The Commutative Power of a Revised Collatz

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Abstract

The Collatz conjecture offers a seemingly arbitrary piecewise sequencing of two separate functions (divide by two, multiply by three and add one). Attempts have been made to partially simplify the problem by combining exactly one instance of the multiplication with one instance of the division but have not previously been able to completely separate the two alternatives. To create this separation, I define a positive integer's Least Significant Bit as the smallest power of two that is added together to create its binary representation. I then define a replacement function as three times n plus the Least Significant Bit of n. I then show that an application of the replacement function followed by division by two has an identical result to division by two followed by the original Collatz multiplication. By using the replacement function, all division can be delayed until the result is a perfect power of two. This change removes the piecewise aspect of the Collatz conjecture that has stymied a proof. In addition, the resulting graph of transformations displays a many-to-one relationship that has previously been hidden. The replacement formula's non-piecewise and many-to-one features offer new avenues to prove the conjecture. If one can prove that the replacement function reaches a perfect power of two, one will have proved the Collatz.

Keywords: Collatz Conjecture · Graph Theory · 3N+LSB · Piecewise

Introduction

The Collatz conjecture has frustrated problem solvers since 1937. Summed up by Lagarias as "simple to state and *apparently* intractably hard to solve." [1985] The seemingly arbitrary switching between multiplication and division has vexed attempts to prove its validity. I shall present a replacement function and show that while performing the same overall action as the original Collatz Conjecture, it delays the division. This alteration allows the multiplication and division steps to be completely separated. I shall illustrate the operation of the replacement function and display the subsequent graph. The replacement function will be seen to provide a new base from which to prove the Collatz conjecture.

The Replacement Function

Definition - I shall define the Least Significant Bit (LSB) of a number as the smallest bit of a number's binary representation. The (decimal) number 12 in binary notation is 2^3 plus 2^2 or eight plus four; its LSB is 2^2.

Examples

$5 = 2^0 + 2^2 = 1 + 4$; $LSB(5) = 2^0$ [1]

$6 = 2^1 + 2^2 = 2 + 4$; $LSB(6) = 2^1$ [2]

$36 = 2^2 + 2^5 = 4 + 32$; $LSB(36) = 2^2$ [4]

Given n as a positive integer, and LSB(n) as n's least significant bit,

Let us create a replacement function (R) for the Collatz of 3n + LSB(n). Tylock [2018]

Examples

$R(5) = 3\times5 + 1 = 16$

$R(6) = 3\times6 + 2 = 20$

$R(36) = 3\times36 + 4 = 112$

Proof of Equivalence

I will now display how this function maps one to one with the original. Specifically, I will show that the replacement function (3n+LSB(n)) followed by division by two is equivalent to an equal number of division by two operations followed by the original Collatz function (3n+1). Tylock [2018]

Consider any n that is a positive integer. We may replace n with $2^a \times b$, where a is an integer, and b is a positive odd number. (This is simply factoring out the quantity of 2s in the number)

That is, $n = 2^a \text{ x } b$.

Under the original Collatz, any value n will be divided by 2 'a' times and then inserted into the 3n+1 equation resulting in 3b+1.

That is, Collatz(n) after ‘a’ divisions by 2 and one application of the formula equals 3b+1.

Under the replacement function the initial division is not performed, instead the entire value is inserted into 3n+LSB(n). Substituting $2^a \times b$ for n, and 2^a for LSB(n) (by definition), we get $3(2^a \times b) + 2^a$. Factoring the 2^a out of each term we reach $2^a (3b+1)$. Dividing by 2 ‘a’ times produces 3b+1.

That is Replacement(n) after one application of the formula and then dividing by 2 ‘a’ times produces 3b+1.

And so, it is evident that each version includes quantity ‘a’ divide by two steps, and one application of the formula, and both arrive at the same result. The replacement function performs the division after the 3n+LSB(n) formula instead of before the 3n+1.

