Expansion of the Number π into a Series of Trigonometric Functions and Its Connection with the Riemann Zeta Function

Bohdan Dykyi
2026

Abstract

In this work, series expansions for negative and positive integer powers of the number π are derived using Viète's infinite product and differentiation techniques. A representation of these powers in terms of trigonometric series involving tangent functions is obtained. Furthermore, a connection between these expansions and the values of the Riemann zeta function at even arguments is established. Explicit formulas for the reciprocal values of the zeta function are presented. Several illustrative examples are provided.

Introduction

I am not particularly concerned with whether the results obtained in this work have practical applications or immediate usefulness in applied sciences. I pursue mathematics primarily as a personal intellectual activity and a source of aesthetic satisfaction, motivated by the search for interesting identities and elegant formulas. From this perspective, I would not be disappointed if the results presented here turned out to have no direct application in technology or engineering.

I might not have published this work at all, but the relations obtained seemed to me sufficiently interesting and beautiful to be worth sharing, and I hope they may appear so to the reader as well. Although the computations themselves are not complicated, the main idea consists simply in taking Viète's formula, applying logarithms and differentiation, and then substituting convenient values in order to obtain explicit formulas involving the number π.

Calculations

Viète's formula:

$$\prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) = \frac{\sin(x)}{x}$$

We apply the natural logarithm and use the basic properties of logarithms.

$$\log_a(x) + \log_a(x) = \log_a(xy)$$

$$\ln\left(\prod_{k=1}^{\infty}\cos\left(\frac{x}{2^k}\right)\right)=\ln\left(\frac{\sin(x)}{x}\right)$$
$$\sum_{k=1}^{\infty}\ln\left(\cos\left(\frac{x}{2^k}\right)\right)=\ln(\sin(x))-\ln(x)$$

Now we differentiate.

$$\frac{d}{dx}\sum_{k=1}^{\infty}\ln\left(\cos\left(\frac{x}{2^k}\right)\right)=-\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)$$
$$\frac{d}{dx}(\ln(\sin(x))-\ln(x))=\cot(x)-\frac{1}{x}$$

We obtain the following equality.

$$\frac{1}{x}-\cot(x)=\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)$$

We substitute $x=\frac{\pi}{2}$

$$\frac{2}{\pi}-\cot\left(\frac{\pi}{2}\right)=\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\left(\frac{\pi}{2^{k+1}}\right)$$

$$\frac{2}{\pi}=\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\left(\frac{\pi}{2^{k+1}}\right)$$

$$\frac{1}{\pi}=\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}\tan\left(\frac{\pi}{2^{k+1}}\right)$$

We obtain the following formula:

$$\frac{1}{\pi}=\sum_{k=2}^{\infty}\frac{1}{2^k}\tan\left(\frac{\pi}{2^k}\right)$$

Now we take the formula

$$\frac{1}{x}-\cot(x)=\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)$$

and differentiate it once again.

$$\csc^2(x) - \frac{1}{x^2} = \sum_{k=1}^{\infty} \frac{1}{4^k} \sec^2\left(\frac{x}{2^k}\right)$$

We substitute $x = \dfrac{\pi}{2}$

$$\frac{1}{\pi^2} = \frac{1}{4} - \sum_{k=2}^{\infty} \frac{1}{4^k} \sec^2\left(\frac{\pi}{2^k}\right)$$

In order to derive a formula for natural powers of the number π, it is necessary to determine the n-th derivative of the following functions

$$f(x) = \frac{1}{x} - \cot(x); \quad g(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \tan\left(\frac{x}{2^k}\right)$$

and substitute. $x = \dfrac{\pi}{2}$

We begin with the simplest case and compute the n-th derivative of $\dfrac{1}{x}$

$$\frac{d^n}{dx^n}\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$$

We substitute $x = \dfrac{\pi}{2}$

$$\frac{d^n}{dx^n}\left(\frac{1}{x}\right)\Big|_{x=\frac{\pi}{2}} = \frac{(-1)^n n!}{\left(\frac{\pi}{2}\right)^{n+1}} = \frac{2^{n+1}(-1)^n n!}{\pi^{n+1}}$$

For the tangent and cotangent functions, there is no simple formula for the n-th derivative. However, in the case of the cotangent this is not required, since only the derivatives at the point π/2 are needed. Therefore, Taylor series may be used to derive them; for this purpose, the Taylor series of the tangent function is required rather than that of the cotangent.

$$\tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}\left(2^{2n}-1\right) B_{2n}\, x^{2n-1}}{(2n)!}$$

We make the substitution $x = t - \dfrac{\pi}{2}$

$$\tan\left(x - \frac{\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}\left(2^{2n}-1\right) B_{2n}}{2n} \frac{\left(x - \frac{\pi}{2}\right)^{2n-1}}{(2n-1)!}$$

$$\cot(x) = -\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}\left(2^{2n}-1\right)B_{2n}}{2n} \frac{\left(x-\frac{\pi}{2}\right)^{2n-1}}{(2n-1)!}$$

We recall the Taylor formula.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We obtain the following formula.

$$\frac{d^n}{dx^n}(\cot(x))|_{x=\frac{\pi}{2}} = -\frac{|B_{n+1}|2^{n+1}\left(2^{n+1}-1\right)}{n+1}$$

It remains to find a formula for g(x). However, there is no general formula for the n-th derivative of the tangent function, and therefore the same method as for the cotangent cannot be applied. Although it is possible that such a formula could be obtained, here we proceed in a different way.

