Two Relativistic Derivations of the Photon Energy-Frequency Relation and Their Pedagogical Significance

Zilu Liu

Abstract

This paper presents two novel methods for deriving the photon energy formula $E = h\nu$. Both methods are strictly rooted in the framework of special relativity, without presupposing quantum theory. Method One establishes the proportionality $E \propto \nu$ rigorously by analyzing the complementarity between the relativistic Doppler effect and energy-momentum conservation in Compton scattering, and subsequently determines the Planck constant h experimentally. Method Two derives the photon energy formula naturally by examining the limiting behavior of a massive particle as its velocity approaches the speed of light, incorporating the frequency ratio relationship. These derivations **compensate for the deficiency in traditional pedagogy where this formula is introduced merely as a prescriptive conjecture fitting experimental results**, providing a profound and unified perspective for understanding this fundamental formula and highlighting the role of relativity as its theoretical foundation, thus holding significant value for physics education.

Keywords: Photon energy; Special relativity; Doppler effect; Compton scattering; Limit process; Physics education

1 Introduction

The photon energy formula $E = h\nu$ is a bridge connecting classical electromagnetic theory with quantum physics and a landmark in the birth of quantum mechanics. In traditional curricula and classic textbooks [Feynman et al., 1965, Halliday et al., 2013], the introduction of this formula typically follows the historical path: first, as a "quantization" hypothesis proposed by Planck to explain the blackbody radiation spectrum; subsequently, borrowed by Einstein to explain the experimental of the photoelectric effect, its status is closer to a **prescriptive conjecture fitting experimental results** or an **empirical assumption**. In later instruction, when explaining Compton scattering, $E = h\nu$ and $p = h/\lambda$ are again used as given conditions to prove the wavelength shift formula.

While this narrative based on great experimental history has its value, from the perspectives of pedagogy and theoretical unity, it renders this core relationship somewhat **isolated**, failing to demonstrate how it **emerges logically** from more fundamental physical theories. Students often wonder: Was this formula "guessed"? What is its theoretical foundation? How is it connected to knowledge we have already mastered, such as relativity?

A more satisfactory approach is: **Can we logically derive this relationship without presupposing the quantum nature of light, but starting from more basic and widely accepted theories?** Special relativity is precisely such a perfect starting point. This paper aims to demonstrate two derivation methods purely based on relativity. Method One combines the wave nature of light (Doppler effect) with its particle nature (collision dynamics), while Method Two presents the photon energy formula as the natural destination of a relativistic particle through an elegant limiting process, incorporating the frequency ratio relationship. **The philosophical and historical paths of these two methods are different**. They do not repeat history (from experiment to theory) but perform a **"retrospection" and "unification"** in theory, demonstrating how this key formula can be logically derived from the pillar of modern physics: space-time symmetry (relativity). This not only possesses theoretical beauty but also effectively helps students integrate the core concepts of modern physics and understand the internal consistency and harmony of physics.

2 Method One: Complementary Derivation via Doppler Effect and Collision Dynamics

This method is divided into two steps: first, proving the proportionality $E \propto \nu$ purely from relativity; second, determining the proportionality constant h by analyzing a collision process.

2.1 Step 1: Deriving $E \propto \nu$ from Relativity

Consider a light source emitting light of frequency ν_0 in its rest frame (S' frame). We **assume** the corresponding photon has energy E_0. When the source moves with velocity u along the x-axis, an observer in the laboratory frame (S frame) measures the frequency ν and energy E.

According to the relativistic Doppler effect, the frequency transformation is:

$$\nu = \nu_0 \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta} \tag{1}$$

where $\beta = u/c$, and θ is the angle between the reception direction and the source's motion direction.

Now, we analyze the transformation of the photon's energy from the S' to the S frame. The transformation of the photon's four-momentum follows the same Lorentz transformation as coordinates. The photon's four-momentum in S' is $(E_0/c, \vec{p_0})$, with $|\vec{p_0}| = E_0/c$. In S, it is $(E/c, \vec{p})$. The transformation of its energy component is analogous to that of the time component, thus:

$$E = \gamma_u(E_0 + up_{0x}) = \gamma_u E_0(1+\beta\cos\theta_0) \tag{1}$$

Here, θ_0 is the emission angle in S'. Using velocity transformation, the relationship between θ_0 and the observation angle θ in S can be established, yielding:

$$E = E_0 \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta} \tag{2}$$

Comparing Eqs. (1) and (2), we immediately obtain:

$$\frac{E}{E_0} = \frac{\nu}{\nu_0} \quad \text{or} \quad \frac{E}{\nu} = \frac{E_0}{\nu_0} \tag{3}$$

Equation (3) shows that for the same photon, the ratio of its energy to frequency is **invariant** across all inertial frames. Therefore, we must have $E \propto \nu$. **This step relies solely on relativistic spacetime concepts, independent of quantum assumptions.**

2.2 Step 2: Determining the Constant h via Compton Scattering

The proportionality $E \propto \nu$ is dictated by relativistic spacetime properties, but the numerical value of the constant h must be determined experimentally. We achieve this by analyzing the collision between a photon and an electron (Compton scattering).

