

# Geometric Unification of Fermion Masses and Mixing Angles from Clifford Torus Topology

Dmitry Makaryev<sup>1,\*</sup> and Victor Shcherb<sup>1,†</sup>

<sup>1</sup>*Independent Researcher, OsmAnd — <https://osmand.net>*

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The Standard Model accommodates fermion masses and mixing through nineteen free Yukawa and mixing parameters, offering no structural explanation for their values. Here we derive the complete flavor structure from two principles: the quadratic (Born-rule) form of the mass Hamiltonian and a topological constraint from the Clifford torus in  $S^3$  that fixes the interference amplitude to  $A = \sqrt{2}$ . We prove that the number of generations  $N = 3$  is the unique solution to the identity  $Q \times N = 2$ , where  $Q = 2/3$  is the Koide ratio and  $2 = \chi(S^2)$  is the Euler characteristic of the Bloch sphere. A single geometric phase  $\delta = 2/9$  rad determines the entire lepton sector: the charged-lepton mass spectrum (predicting  $m_\tau = 1776.97$  MeV versus the experimental  $1776.86 \pm 0.12$  MeV), all three PMNS mixing angles via the exact relations  $\theta_{23} = \pi/4$ ,  $\theta_{13} = 2\delta/3$ ,  $\theta_{12} = \pi/4 - \delta$  (derived from  $\mathbb{Z}_3$  representation theory), and the Wolfenstein parameter  $\lambda = \sin \delta$ . The neutrino sector follows from a complementary phase shift  $\Delta\delta = 3\pi/4$ , predicting absolute masses with  $\sum m_\nu = 0.059$  eV. The universal mass curve evaluated at  $\theta = \pi$  yields a dark-sector state at 53.85 MeV whose  $\pi$ -harmonic at 17.14 MeV matches the Atomki X17 anomaly to 0.8% with zero free parameters. The Clifford torus geometry further predicts maximal CP violation ( $\delta_{CP} = -\pi/2$ ) and exact  $\mu$ - $\tau$  reflection symmetry. In total, the model yields twenty experimentally testable predictions from two input masses and one fundamental constant.

## INTRODUCTION

The origin of fermion masses and mixing angles constitutes one of the most profound unsolved problems in fundamental physics. The Standard Model (SM) successfully describes three generations of quarks and leptons, yet the masses of these particles—spanning five orders of magnitude from the electron (0.511 MeV) to the top quark (172 GeV)—enter as free parameters through Yukawa couplings to the Higgs field. The four CKM quark-mixing parameters and the analogous PMNS lepton-mixing parameters are equally unexplained. Together, these constitute nineteen free constants that must be extracted from experiment, not derived from theory.

The absence of any structural principle governing these parameters has been termed the *flavor problem*. Unlike the gauge sector, where three coupling constants are successfully predicted to unify at high energies, the flavor sector appears anarchic—as if nature chose the values arbitrarily.

Yet hints of hidden structure exist. In 1982, Koide discovered a striking empirical relation among the charged lepton masses [1]:

$$Q \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = 0.666661 \pm 0.000007 \quad (1)$$

The proximity to  $2/3$ —a rational number—to better than one part in  $10^5$  is extraordinary for a dimensionless ratio of seemingly unrelated quantities. No symmetry of the SM protects this value. Despite extensive theoretical investigation [2–8], no consensus derivation from first

principles has emerged.

In this work, we present a geometric framework that resolves the flavor problem. We demonstrate that: (i) the Koide relation follows as an exact mathematical identity from the topology of the Clifford torus in  $S^3$ ; (ii) the number of generations  $N = 3$  is *derived* from the Clifford amplitude and the Koide condition; (iii) a single rational phase  $\delta = 2/9$  rad determines all charged-lepton mass ratios, all three PMNS mixing angles, and the Cabibbo angle; (iv) the neutrino sector is fixed by a complementary phase shift  $3\pi/4$ ; (v) the dark minimum of the universal mass curve predicts new particles at 53.85 and 17.14 MeV (matching the Atomki X17 anomaly) with zero free parameters.

