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Number Triangles (Triangular Arrays of Numbers): Pascal's Triangle, Others, and The Birth of a New One

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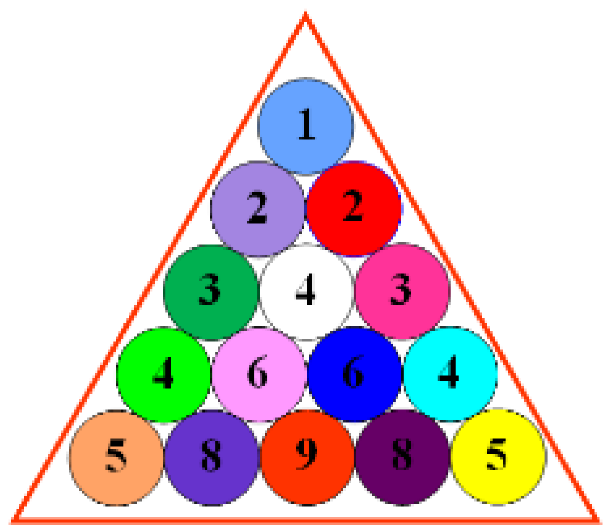
Abstract

Throughout mathematics history, mathematicians had created triangular array of numbers. Famous among these number triangles is the Pascal's Triangle which had marked its prominence in many areas of mathematics and even extends its usefulness in the sciences. This paper presents an inventory of number triangles known and recognized in the mathematics world and takes a look to newly-found triangular array of numbers generated by the function,

$$\Delta_{\Pi}(n) = \prod_{m_{i=1}}^{n}[m_i(n+1-m_i)] = n!^2,$$

and its link to the Pascal's Triangle particularly to the Tetrahedral Numbers.



Keywords and phrases: number triangle/triangular array of numbers, Pascal's triangle, tetrahedral number.

I. Introduction

Mathematics is as varied as the mathematicians themselves. But all of them converge in filling up the “general stock of mathematical concepts” [7]. In ancient times, Pythagoras (c. 570-495 BC) mathematical works were related to arts and mystics. Aside from his famous Pythagorean Theorem, he was also credited for devising the tetractys, a sacred symbol in triangular arrangement of points [9]. Tetractys, defined as a triangular figure consisting of ten points arranged in four rows … which is the geometrical representation of the fourth triangular number [12].

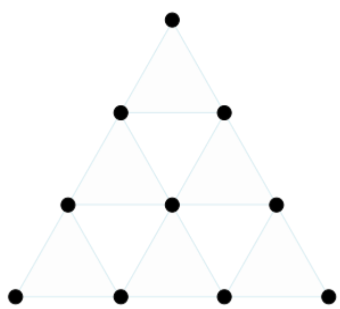


Figure 1: Tetractys
(Photo courtesy of Wikipedia)

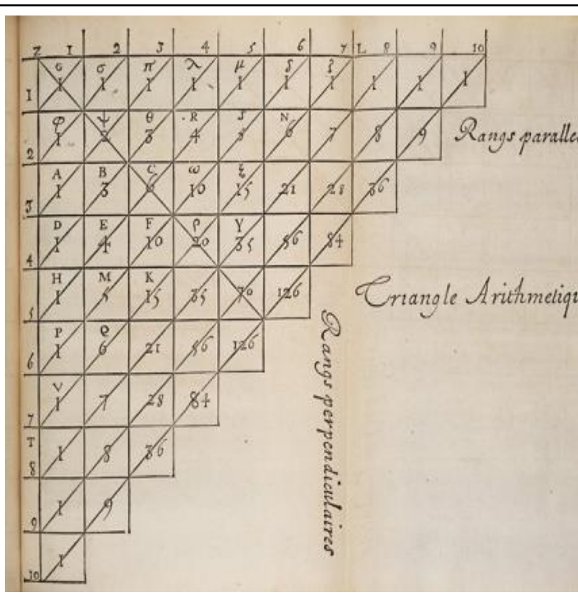


Figure 2: Blaise Pascal’s version of the triangle
(Photo courtesy of Wikipedia)