Consider an electron of rest mass m_e initially at rest. A photon with energy E and momentum p collides with it elastically. After the collision, the photon's energy becomes E', and the electron acquires relativistic momentum $\vec{p_e}$ and energy K_e. Applying energy and momentum conservation:

$$E + m_e c^2 = E' + \sqrt{(p_e c)^2 + (m_e c^2)^2} \tag{4}$$

$$\vec{p} = \vec{p'} + \vec{p_e} \tag{5}$$

Let ϕ be the angle between the incident and scattered photon directions. Through rigorous vector operations and algebraic derivation (see Compton 1923 for details), the electron terms can be eliminated from Eqs. (4) and (5) to obtain:

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2}(1 - \cos\phi) \tag{6}$$

Now, we **introduce** a proportionality constant h and **define** the photon's energy as $E = h\nu$. Substituting this into Eq. (6) and using $\nu = c/\lambda$, we get:

$$\frac{\lambda'}{hc} - \frac{\lambda}{hc} = \frac{1}{m_e c^2}(1 - \cos\phi) \tag{2}$$

Rearranging gives:

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\phi) \tag{7}$$

Compton's experimental observations perfectly verified Eq. (7) and precisely measured the constant h in the formula, finding it identical to the Planck constant [Compton, 1923]. Therefore, combining the conclusion $E \propto \nu$ from Step 1 and the experimental calibration from Step 2, we completely derive:

$$E = h\nu \tag{3}$$

Thus, the photon energy formula transitions from a "prescriptive conjecture fitting experimental results" to a "theorem" that can be derived from relativity and verified experimentally.

3 Method Two: Relativistic Limit Derivation from Particle Behavior with Frequency Ratio Relationship

This method provides a more direct perspective, treating the photon as the limit of a massive particle as its velocity approaches the speed of light, thereby seamlessly connecting its energy formula with relativistic mechanics.

3.1 Physical Scenario and Energy Ratio Function

Consider a particle of rest mass m_0 emitted from a source in its rest frame (S' frame). Its speed is v_0, and its total energy is $\epsilon_0 = \gamma_0 m_0 c^2$, where $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$. In the laboratory frame (S frame), the source moves with velocity u, and the particle is observed to have speed v, energy $\epsilon = \gamma_v m_0 c^2$, and its velocity direction makes an angle θ with the x-axis.

The particle's energy in S' and S is related by Lorentz transformation:

$$\epsilon = \gamma_u(\epsilon_0 + u p_{0x}) \tag{4}$$

where p_{0x} is the x-component of the particle's momentum in S'. Using $p_{0x} = \gamma_0 m_0 v_{0x} = (\epsilon_0/c^2) v_{0x}$, we get:

$$\frac{\epsilon_0}{\epsilon} = \frac{1}{\gamma_u(1 + \frac{u v_{0x}}{c^2})} \tag{8}$$

Using the relativistic velocity inverse transformation formula $v_{0x} = (v_x - u)/(1 - u v_x/c^2)$, where $v_x = v\cos\theta$, and substituting into Eq. (8). After algebraic simplification (see Appendix for details), we obtain the concise expression:

$$f(v) \equiv \frac{\epsilon_0}{\epsilon} = \gamma_u \left(1 - \frac{uv\cos\theta}{c^2}\right) \tag{9}$$

This function $f(v)$ describes how the energy ratio of a massive particle varies with its speed v in the S frame.

3.2 Frequency Ratio Relationship and Limit Process

In the S' frame, the particle possesses some intrinsic periodic characteristic with frequency ν_0. In the S frame, the observed frequency of this periodic characteristic is ν. According to the relativistic Doppler effect, the relationship between these two frequencies is:

$$\frac{\nu}{\nu_0} = \frac{\sqrt{1-\beta^2}}{1 - \beta\cos\theta} \tag{10}$$

where $\beta = u/c$.

Now, we examine the limiting behavior as the particle's speed approaches the speed of light ($v \to c$). At this limit, the particle's behavior approaches that of a photon. Taking the limit of Eq. (9):

$$\lim_{v\to c} f(v) = \lim_{v\to c} \gamma_u \left(1 - \frac{uv\cos\theta}{c^2}\right) = \gamma_u(1 - \beta\cos\theta) \tag{11}$$

Comparing Eq. (10) with Eq. (11), we find:

$$\frac{\nu}{\nu_0} = \frac{1}{\gamma_u(1 - \beta\cos\theta)} = \frac{1}{\lim_{v\to c} f(v)} = \lim_{v\to c}\frac{\epsilon}{\epsilon_0} \tag{12}$$