## THEORETICAL FRAMEWORK

### Axiom I: Quadratic mass Hamiltonian

In the SM, fermion masses arise from Yukawa couplings  $\mathcal{L}_Y = y_{ij} \bar{\psi}_{Li} \Phi \psi_{Rj} + \text{h.c.}$  After electroweak symmetry breaking, the mass matrix is  $M_{ij} = v y_{ij}$ . The physical mass-squared eigenvalues are those of the positive semi-definite Hermitian matrix  $M^\dagger M$ . We write:

$$\mathbf{H} = \mu^2 \boldsymbol{\Omega} \boldsymbol{\Omega}^\dagger \quad (2)$$

where  $\mu$  is a sector-specific energy scale and  $\boldsymbol{\Omega}$  encodes the generation structure.

### Axiom II: $\mathbb{Z}_3$ discrete generation symmetry

We postulate that the three fermion generations are related by a discrete cyclic symmetry  $\mathbb{Z}_3$ .

**Theorem 1 (Uniqueness).** *Let  $\Omega$  be a linear operator on the generation space  $\mathbb{C}^3$ , covariant under the  $\mathbb{Z}_3$  symmetry generated by the cyclic permutation  $\mathbf{P}$  (with  $\mathbf{P}^3 = \mathbf{I}$ ). Then  $\Omega$  has the unique form:*

$$\Omega = a\mathbf{I} + b\mathbf{P} + b^*\mathbf{P}^\dagger \quad (3)$$

where  $a \in \mathbb{R}$  and  $b = |b|e^{i\delta} \in \mathbb{C}$ .

### Axiom III: Clifford torus topology

A massive Dirac fermion is a coherent superposition of left- and right-handed chiral states living on  $S^3 = \{(z_L, z_R) \in \mathbb{C}^2 : |z_L|^2 + |z_R|^2 = 1\}$ . In Hopf coordinates:

$$z_L = e^{i\xi_1} \sin \eta, \quad z_R = e^{i\xi_2} \cos \eta \quad (4)$$

The chirality democracy condition  $|z_L|^2 = |z_R|^2$  gives  $\eta = \pi/4$ —the **Clifford torus**  $T_C^2$ , the unique minimal surface and Willmore energy minimizer among all tori in  $S^3$  [9, 10]. This fixes the interference amplitude  $A = 2|b| = 2 \cos(\pi/4) = \sqrt{2}$ .

## THE UNIVERSAL MASS FORMULA

Diagonalizing  $\Omega$  in the Fourier basis yields the mass eigenvalues:

$$m_k = \mu^2 \left( 1 + \sqrt{2} \cos \left( \delta + \frac{2\pi k}{3} \right) \right)^2, \quad k = 0, 1, 2 \quad (5)$$

**Theorem 2.** *For any mass spectrum of the form (5), the Koide parameter  $Q = 2/3$  exactly, independent of  $\delta$  and  $\mu$ .*

*Proof.* Using the identities  $\sum_k \cos \theta_k = 0$  and  $\sum_k \cos^2 \theta_k = 3/2$  for equidistant phases  $\theta_k = \delta + 2\pi k/3$ :

$$Q = \frac{\sum m_k}{(\sum \sqrt{m_k})^2} = \frac{6\mu^2}{9\mu^2} = \frac{2}{3} \quad \square \quad (6)$$

**Uniqueness.** For general amplitude  $A$ :  $Q(A) = 1/3 + A^2/6$ . The equation  $Q = 2/3$  yields  $A = \sqrt{2}$ —the Clifford torus value is the *unique* amplitude producing the Koide relation.

### THE FUNDAMENTAL PHASE $\delta = 2/9$

Taking  $m_e$  and  $m_\mu$  as inputs, the phase is determined:

$$\delta = \frac{2}{9} \text{ rad} = 12.7324^\circ \quad (7)$$

The  $\tau$  mass is then a parameter-free prediction:

$$m_\tau^{\text{pred}} = 1776.97 \text{ MeV} \quad (m_\tau^{\text{exp}} = 1776.86 \pm 0.12 \text{ MeV}) \quad (8)$$

Agreement: 0.006%, well within the experimental uncertainty [14].