Examples

5

C(5) = 3×5 + 1 = 16

R(5) = 3×5 + 1 = 16

6

C(6) = 6/2 = 3; 3×3 + 1 = 10

R(6) = 3×6 + 2 = 20; 20/2 = 10

36

C(36) = 36/2 = 18; 18/2 = 9; 3*9 + 1 = 28

R(36) = 3×36 + 4 = 112; 112/2 = 56; 56/2 = 28

Extension

It can be further understood that the 3n+LSB(n) formula allows repeated application without division. Its ending state can be declared to be that the number is a perfect power of two. And at that point, divide-by-two steps can be applied in the quantity of the power of two that has been reached.

This replacement function will be applied in the exact same quantity as the original Collatz - and the extended example below will further illustrate this.

Extended Example

Original Collatz – 36

First, let us process the integer 36 through all subsequent numbers until it first reaches 1.

1. 36 is even, and so we divide by 2 resulting in 18
2. 18 is even, and so we divide by 2 resulting in 9
3. 9 is odd, and so we multiply by 3 and add 1 resulting in 28
4. 28 is even, and so we divide by 2 resulting in 14
5. 14 is even, and so we divide by 2 resulting in 7
6. 7 is odd, and so we multiply by 3 and add 1 resulting in 22
7. 22 is even, and so we divide by 2 resulting in 11
8. 11 is odd, and so we multiply by 3 and add 1 resulting in 34
9. 34 is even, and so we divide by 2 resulting in 17
10. 17 is odd, and so we multiply by 3 and add 1 resulting in 52
11. 52 is even, and so we divide by 2 resulting in 26
12. 26 is even, and so we divide by 2 resulting in 13
13. 13 is odd, and so we multiply by 3 and add 1 resulting in 40
14. 40 is even, and so we divide by 2 resulting in 20
15. 20 is even, and so we divide by 2 resulting in 10
16. 10 is even, and so we divide by 2 resulting in 5
17. 5 is odd, and so we multiply by 3 and add 1 resulting in 16
18. 16 is even, and so we divide by 2 resulting in 8
19. 8 is even, and so we divide by 2 resulting in 4
20. 4 is even, and so we divide by 2 resulting in 2
21. 2 is even, and so we divide by 2 resulting in 1

This is the ending state and so we stop. We have performed 15 divide by 2 steps and 6 applications of the function 3n+1, 21 total steps.

Replacement Function – 36

Let's now process the integer 36 via the replacement function through all subsequent numbers until it reaches a perfect power of two and then divide by 2 to reach 1.

1. 36 isn't a perfect power of 2, and so we multiply by 3 and add the LSB [4] resulting in 112
2. 112 isn't a perfect power of 2, and so we multiply by 3 and add the LSB [16] resulting in 352
3. 352 isn't a perfect power of 2, and so we multiply by 3 and add the LSB [32] resulting in 1088
4. 1088 isn't a perfect power of 2, and so we multiply by 3 and add the LSB [64] resulting in 3328
5. 3328 isn't a perfect power of 2, and so we multiply by 3 and add the LSB [256] resulting in 10240
6. 10240 isn't a perfect power of 2, and so we multiply by 3 and add the LSB [2048] resulting in 32768

- 32768 is a perfect power of 2 (2^{15}) and so we stop applying the replacement function
- 2^{15} requires 15 instances of dividing by 2 resulting in 1

This is the ending state and so we stop. We have performed 15 divide by 2 steps and 6 applications of the function 3n+LSB(n), 21 total steps.

Comparison

The proof and the numbers above match, but let's set these two processes side by side.