First, several initial derivatives of the tangent function are computed.

$$1+\tan^2 x; \quad 2\tan x + 2\tan^3 x; \quad 2 + 8\tan^2 x + 6\tan^4 x;$$

$$16\tan x + 40\tan^3 x + 24\tan^5 x; \quad 16 + 136\tan^2 x + 240\tan^4 x + 120\tan^6 x$$

Let the coefficients of the powers of the tangent function be denoted by, $T_{n,j}$ where nnn indicates the order of the derivative of the tangent function and jjj denotes the power of the tangent associated with the coefficient. Using this notation, we can construct the following table:

n \ j	0	1	2	3	4	5	6
0	0	1	0	0	0	0	0
1	1	0	1	0	0	0	0
2	0	2	0	2	0	0	0
3	2	0	8	0	6	0	0
4	0	16	0	40	0	24	0
5	16	0	136	0	240	0	120

The coefficients satisfy the following rule:

$$T_{n,j} = (j-1)T_{n-1,j-1} + (j+1)T_{n-1,j+1}$$

The sequence of coefficients can be extended indefinitely. For the coefficients located at the boundary of the table, the following formula holds:

$$T_{n,0} = T_{n-1,1}$$

Thus, we obtain a general formula for the n-th derivative of the tangent function:

$$\frac{d^n}{dx^n}(\tan x) = \sum_{j=0}^{n+1} T_{n,j} \tan^j x$$

From this, we derive formulas for $g^n(x)$and for $g^n\left(\frac{\pi}{2}\right)$

$$\frac{d^n}{dx^n}\left(\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)\right) = \sum_{k=1}^{\infty}\frac{1}{2^{(n+1)k}}\sum_{j=0}^{n+1} T_{n,j}\tan^j\left(\frac{x}{2^k}\right)$$

$$\frac{d^n}{dx^n}\left(\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)\right)\Big|_{x=\frac{\pi}{2}} = \sum_{k=1}^{\infty}\frac{1}{2^{(n+1)k}}\sum_{j=0}^{n+1} T_{n,j}\tan^j\left(\frac{\pi}{2^{k+1}}\right)$$

We substitute this into our formula.

$$\frac{2^{n+1}(-1)^n n!}{\pi^{n+1}} + \frac{|B_{n+1}|2^{n+1}\left(2^{n+1}-1\right)}{n+1} = \sum_{k=1}^{\infty}\frac{1}{2^{(n+1)k}}\sum_{j=0}^{n+1} T_{n,j}\tan^j\left(\frac{\pi}{2^{k+1}}\right)$$

We now express $\frac{1}{\pi^{n+1}}$

$$\frac{1}{\pi^{n+1}} = \frac{(-1)^n}{2^{n+1}n!}\sum_{k=1}^{\infty}\frac{1}{2^{k(n+1)}}\sum_{j=0}^{n+1} T_{n,j}\tan^j\left(\frac{\pi}{2^{k+1}}\right) - \frac{(-1)^n\left|B_{n+1}\right|\left(2^{n+1}-1\right)}{(n+1)!}$$

Replacing n+1 by n, we obtain the following formula, which is valid only for natural numbers n>1.

$$\frac{1}{\pi^n} = \frac{\left|B_n\right|\left(2^n-1\right)}{n!} - \frac{(-1)^n}{2^n\,(n-1)!}\sum_{k=1}^{\infty}\frac{1}{2^{nk}}\sum_{j=0}^{n} T_{n-1}\tan^j\left(\frac{\pi}{2^{k+1}}\right);\quad n\in\mathbb{N}$$

$$n \geqslant 2$$

$$T_{0,j}=0\,;\quad j\neq 1;\quad T_{0,1}=1;$$

$$T_{n,j}=(j-1)T_{n-1,j-1}+(j+1)T_{n-1,j-1};\quad T_{n,0}=T_{n-1,1}$$

For n=1, the following formula holds.

$$\frac{1}{\pi}=\sum_{k=2}^{\infty}\frac{1}{2^k}\tan\left(\frac{\pi}{2^k}\right)$$

Connection with the Riemann zeta function

In this formula, it is natural to substitute the Riemann zeta function due to its connection with the number pi

$$\zeta\left(2n\right)=\frac{\left(2\pi\right)^{2n}\left|B_{2n}\right|}{2\left(2n\right)!}$$

We express the number pi raised to the power -2n.

