=(b^4+c^4-2b^2c^2-a^4)(16\Delta^2+2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$=(b^4+c^4-2b^2c^2-a^4)(2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4+2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4)$$
$$+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$=(b^4+c^4-2b^2c^2-a^4)(4a^2b^2+4a^2c^2-4a^4)+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$=4a^2((b^2-c^2)^2-a^4)(b^2+c^2-a^2)+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$=4a^2(b^2-c^2+a^2)(b^2-c^2-a^2)(b^2+c^2-a^2)+8a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$=4a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(b^2-c^2-a^2+2a^2)$$

$$\left(32\sqrt{2}a^2\Delta\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}\right)l=4a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

$$l=\frac{4a^2(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{32\sqrt{2}a^2\Delta\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}$$

$$l=\frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{8\sqrt{2}\Delta}$$

$$l=\frac{3\sqrt{\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{72}}}{4\Delta}$$

$$\boldsymbol{l}=\frac{\boldsymbol{3V}}{\boldsymbol{4\Delta}}=\boldsymbol{r}=\text{Radius of inscribed sphere}\qquad \dots\dots\dots\dots(8)$$

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The above result (Eq.(8)) shows that

1. Distance of circum-centre P from the acute-triangular face ABC (lying on the XY-plane) is equal to the radius r of inscribed sphere of disphenoid ABCD
2. The perpendicular drawn from the circum-centre P of disphenoid ABCD falls at the circum-centre of acute-triangular face ABC
3. By symmetry of disphenoid that four faces are acute angled triangles, the circum-centre of disphenoid is at an equal normal distance from each acute-triangular face.
4. The normal distance of circum-centre from each face is equal to the in-radius of disphenoid, hence by symmetry the circum-centre must be coincident with the in-centre of a disphenoid.
5. Inscribed sphere touches each of four congruent acute-triangular faces at circum-centre of that face

Now, substituting the values of h, k & l from (6), (7) & (8) respectively into (5), the circum-radius R is given as

$$R = \sqrt{h^2 + k^2 + l^2}$$

$$= \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{ab^2 + ac^2 - a^3}{8\Delta}\right)^2 + \left(\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{8\sqrt{2}\Delta}\right)^2}$$

$$= \sqrt{\frac{a^2}{4} + \frac{(ab^2 + ac^2 - a^3)^2}{64\Delta^2} + \frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{128\Delta^2}}$$

$$= \sqrt{\frac{32a^2\Delta^2 + 2(ab^2 + ac^2 - a^3)^2 + (a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{128\Delta^2}}$$

$$= \sqrt{\frac{32a^2\Delta^2 + 2a^2(b^2 + c^2 - a^2)^2 + (b^2 + c^2 - a^2)(a^4 - b^4 - c^4 + 2b^2c^2)}{128\Delta^2}}$$

$$= \sqrt{\frac{32a^2\Delta^2 + (b^2 + c^2 - a^2)(2a^2(b^2 + c^2 - a^2) + a^4 - b^4 - c^4 + 2b^2c^2)}{128\Delta^2}}$$

$$= \sqrt{\frac{32a^2\Delta^2 + (b^2 + c^2 - a^2)(2a^2b^2 + 2a^2c^2 - 2a^4 + a^4 - b^4 - c^4 + 2b^2c^2)}{128\Delta^2}}$$

$$= \sqrt{\frac{32a^2\Delta^2 + (b^2 + c^2 - a^2)(2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4)}{128\Delta^2}}$$

$$= \sqrt{\frac{32a^2\Delta^2 + (b^2 + c^2 - a^2)(16\Delta^2)}{128\Delta^2}} \quad \left(\text{since, } 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4 = 16\Delta^2\right)$$

$$= \sqrt{\frac{2a^2 + b^2 + c^2 - a^2}{8}} = \sqrt{\frac{a^2 + b^2 + c^2}{8}}$$

For a disphenoid (isosceles tetrahedron) having three unequal edges a, b & c, radius of circumscribed sphere

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$$\text{Circumradius of disphenoid,}\quad \boldsymbol{R}=\sqrt{\frac{\boldsymbol{a^2+b^2+c^2}}{\boldsymbol{8}}}\quad \dots\dots\dots\dots(9)$$

2.7. Coordinates of circum-centre or in-centre of disphenoid (isosceles tetrahedron): The coordinates of in-centre or circum-centre $P(h,k,l)$ of a disphenoid ABCD having vertices (for the optimal configuration in 3D space) $A(0,0,0), B(a,0,0)$, $C\left(\frac{a^2+c^2-b^2}{2a},\frac{2\Delta}{a},0\right)$ & $D\left(\frac{a^2+b^2-c^2}{2a},\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta},\frac{3V}{\Delta}\right)$ is given by substituting the values of h, k & l from (6), (7) & (8) respectively as follows

$$\text{Incentre or Circumcentre of disphenoid,}\quad \boldsymbol{P(h,k,l)}\equiv\left(\frac{\boldsymbol{a}}{\boldsymbol{2}},\frac{\boldsymbol{ab^2+ac^2-a^3}}{\boldsymbol{8\Delta}},\frac{\boldsymbol{3V}}{\boldsymbol{4\Delta}}\right)$$

Where, V is the volume of disphenoid & Δ is the area of each acute-angled triangular face with sides a, b & c

It is clear from the above result that the in-centre & the circum-centre of a disphenoid (isosceles tetrahedron) are coincident.

