

Solid Angle of Torus on Normal Geometrical Axis Using HCR's Method of Concentric Cones

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1. Introduction

An analytical method is developed for the exact evaluation of the solid angle subtended by a torus at a point lying on its axis of symmetry, i.e., on the line perpendicular to the mid-plane of the torus and passing through its centre. The method is based on a geometric enclosure of the torus between two coaxial right circular cones with a common apex at the observation point and with axes coincident with the torus axis. It is shown that the solid angle associated with the torus equals the algebraic sum i.e. the difference between the solid angles subtended by the outer and inner bounding cones at the apex [1,2,3]. This construction leads to closed-form expressions for the solid angle as a function of the torus radii and the axial distance of the observation point, without recourse to surface integration. The resulting formulation provides a concise geometric characterization of toroidal visibility and is well suited for applications in geometric modeling, where exact angular measures are required, and in photometry and radiative transfer, particularly for the evaluation of irradiance, flux, and angular response of axially symmetric toroidal sources and apertures.

2. Solid angle subtended by a torus at a point on its normal geometric axis

Let there be a torus with inner and outer radii r and R respectively, and a given point say $P(0, 0, h)$ lying on the axis at a normal height h from the centre O (point $P(0, 0, h)$ is lying perpendicularly outwards to the plane of paper at a height h from the centre O , as shown by Top view in Figure 1 below).

Now, draw the tangents PA & PB from the point $P(0, 0, h)$ at the surface of torus (upper part of Figure 1 is showing the Front view i.e. cross-sectional view of torus).

$$\Rightarrow FD = OD - OB = R - r$$

$$\Rightarrow FC = CD = CA = CB = \frac{FD}{2} = \frac{R - r}{2}$$

$$\Rightarrow OC = OF + FC = r + \frac{R - r}{2} = \frac{R + r}{2}$$

In right $\triangle COP$ (Fig. 1),

$$\begin{aligned} \Rightarrow CP &= \sqrt{OC^2 + OP^2} = \sqrt{\left(\frac{R + r}{2}\right)^2 + (h)^2} \\ &= \frac{1}{2}\sqrt{(R + r)^2 + 4h^2} \end{aligned}$$

$$\Rightarrow \sin \angle OPC = \frac{OC}{CP}$$

$$\sin \beta = \frac{\left(\frac{R + r}{2}\right)}{\left(\frac{1}{2}\sqrt{(R + r)^2 + 4h^2}\right)} = \frac{R + r}{\sqrt{(R + r)^2 + 4h^2}} \quad \dots \dots (1)$$

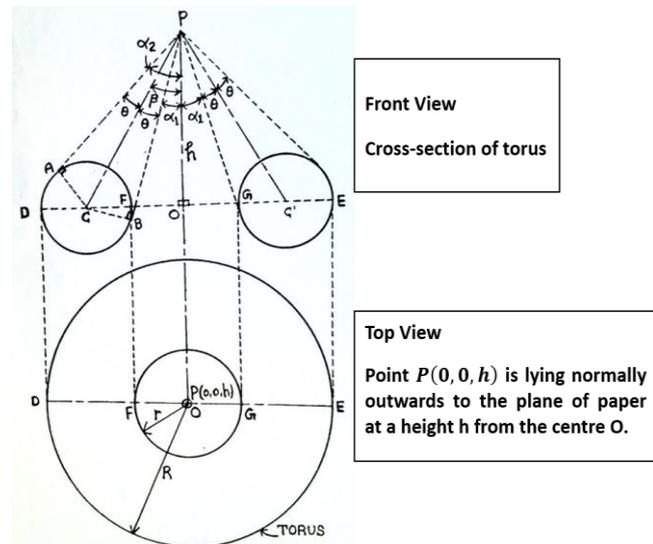


Figure 1: Two imaginary concentric right conical surfaces, inner PB & outer PA , with apex angles $2\alpha_1$ and $2\alpha_2$ respectively encloses the torus completely (as shown by the dotted lines in the front view (cross-sectional view)).

In right ΔPAC (Fig. 1),

$$\Rightarrow \sin \angle APC = \frac{CA}{CP} \Rightarrow \sin \theta = \frac{\left(\frac{R-r}{2}\right)}{\left(\frac{1}{2}\sqrt{(R+r)^2 + 4h^2}\right)} = \frac{R-r}{\sqrt{(R+r)^2 + 4h^2}} \dots \dots (2)$$

In this work, the concentric cones are defined as the right circular cones (or conical surfaces) having common vertex (apex point) and the geometrical axis and the equal vertical height. Dotted lines PB and PA are showing inner and outer conical surfaces respectively in top view in above Figure 1.