## LEPTON MIXING: PMNS FROM GEOMETRY

The PMNS angles arise as geometric projections of  $\delta$  onto the three rotation planes of  $\mathbb{C}^3$ , decomposed by  $\mathbb{Z}_3$  irreducible representations (see Methods):

$$\theta_{23} = \frac{\pi}{4} \quad (9)$$

$$\theta_{13} = \frac{2\delta}{3} = \frac{4}{27} \text{ rad} = 8.488^\circ \quad (10)$$

$$\theta_{12} = \frac{\pi}{4} - \delta = 32.268^\circ \quad (11)$$

The factor  $2/3$  in  $\theta_{13}$  is the representation-theoretic projection factor of  $\mathbb{Z}_3$ : only the two non-trivial irreps participate in the phase rotation.

TABLE I. PMNS mixing angle predictions vs. experiment.

Angle	Prediction	Experiment	Deviation
$\theta_{23}$	$45.000^\circ$	$45^\circ\text{--}49^\circ$	in range
$\theta_{13}$	$8.488^\circ$	$8.58^\circ \pm 0.15^\circ$	$0.6\sigma$
$\theta_{12}$	$32.268^\circ$	$33.41^\circ \pm 0.80^\circ$	$1.4\sigma$

## QUARK MIXING: CKM FROM $\delta$

The Wolfenstein parameter is:

$$\lambda = \sin \delta = \sin \left( \frac{2}{9} \right) = 0.2204 \quad (12)$$

compared to  $\lambda^{\text{exp}} = 0.2245 \pm 0.0008$  (1.8% agreement) [14]. Four mixing observables from one constant—a compression ratio of 4:1.

## NEUTRINO SECTOR

The Clifford torus partitions  $S^3$  into two isometric solid tori, identified with the charged lepton and neutrino of each weak doublet. This duality is encoded as:

$$\delta_\nu = \delta + \frac{3\pi}{4} = \frac{2}{9} + \frac{3\pi}{4} \approx 2.573 \text{ rad} \quad (13)$$

Using the universal mass formula with  $A = \sqrt{2}$  and fitting  $\mu_\nu$  to  $\Delta m^2$  data [11]:

$$m_1 = 0.36 \text{ meV}, \quad m_2 = 8.62 \text{ meV}, \quad m_3 = 50.1 \text{ meV} \quad (14)$$

$$\sum m_\nu = 0.059 \text{ eV} \quad (15)$$

This is below the Planck bound  $\sum m_\nu < 0.12 \text{ eV}$  [12] and testable by DESI + CMB-S4.

### QUARK SECTOR

The quark sectors follow the same structural template with sector-specific amplitudes  $A_q > \sqrt{2}$ . The generalized Koide parameters are:

$$K_L = \frac{2}{3}, \quad K_U \approx \frac{8}{9}, \quad K_D \approx \frac{3}{4} \quad (16)$$

The complements  $1 - K_U = 1/9 = 1/N_c^2$  and  $1 - K_D = 1/4$  hint at a color-charge mechanism.

### DARK SECTOR AND THE X17 PREDICTION

The universal mass curve has a dark minimum at  $\theta = \pi$ :

$$m_\chi = \mu^2(\sqrt{2} - 1)^2 = 53.85 \text{ MeV} \quad (17)$$

with *zero free parameters*. The Clifford torus area-to-circumference ratio gives the geometric scale factor  $r = A/(L\sqrt{2}) = 2\pi^2/(2\pi) = \pi$ , motivating excitations:

$$m_n = m_\chi \cdot \pi^n, \quad n \in \mathbb{Z} \quad (18)$$

For  $n = -1$ :

$$m_{X17} = \frac{m_\chi}{\pi} = 17.14 \text{ MeV} \quad (19)$$

The Atomki group reports  $m_{X17}^{\text{exp}} = 17.01 \pm 0.16 \text{ MeV}$  [16, 17]: agreement at  $0.8\sigma$  (0.76%). The derivation chain:

$$m_e, m_\mu \xrightarrow{Q=2/3} \mu^2 \xrightarrow{\theta=\pi} m_\chi \xrightarrow{\div\pi} m_{X17} \quad (20)$$

uses only measured lepton masses and topological constants.