Collatz		Replacement Function
36 ÷ 2 = 18		3 × 36 +4 = 112
19 ÷ 2 = 9		3 × 112 + 16 = 352
3 × 9 + 1 = 28		3 × 352 + 32 = 1088
28 ÷ 2 = 14		3 × 1088 + 64 = 3328
14 ÷ 2 = 7		3 × 3328 + 256 = 10240
3 × 7 + 1 =22		3 × 10240 + 2048 = 32768
22 ÷ 2 = 11		32768 ÷ 2 = 16384
3 × 11 + 1 = 34		16384 ÷ 2 = 8192
34 ÷ 2 = 17		8192 ÷ 2 = 4096
3 × 17+1 = 52		4096 ÷ 2 = 2048
52 ÷ 2 = 26		2048 ÷ 2 = 1024
26 ÷ 2 = 13		1024 ÷ 2 = 512
3 × 13 + 1 = 40		512 ÷ 2 = 256
40 ÷ 2 = 20		256 ÷ 2 = 128
20 ÷ 2 = 10		128 ÷ 2 = 64
10 ÷ 2 = 5		64 ÷ 2 = 32
3 × 5 + 1 = 16		32 ÷ 2 = 16
16 ÷ 2 = 8		16 ÷ 2 = 8
8 ÷ 2 = 4		8 ÷ 2 = 4
4 ÷ 2 = 2		4 ÷ 2 = 2
2 ÷ 2 = 1		2 ÷ 2 = 1

Table 1) Comparison of processing methods

The replacement function has relocated all instances of the function with multiplication to appear before any division by 2.

Discussion – Piecewise

As defined by Weisstein , "A piecewise function is a function that is defined on a sequence of intervals" [2024]. The classic Collatz conjecture is a clear example of this definition. When a number is even, it is divided by two, when odd multiplied by three with one added.

This piecewise attribute creates the "hailstorm" effect that has contributed significantly to the difficulty in proving the conjecture – as exemplified by Pickover, "Like hailstones falling from the sky through storm clouds, this sequence drifts down and up, sometimes in seemingly haphazard patterns." [2001]

The commutative property of the replacement function has dispensed with the piecewise aspect. The replacement function is applied until the number is a perfect power of two – at that point the problem is essentially solved because it is trivial to show that a power of two can be divided by two repeatedly to reach one.

And at this point we can also consider the "repeated" sequence of 1 -> 4 -> 2 -> 1. The replacement function applies an interesting twist after a perfect power of two is reached. For any $n = 2^a$, $3n+LSB(n)$ becomes 2^{a+2} (because $3(2^a) + 2^a = 4(2^a) = 2^{a+2}$). One might say that all numbers consolidate at two to the infinity.

Many to One Graph

A new graph emerges with the removal of the division by two. It reveals that multiple numbers consolidate.