$$\pi^{-2n}=\frac{|B_{2n}|2^{2n-1}}{\zeta\left(2n\right)\left(2n\right)!}$$

We take the previously derived expression, but only for even values of n

$$\pi^{-2n}=\frac{|B_{2n}|\left(2^{2n}-1\right)}{\left(2n\right)!}-\frac{1}{2^{2n}\left(2n-1\right)!}\sum_{k=1}^{\infty}\frac{1}{4^{nk}}\sum_{j=0}^{2n}T_{2n-1}\tan^{j}\left(\frac{\pi}{2^{k+1}}\right)$$

We equate these expressions.

$$\frac{|B_{2n}|2^{2n-1}}{\zeta\left(2n\right)\left(2n\right)!}=\frac{|B_{2n}|\left(2^{2n}-1\right)}{\left(2n\right)!}-\frac{1}{2^{2n}\left(2n-1\right)!}\sum_{k=1}^{\infty}\frac{1}{4^{nk}}\sum_{j=0}^{2n}T_{2n-1}\tan^{j}\left(\frac{\pi}{2^{k+1}}\right)$$

We express $\dfrac{1}{\zeta\left(2n\right)}$ and immediately simplify the result.

$$\frac{1}{\zeta\left(2n\right)}=2-2^{1-2n}-\frac{4^{1-2n}n}{|B_n|}\sum_{k=1}^{\infty}\frac{1}{4^{nk}}\sum_{j=0}^{2n}T_{2n-1}\tan^{j}\left(\frac{\pi}{2^{k+1}}\right)$$

Thus, we obtain a formula that represents the reciprocal values of the Riemann zeta function in terms of a series of trigonometric functions.

Explicit examples

We now expand the tangent functions. We begin with the formula for n = 1.

$$\frac{1}{\pi}=\frac{1}{4}+\frac{1}{8}\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}+\frac{1}{16}\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}+\frac{1}{32}\sqrt{\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}+\ldots$$

For n = 2.

$$\frac{1}{\pi^2}=\frac{1}{2}-\frac{1}{2^3+\sqrt{2^5}}-\frac{1}{2^5+\sqrt{2^9+\sqrt{2^{17}}}}-\frac{1}{2^7+\sqrt{2^{13}+\sqrt{2^{25}+\sqrt{2^{49}}}}}-\ldots$$

At the same time, we evaluate the zeta function at argument 2.

$$\frac{1}{\zeta(2)}=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{2^3+\sqrt{2^5}}-\frac{1}{2^5+\sqrt{2^9+\sqrt{2^{13}}}}-\ldots\right)$$

$$\frac{1}{1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots}=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{2^3+\sqrt{2^5}}-\frac{1}{2^5+\sqrt{2^9+\sqrt{2^{13}}}}-\ldots\right)$$

Here, successive powers of square roots of two with odd exponents appear, and in the factors they increase according to the rule 2n − 1. In the same way, a formula for ζ(4) and higher values can be derived. In this work, it is sufficient to present the formula for ζ(4); further cases can be obtained analogously using the formulas derived above. Most likely, they will also involve infinite products of square roots of two in various combinations. From the formula derived above, we obtain the following equality for n = 2.

$$\frac{1}{\zeta(4)}=15\left(\frac{1}{2^4}-\frac{1}{2^8}\frac{4-\sqrt{2}}{\left(2+\sqrt{2}\right)^2}-\frac{1}{2^{12}}\frac{4-\sqrt{2+\sqrt{2}}}{\left(2+\sqrt{2+\sqrt{2}}\right)^2}-\ldots\right)$$

$$\frac{1}{1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots}=15\left(\frac{1}{2^4}-\frac{1}{2^8}\frac{4-\sqrt{2}}{\left(2+\sqrt{2}\right)^2}-\frac{1}{2^{12}}\frac{4-\sqrt{2+\sqrt{2}}}{\left(2+\sqrt{2+\sqrt{2}}\right)^2}-\ldots\right)$$

Conclusion

In this work, formulas for negative powers of the number π were derived. Formulas for positive integer powers can also be obtained simply by taking the reciprocal expression.

$$\pi^n=\frac{(-2)^n n!}{|B_n|2^n\left(2^n-1\right)-n\sum_{k=1}^{\infty}\frac{1}{2^{nk}}\sum_{j=0}^{\infty}T_{n-1}\tan^j\left(\frac{\pi}{2^{k+1}}\right)};\quad n\in\mathbb{N}$$

$$n\geqslant 2$$

For n = 1.

$$\pi=\frac{1}{\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\left(\frac{\pi}{2^{k+1}}\right)}$$

Explicit formulas for the Riemann zeta function were also obtained. Although these series converge slowly for small values of n, the rate of convergence increases as n grows. For example, in the formula for π^(-n) with n = 10, the first term of the series already agrees with the exact value to nine decimal places. For larger n, the convergence continues to improve. A similar behavior is observed for the reciprocal of the Riemann zeta function. However, for π^n the convergence rate increases more slowly. This concludes the present work.