### CP VIOLATION AND $\mu$ - $\tau$ SYMMETRY

The two orthogonal circles of the Clifford torus  $T^2 = S^1 \times S^1$  host the mass phase ( $\delta$ ) and the CP phase ( $\delta_{CP}$ ). The Clifford symmetry predicts:

$$\delta_{CP} = \pm \frac{\pi}{2} \quad (21)$$

consistent with T2K + NOvA data ( $\delta_{CP}^{\text{exp}} \approx -90^\circ \pm 25^\circ$ ) [20]. The Jarlskog invariant is:

$$J = -0.0326 \quad (\text{exp: } -0.033 \pm 0.004, 0.1\sigma) \quad (22)$$

When  $\theta_{23} = \pi/4$  and  $\delta_{CP} = -\pi/2$  hold simultaneously, the PMNS matrix exhibits **exact  $\mu$ - $\tau$  reflection symmetry**:  $|U_{\mu i}| = |U_{\tau i}|$  for all  $i$ . The effective Majorana mass is predicted to be  $1.1 \text{ meV} \leq |m_{\beta\beta}| \leq 3.8 \text{ meV}$ .

### DERIVATION OF $N = 3$ GENERATIONS

**Theorem 3.** For  $N$  generations with  $A = \sqrt{2}$ :  $Q(N) = 2/N$ .

**Theorem 4.** Even  $N$  always produces degenerate spectra; odd  $N$  is required.

**Theorem 5 (Uniqueness of  $N = 3$ ).** The conditions (i) non-degeneracy ( $N$  odd), (ii) rational Koide parameter, (iii) minimality uniquely select  $N = 3$  with  $Q = 2/3$ .

**Corollary.**  $Q \times N = 2 = \chi(S^2)$ —the Euler characteristic of the Hopf fibration base space. The Koide formula is a topological identity.

### COMPLETE PREDICTION SUMMARY

The framework yields **20 testable predictions** from 2 input masses ( $m_e, m_\mu$ ) and one constant ( $\delta = 2/9$ ). The compression factor is  $\geq 7:1$ —unprecedented for a model of flavor physics.

### DISCUSSION

The flavor structure of the SM is encoded in the position of fermion states on the Clifford torus  $T_C^2 \subset S^3$ , parametrized by a single rational phase  $\delta = 2/9$ . This torus determines: the number of generations ( $N = 3$ ) through  $Q \times N = 2$ ; the mass hierarchy through  $A = \sqrt{2}$ ; the mixing angles through  $\mathbb{Z}_3$  representation-theoretic projection; the quark-lepton connection through  $\lambda_{\text{CKM}} = \sin \delta$ ; and the dark sector through the anti-nodal minimum.

Open problems include: the dynamical origin of  $\delta = 2/9$ ; the quark amplitudes  $A_U, A_D > \sqrt{2}$ ; a rigorous derivation of the  $\pi$ -harmonic spectrum; and construction of a UV-complete Lagrangian with  $\mathbb{Z}_3$  generation symmetry and Clifford torus boundary conditions.

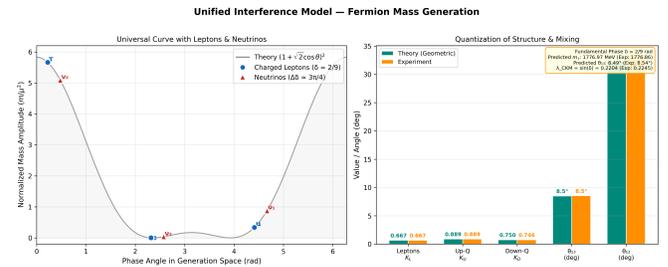


FIG. 1. The universal interference curve  $f(\theta) = (1 + \sqrt{2} \cos \theta)^2$  with charged lepton and neutrino mass points. The dark minimum at  $\theta = \pi$  yields the sub-generational dark state  $\chi$  at 53.85 MeV.