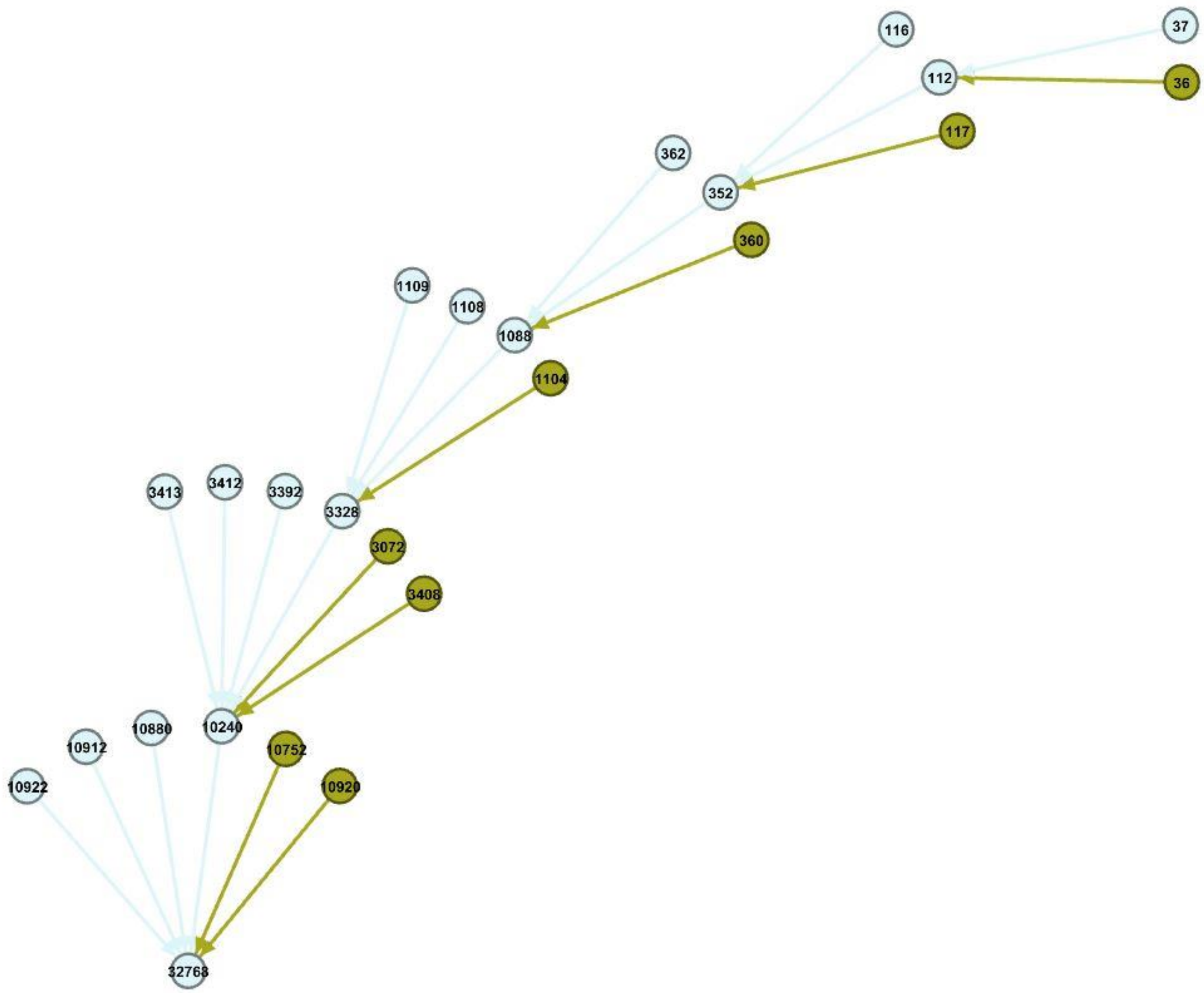

Figure 1) The replacement function as applied to 36 (and peer nodes that also transform to the same values)

Combining the prior graph with the concept of continuing the 3n +LSB(n) transformation even after a number reaches a perfect power of two, the following graph shows all numbers reaching 2^{15}.