The solid angle subtended by a circular right cone with apex angle 2α at its apex point is given by generalized equation [1] as follows

$$\omega = 2\pi(1 - \cos\alpha) \dots \dots (3)$$

Now, consider two imaginary concentric right circular conical surfaces (i.e. inner conical surface PB & outer conical surface PA) having common apex at the given point $P(0, 0, h)$, equal normal height $OP = h$, minimum apex angle $2\angle BPO = 2\alpha_1$ and maximum apex angle $2\angle APO = 2\alpha_2$ respectively such that the torus is completely enclosed by both the conical surfaces i.e. outer PA and inner PB (see Fig. 1).

In such case, since the torus is completely covered/enclosed by inner and outer conical surfaces hence the solid angle subtended by the torus will be equal to the solid angle subtended by the conical annular-space i.e. equal to the algebraic sum which is the difference of solid angles subtended by the outer and inner conical surfaces respectively [2,3]. Hence, the solid angle (ω) subtended by the torus at the given point P lying on the geometrical axis is obtained by substituting the apex angles in (3) and taking the difference as follows

$$\begin{aligned} \omega &= \omega_{(\text{outer right cone (with apex angle } 2\alpha_2)} - \omega_{(\text{inner right cone (with apex angle } 2\alpha_1)} \\ \Rightarrow \omega &= 2\pi(1 - \cos\alpha_2) - 2\pi(1 - \cos\alpha_1) \\ &= 2\pi(1 - \cos(\beta + \theta)) - 2\pi(1 - \cos(\beta - \theta)) \\ &= 2\pi(\cos(\beta - \theta) - \cos(\beta + \theta)) \\ &= 2\pi(\cos\beta\cos\theta + \sin\beta\sin\theta - \cos\beta\cos\theta + \sin\beta\sin\theta) \\ &= 4\pi\sin\beta\sin\theta \end{aligned}$$

Substituting the values of $\sin\beta$ and $\sin\theta$ from (1) and (2) in the above expression, we obtain

$$\begin{aligned} \omega &= 4\pi \times \frac{R+r}{\sqrt{(R+r)^2 + 4h^2}} \times \frac{R-r}{\sqrt{(R+r)^2 + 4h^2}} \\ &= 4\pi \times \frac{(R+r)(R-r)}{(R+r)^2 + 4h^2} = 4\pi \frac{R^2 - r^2}{(R+r)^2 + 4h^2} \\ \Rightarrow \omega &= 4\pi \left(\frac{R^2 - r^2}{(R+r)^2 + 4h^2} \right) \quad \forall R \geq r, R, r, h \in [0, \infty) \dots \dots \dots (3) \end{aligned}$$

3. Important deductions

3.1. Corollary 1: Solid angle subtended by a torus at its centre is given by

$$\omega = 4\pi \left(\frac{R-r}{R+r} \right) \quad \dots \dots (4)$$

Proof. In this case, the solid angle subtended by the torus is obtained by setting $h = 0$ in the above Eq.(3) as follows

$$\begin{aligned} \omega &= 4\pi \left(\frac{R^2 - r^2}{(R+r)^2 + 4(0)^2} \right) \\ &= 4\pi \left(\frac{R^2 - r^2}{(R+r)^2} \right) = 4\pi \frac{(R+r)(R-r)}{(R+r)^2} = 4\pi \left(\frac{R-r}{R+r} \right) \\ \Rightarrow \omega &= 4\pi \left(\frac{R-r}{R+r} \right) \end{aligned}$$

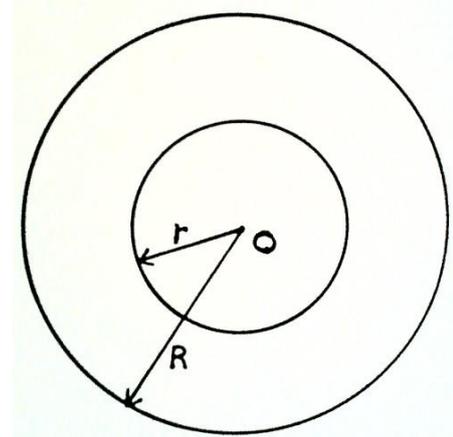


Figure 2: Solid angle subtended by the torus at its centre O.