2.8. Coordinates of centroid of disphenoid (isosceles tetrahedron): The coordinates of centroid $G(\bar{x},\bar{y},\bar{z})$ of a disphenoid ABCD having vertices (for the optimal configuration in 3D space) $A(0,0,0), B(a,0,0)$, $C\left(\frac{a^2+c^2-b^2}{2a},\frac{2\Delta}{a},0\right)$ & $D\left(\frac{a^2+b^2-c^2}{2a},\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta},\frac{3V}{\Delta}\right)$ is given as follows

$$G(\bar{x},\bar{y},\bar{z})\equiv\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$$

$$\bar{\boldsymbol{x}}=\frac{0+a+\frac{a^2+c^2-b^2}{2a}+\frac{a^2+b^2-c^2}{2a}}{4}=\frac{2a^2+a^2+c^2-b^2+a^2+b^2-c^2}{8a}=\frac{4a^2}{8a}=\frac{\boldsymbol{a}}{\boldsymbol{2}}$$

$$\bar{\boldsymbol{y}}=\frac{0+0+\frac{2\Delta}{a}+\frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}}{4}$$

$$\bar{y}=\frac{16\Delta^2+2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{32a\Delta}$$

But from Heron's formula, $\Delta=\sqrt{s(s-a)(s-b)(s-c)}\ \Rightarrow\ 16\Delta^2=2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4$
Setting the value of $16\Delta^2$ in above equation, we should get

$$\bar{y}=\frac{2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4+2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{32a\Delta}$$

$$\bar{\boldsymbol{y}}=\frac{4a^2b^2+4a^2c^2-4a^4}{32a\Delta}=\frac{\boldsymbol{ab^2+ac^2-a^3}}{\boldsymbol{8\Delta}}$$

$$\bar{\boldsymbol{z}}=\frac{0+0+0+\frac{3V}{\Delta}}{4}=\frac{\boldsymbol{3V}}{\boldsymbol{4\Delta}}$$

$$\textbf{Centroid of disphenoid},\quad \boldsymbol{G(\bar{x},\bar{y},\bar{z})}\equiv\left(\frac{\boldsymbol{a}}{\boldsymbol{2}},\frac{\boldsymbol{ab^2+ac^2-a^3}}{\boldsymbol{8\Delta}},\frac{\boldsymbol{3V}}{\boldsymbol{4\Delta}}\right)$$

The above result shows that the coordinates of centroid $G(\bar{x},\bar{y},\bar{z})$ are same as that of in-centre or circum-centre of a disphenoid hence we can conclude that in-centre, circum-centre & centroid of a disphenoid (isosceles tetrahedron with four congruent faces each as an acute-angled triangle) are always coincident.

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2.9. Solid angle subtended by disphenoid (isosceles tetrahedron) at its vertex: All four acute-angled triangular faces of the disphenoid are congruent, hence the solid angle subtended by each face at its opposite vertex will be equal to the solid angle subtended by disphenoid at any of its four vertices. The solid angle subtended by any tetrahedron (or disphenoid) at its vertex is given by the general formula [3-6] as follows

$$\omega = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) - \sin^{-1}\left(\frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}\right) - \sin^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) \quad \ldots\ldots\ldots.(10)$$

Where, $\alpha, \beta \,\&\, \gamma$ are the angles between consecutive lateral edges meeting at the concerned vertex of disphenoid. The angles $\alpha, \beta \,\&\, \gamma$ meeting at any vertex of a disphenoid can be easily computed by using the cosine formula as follows

$$\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right), \ \beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right), \ \gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$(\forall \ \ 0 < \alpha, \beta, \gamma < \pi \ \ \& \ \ \alpha + \beta + \gamma < 2\pi)$$

By substituting the values of α, β, and γ in the Eq(10), the solid angle subtended by disphenoid at its vertex can be determined.