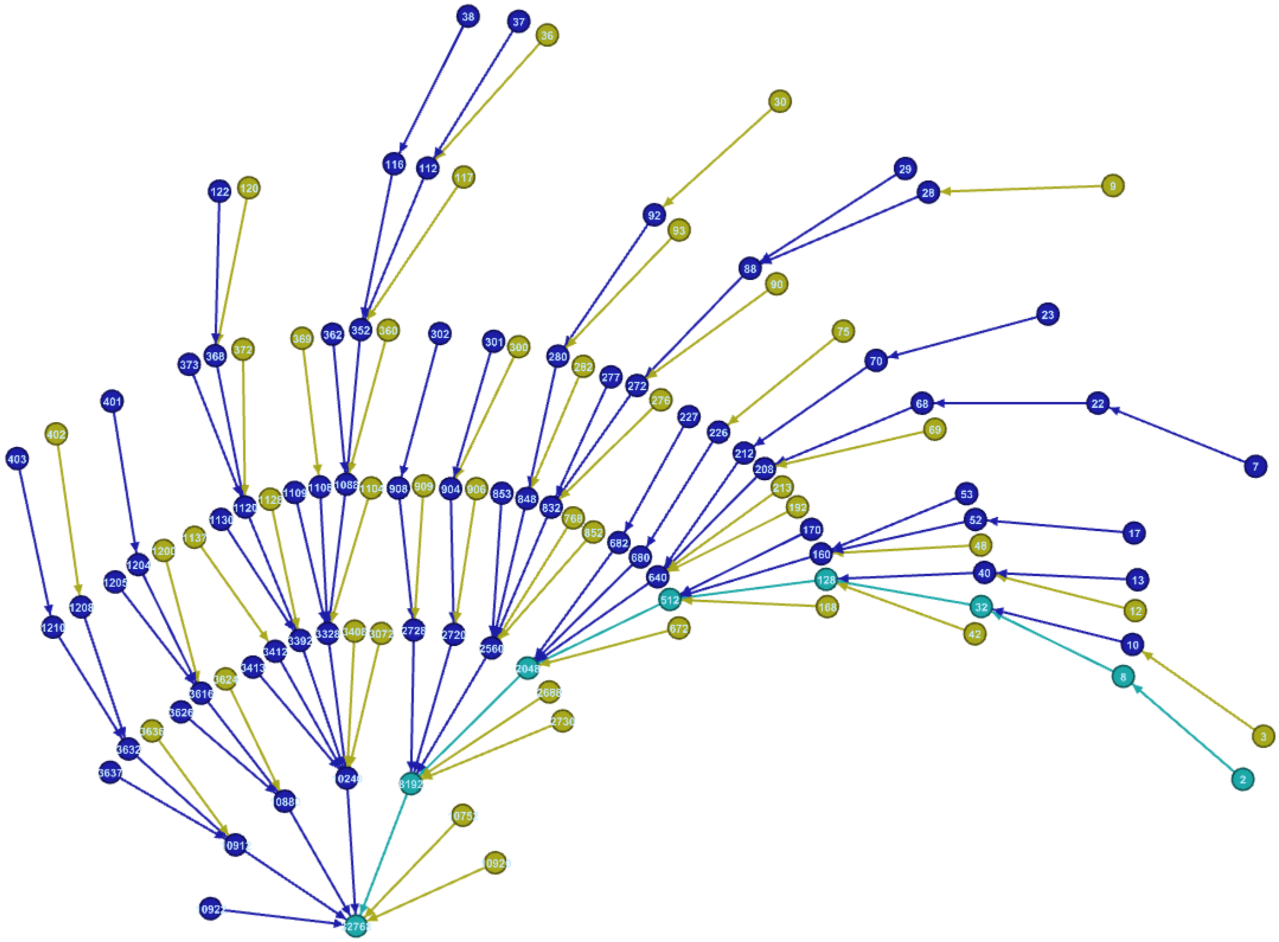

Figure 2) Graph of all numbers that transform into 2^{15}

As a side effect of this replacement function, in addition to the successor of a number we can also calculate all potential predecessors of a number.

Let us consider 2^{15}. When divided by three in binary we may stop the division by restricting the precision of the result.

Using just one bit of precision, 2^{15} divided by three is 2^{13} with a remainder of 2^{13}. Using 3 bits, we find $2^{13} + 2^{11}$ with a remainder of 2^{11}, 5 bits extends this to $2^{13} + 2^{11} + 2^{9}$ with a remainder of 2^{9}. One can see in the graph above that there are 7 positive integers that match this pattern.

Conclusion

The Replacement Function presented transforms the landscape of the Collatz. It resolves the piecewise attribute by completely shuffling the 3n portion away from the divide by two portion. The subsequent change in the graph of values rewrites the script on proving the Collatz.

If one can prove that the replacement function reaches a perfect power of two, one will have proved the Collatz. A path to a proof has been made easier by the illustration of the revised graph. The new graph shows a many-to-one mapping that may hold a key.

That proof must still show that bit patterns cannot be repeated and that every number must resolve to a power of two, but the effect of dividing by two can now be seen as a non-essential step. In fact, it may well have disguised the true progression of numbers all along.

If one is intent on proving the Collatz, one should work with 3n + LSB(n).

References

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Subject Classification

Subject Level 1: 11 Number theory

Subject Level 2: 11B - Sequences and sets

Subject Level 3: 11B83 - Special sequences and polynomials