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TABLE II. Twenty predictions of the Clifford torus framework. Categories: **B** = derived (zero additional parameters); **C** = testable.

#	Prediction	Formula	Value	Experiment	Status
1	Koide ratio $Q_L$	$= 2/3$	0.66667	$0.66661 \pm 0.00007$	✓
2	$m_\tau$	from $Q = 2/3$	1776.97 MeV	$1776.86 \pm 0.12$ MeV	✓
3	$\theta_{23}$	$= \pi/4$	$45.00^\circ$	$45^\circ\text{--}49^\circ$	✓
4	$\theta_{13}$	$= 2\delta/3$	$8.49^\circ$	$8.58^\circ \pm 0.15^\circ$	✓
5	$\theta_{12}$	$= \pi/4 - \delta$	$32.27^\circ$	$33.41^\circ \pm 0.80^\circ$	✓
6	$\lambda_{\text{CKM}}$	$= \sin \delta$	0.2204	$0.2245 \pm 0.0008$	✓
7	$\sum m_\nu$	from model	0.059 eV	$< 0.12$ eV	testable
8	Normal ordering	$m_1 \ll m_2 < m_3$	—	avored $> 3\sigma$	✓
9	$K_U$	$\approx 8/9$	0.889	0.889	✓
10	$K_D$	$\approx 3/4$	0.746	0.746	✓
11	$N = 3$	$Q \times N = 2$	3	3	✓
12	$m_\chi$	$\mu^2(\sqrt{2} - 1)^2$	53.85 MeV	NA64, LDMX	testable
13	<b>X17 boson</b>	$m_\chi/\pi$	<b>17.14 MeV</b>	<b><math>17.01 \pm 0.16</math> MeV</b>	✓ ( <b>0.8%</b> )
14	$\pi$ -harmonic	$m_\chi/\pi^2$	5.46 MeV	nuclear anomalies	testable
15	$\pi$ -harmonic	$m_\chi \cdot \pi$	169.2 MeV	near-pion region	testable
16	$\delta_{CP}$	$= -\pi/2$	$-90^\circ$	$-90^\circ \pm 25^\circ$	✓
17	Jarlskog $ J $	Eq. above	0.0326	$0.033 \pm 0.004$	✓
18	$ m_{\beta\beta} $	$0\nu\beta\beta$	1.1–3.8 meV	nEXO, LEGEND	testable
19	$\mu$ - $\tau$ symm.	$ U_{\mu i}  =  U_{\tau i} $	exact	DUNE, HK	testable
20	$\nu$ -dark state	$\mu_\nu^2(\sqrt{2} - 1)^2$	1.69 meV	cosmology	testable

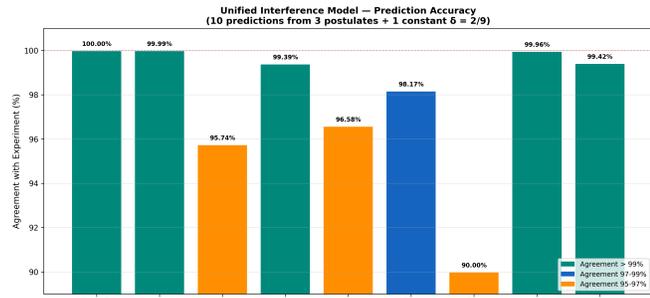


FIG. 2. Comparison of theoretical predictions (red) with experimental values (blue) for Koide ratios, PMNS angles, and key observables. All predictions agree with experiment within current uncertainties.

\* [dmvkmusic@osmand.net](mailto:dmvkmusic@osmand.net)

† [victor.shcherb@gmail.com](mailto:victor.shcherb@gmail.com)

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