2.10. Formula (Governing equation) of disphenoid: Consider a disphenoid (isosceles tetrahedron) having three unequal edges a, b, c meeting at the same vertex. Now, the area Δ of each of four congruent acute-triangular faces with sides $a, b \,\&\, c$ is given by Heron's formula as follows

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \quad \forall \ s = \frac{a+b+c}{2}$$

$$\Delta^2 = s(s-a)(s-b)(s-c)$$

$$\Delta^2 = \left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right)$$

$$\Delta^2 = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{16}$$

$$16\Delta^2 = \{(b+c+a)(b+c-a)\}\{\big(a-(b-c)\big)(a+b-c)\}$$

$$16\Delta^2 = \{(b+c)^2 - a^2\}\{a^2 - (b-c)^2\}$$

$$16\Delta^2 = \{b^2 + c^2 + 2bc - a^2\}\{a^2 - b^2 - c^2 + 2bc\}$$

$$16\Delta^2 = \{2bc + (b^2 + c^2 - a^2)\}\{2bc - (b^2 + c^2 - a^2)\}$$

$$16\Delta^2 = (2bc)^2 - (b^2 + c^2 - a^2)^2$$

$$16\Delta^2 = 4b^2c^2 - b^4 - c^4 - a^4 - 2b^2c^2 + 2a^2c^2 + 2a^2b^2$$

$$16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$$

$$16\Delta^2R^2 = (2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4)R^2 \qquad \text{(multiplying } R^2 \text{ on both sides)}$$

$$16\Delta^2R^2 = (2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4)\left(\frac{a^2 + b^2 + c^2}{8}\right) \qquad \text{(setting value of } R^2\text{)}$$

$$16\Delta^2R^2 = \frac{2a^4b^2 + 2a^2b^2c^2 + 2a^4c^2 - a^6 - a^2b^4 - a^2c^4 + 2a^2b^4 + 2b^4c^2 + 2a^2b^2c^2 - a^4b^2 - b^6 - b^2c^4 + 2a^2b^2c^2 + 2b^2c^4 + 2a^2c^4 - a^4c^2 - b^4c^2 - c^6}{8}$$

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$$16\Delta^2R^2 = \frac{6a^2b^2c^2 + a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4 - a^6 - b^6 - c^6}{8}$$

$$16\Delta^2R^2 = \frac{8a^2b^2c^2 + (a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4 - a^6 - b^6 - c^6 - 2a^2b^2c^2)}{8}$$

$$16\Delta^2R^2 = a^2b^2c^2 + \frac{a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4 - a^6 - b^6 - c^6 - 2a^2b^2c^2}{8}$$

$$16\Delta^2R^2 = a^2b^2c^2 + \frac{(a^2 + b^2 - c^2)(2a^2b^2 + c^4 - a^4 - b^4)}{8}$$

$$16\Delta^2R^2 = a^2b^2c^2 + \frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{8}$$

$$16\Delta^2R^2 = a^2b^2c^2 + 9\left(\frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{72}\right)$$

$$16\Delta^2R^2 = a^2b^2c^2 + 9(V^2) \qquad \text{(substituting by } V^2\text{)}$$

$$\boldsymbol{16\Delta^2R^2 = a^2b^2c^2 + 9V^2} \qquad \ldots\ldots\ldots\ldots(11)$$

Above Eq.(11) (called formula of disphenoid) is always satisfied by a disphenoid (isosceles tetrahedron) and also very important for directly computing the volume V or circum-radius R when other parameters a, b & c are known**.**

Conclusions: Let there be a disphenoid (isosceles tetrahedron) having four congruent faces each as an acute-angled triangle with sides $a, b \,\&\, c$ then all the important parameters of disphenoid can be computed as tabulated below.

Volume	$V = \sqrt{\frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{72}}$
Surface area	$4\Delta = 4\sqrt{s(s-a)(s-b)(s-c)}$
Vertical height	$\frac{3V}{\Delta}$
In-radius (radius of inscribed sphere)	$\frac{3V}{4\Delta}$
circum-radius (radius of circumscribed sphere)	$\sqrt{\frac{a^2 + b^2 + c^2}{8}}$
Coordinates of four vertices **(Optimal configuration of a disphenoid in 3D space)**	$(0,0,0),\ (a,0,0),\ \left(\frac{a^2 + c^2 - b^2}{2a}, \frac{2\Delta}{a}, 0\right)$ & $\left(\frac{a^2 + b^2 - c^2}{2a}, \frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}, \frac{3V}{\Delta}\right)$

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Coordinates of coincident in-centre, circum-centre & centroid	$\left(\frac{a}{2}, \frac{ab^2+ac^2-a^3}{8\Delta}, \frac{3V}{4\Delta}\right)$
A **disphenoid (isosceles tetrahedron)** always satisfies the relation: $\mathbf{16\Delta^2 R^2 = a^2 b^2 c^2 + 9V^2}$ Where, V is the volume, R is the radius of circumscribed sphere & Δ is the area of each acute-angled triangular face with sides a, b, c in a disphenoid.	

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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