That is:

$$\frac{\nu}{\nu_0} = \lim_{v\to c}\frac{\epsilon}{\epsilon_0} \tag{13}$$

This means that as the velocity of a massive particle approaches the speed of light, its energy ratio ϵ_0/ϵ approaches the frequency ratio ν_0/ν. Therefore, for a photon ($v = c$), we must have:

$$\frac{E_0}{E} = \frac{\nu_0}{\nu} \quad \text{or} \quad \frac{E}{\nu} = \frac{E_0}{\nu_0} \tag{14}$$

Since E_0/ν_0 is a constant (in the S' frame, the characteristics of photons emitted by the source are fixed), E/ν is also a constant invariant across reference frames, denoted as h. Thus, we obtain:

$$E = h\nu \tag{5}$$

Therefore, through this limiting process, we naturally derive the photon energy formula from the mechanical behavior of relativistic particles.

4 Discussion and Pedagogical Significance

The two derivation methods proposed in this paper have significant teaching advantages and theoretical value.

1. **Compensating for the Logical Gap in Traditional Teaching**: Traditional teaching presents $E = h\nu$ as the starting point, whereas this paper treats it as the end-point. This approach fills the logical chain from "relativity" to "quantum theory," helping students recognize that this formula is not without foundation but is deeply connected to relativistic spacetime concepts. It answers the student's question of "Why $E = h\nu$?" transforming the formula from "memorized knowledge" to "understood knowledge."

2. **Strengthening the Unity of the Theoretical System**: Method One demonstrates how relativity (Doppler effect) combines with experimental physics (Compton scattering) to jointly establish a physical law. Method Two brilliantly shows how relativistic mechanics, through a limiting process, encompasses the behavior of photons, promoting students' understanding of the grand unification of physical theories.

3. **Highlighting the Power of Limit Thinking in Physics**: Method Two is a perfect demonstration of limit thinking. It shows that the core formula of new theory (quantum theory) can be seen as the behavior of entities from old theory (relativistic mechanics) under a certain limit. This idea recurs throughout the development of physics (e.g., classical mechanics as the low-speed limit of relativistic mechanics).

4. **Application in Teaching Practice**: These two derivations are particularly suitable for teaching after completing special relativity and before entering quantum mechanics. They can serve as a bridge connecting these two parts, stimulating students' interest in learning and inquiry. Teachers can choose to explain one in detail or compare both to cultivate students' ability for comparison and integration.

5 Conclusion

This paper successfully derives the photon energy formula $E = h\nu$ from special relativity through two distinct paths. 1. **Method One** combines the relativistic Doppler effect and Compton scattering, clearly dividing the derivation into "proportionality relation" and "constant determination," with rigorous logic. 2. **Method Two** elegantly demonstrates how the photon energy formula emerges from relativistic mechanics by analyzing the limiting behavior of a massive particle as its velocity approaches the speed of light, incorporating the frequency ratio relationship.

Together, these methods show that $E = h\nu$ is not a purely empirical assumption but an inevitable relationship that can be deeply understood and strictly derived within the relativistic framework. They **transcend the historical narrative order**, providing a more theoretically unified and pedagogically inspiring way to teach this cornerstone formula of quantum mechanics, helping students construct a more complete and self-consistent modern physical picture.

Appendix: Simplification of Formula (9)

From Eq. (8):

$$\frac{\epsilon_0}{\epsilon} = \frac{1}{\gamma_u(1 + \frac{uv_{0x}}{c^2})}$$

Substitute the velocity inverse transformation $v_{0x} = \frac{v\cos\theta - u}{1 - \frac{uv\cos\theta}{c^2}}$:

$$\begin{aligned}
\frac{\epsilon_0}{\epsilon} &= \frac{1}{\gamma_u\left[1 + \frac{u}{c^2}\cdot\frac{v\cos\theta - u}{1 - \frac{uv\cos\theta}{c^2}}\right]} \\
&= \frac{1}{\gamma_u}\cdot\frac{1 - \frac{uv\cos\theta}{c^2}}{1 - \frac{uv\cos\theta}{c^2} + \frac{u}{c^2}(v\cos\theta - u)} \\
&= \frac{1}{\gamma_u}\cdot\frac{1 - \frac{uv\cos\theta}{c^2}}{1 - \frac{u^2}{c^2}} \\
&= \frac{1 - \frac{uv\cos\theta}{c^2}}{\gamma_u(1 - \beta^2)}
\end{aligned}$$

Note that $\gamma_u = \frac{1}{\sqrt{1-\beta^2}}$, so $\gamma_u(1-\beta^2) = \frac{1-\beta^2}{\sqrt{1-\beta^2}} = \sqrt{1-\beta^2} = \frac{1}{\gamma_u}$. Therefore:

$$\frac{\epsilon_0}{\epsilon} = \gamma_u\left(1 - \frac{uv\cos\theta}{c^2}\right)$$

Q.E.D.

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