3.2. Corollary 2: Solid angle subtended by a circle at its centre is always zero.

Proof. In this case, the torus becomes a circle when its inner and outer radii are equal i.e. $R = r$. Hence, substituting $R = r \neq 0$ in the above Eq.(3) or (4), we get

$$\omega = 4\pi \left(\frac{r^2 - r^2}{(r+r)^2 + 4h^2} \right) = 0$$

Or,

$$\omega = 4\pi \left(\frac{r-r}{r+r} \right) = 0$$

It is clear from the above result that solid angle subtended a torus with diameter $R - r = 0$ i.e. circle at any point on its axis is always zero.

3.3. Corollary 3: Solid angle subtended by a finite torus at a point at infinite distance is always zero.

Proof. In this case, the given point lies at an infinite distance from the torus of finite radii i.e. $h \rightarrow \infty$. Therefore, the solid angle subtended by a torus with finite radii at any point lying at the infinity is obtained by taking limit of ω from Eq.(3) as $h \rightarrow \infty$

$$\omega = \lim_{h \rightarrow \infty} 4\pi \left(\frac{R^2 - r^2}{(R+r)^2 + 4h^2} \right) = 0$$

It is obvious from the above result that the torus of finite radii appear like a point to an observer located at an infinite distance.

3.4. Corollary 4: The solid angle, subtended by a Simply Closed Torus (i.e. a torus with zero inner radius, ($r = 0$)) with outer radius R , at any point lying at a normal height h on its normal geometrical axis, is given by

$$\omega = 4\pi \left(\frac{R^2}{R^2 + 4h^2} \right) \quad \dots \dots (5)$$

Proof. A simply closed torus is a torus with zero inner radius ($r = 0$) or it can be defined as a solid or a surface generated simply by rotating a circle about its tangent at any point (as shown in Figure 3).

The solid angle subtended by a torus at any point $P(0, 0, h)$ lying on the vertical axis at a height h from the centre O of the torus, is obtained by setting $r = 0$ in the above Eq.(3) as follows

$$\omega = 4\pi \left(\frac{R^2 - (0)^2}{(R + 0)^2 + 4h^2} \right)$$

$$\omega = 4\pi \left(\frac{R^2}{R^2 + 4h^2} \right)$$

3.5. Corollary 5: The solid angle, subtended by a Simply Closed Torus (i.e. a torus with zero inner radius) at its centre is always $4\pi sr$.

Proof. In this case, the solid angle subtended by a simply closed torus at its centre is obtained by substituting $r = 0$ & $h = 0$ in the above Eq.(3) as follows

$$\omega = 4\pi \left(\frac{R^2 - (0)^2}{(R + 0)^2 + 4(0)^2} \right)$$

$$= 4\pi \left(\frac{R^2}{R^2} \right)$$

$$= 4\pi sr$$

It is clear from the above result that the centre of any torus having zero inner radius ($r = 0$) is completely surrounded by its surface hence a simply closed torus always subtends a solid angle of $4\pi sr$ at its centre.

Conclusions

A generalized closed-form expression has been obtained for the solid angle subtended by a torus at any point on its axis of symmetry. Several corollaries follow naturally from this result, yielding explicit formulas for the solid angle at the geometric center of the torus, for a simply closed torus, and for the asymptotic case in which the observation point recedes to infinity. The unified formulation presented herein provides a rigorous geometric framework for solid-angle evaluation and may serve as a useful analytical tool in geometric modeling, photometry, and related applications involving axially symmetric geometries.

Note: Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)

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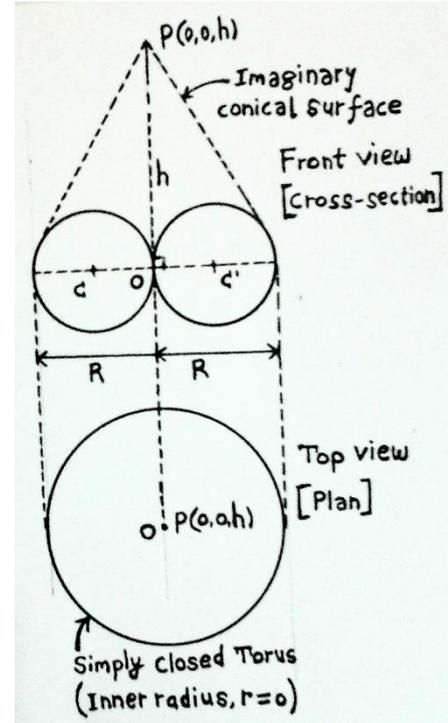


Figure 3: A simply closed torus having its inner radius zero.

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