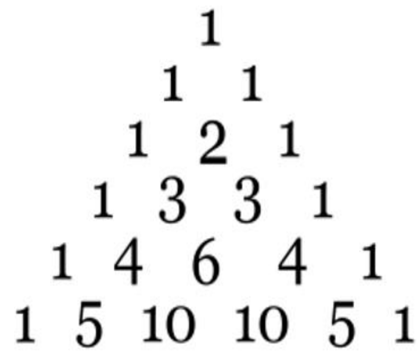


Figure 3: A diagram that shows Pascal’s Triangle with rows 0 to 5.
(Photo courtesy of Wikipedia)

One of the famous triangular arrays if not the most famous is the Pascal’s Triangle. Though it was named after Blaise Pascal (19 June 1623 – 19 August 1662), it was already known before his time and other mathematicians studied it centuries before him in China, Germany, India, Italy, and Persia (Iran). Pascal’s innovations were described comprehensively in his *Traité du triangle arithmétique* (*Treatise on Arithmetical Triangle, 1653*). However it was published posthumously in 1665. The triangle was later named after Pascal by the two mathematicians, Pierre Raymond de Montmort (1708) and Abraham de Moivre (1730) which became the modern Western name, the Pascal’s Triangle [8].

Other number triangles had followed as enumerated by Wikipedia and Wolfram MathWorld, viz. [16][17]:

1. Leibniz Harmonic Triangle (c. 1692)

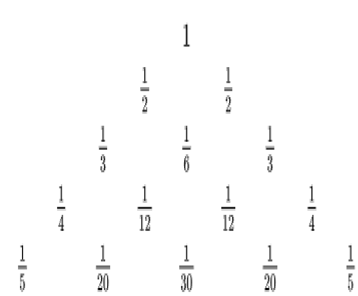


Figure 4: Leibniz Harmonic Triangle
(Photo courtesy of Wikipedia)

is a triangular arrangement of unit fractions where the outmost diagonals, consist of the reciprocals of the row numbers and each inner cell, are the cells diagonally above and to the left minus the cell to the left.

2. Bernoulli Triangle (1713)

1
1 2
1 3 4
1 4 7 8
1 5 11 15 16
1 6 16 26 31 32

Figure 5: Bernoulli Triangle
(Photo courtesy of Wikipedia)

is an array of partial sums of the binomial coefficients. For any non-negative integer *n* and for any integer *k* included between 0 and *n*, the component in row *n* and column *k* is given by $\sum_{p=0}^{k}\binom{n}{p}$, i.e. the sum of the first *k* *n*th order binomial coefficients.

3. Euler's Number Triangle (1755)

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1
1 57 302 302 57 1

Figure 6: Euler's Number Triangle
(Photo courtesy of Wikipedia)

is a triangular array that shares some common characteristics with Pascal's triangle. The sum of row *n* is the factorial *n*!. From Eulerian number (*n, m*) in combinatorics, the number of permutations of the numbers 1 to *n* in which exactly *m* elements are greater than the previous element (permutations with *m* "ascents"). They are the coefficients of the Eulerian polynomials.

4. Catalan's Triangle (1855)

1
1 1
1 2 2
1 3 5 5
1 4 9 14 14
1 5 14 28 42 42

Figure 7: Catalan's Triangle
(Photo courtesy of Wikipedia)

is a generalization of Catalan numbers, and is named after Eugene Charles Catalan (1814-1894), a French and Belgian mathematician. The earliest appearance of Catalan triangle along with the recursion formula, was in 1800 and published on the "Treatise of Calculus" by Louis Francois Antoine Arbogast.

n \ k	1	2	3	4	5	6	
1	1						...
2	2	1					...
3	5	4	1				...
4	14	14	6	1			...
5	42	48	27	8	1		...
6	132	165	110	44	10	1	...
	⋮	⋮	⋮	⋮	⋮	⋮	

Figure 7.1: Catalan's Triangle (L.W. Sharpio)
(Photo courtesy of L.W. Sharpio)

Another Catalan's triangle was published in 1974 by L. W. Shapiro which is different from the previous one [11].

5. Tribonacci Triangle (c. 1879)

```
          1
        1   1
      1   3   1
    1   5   5   1
  1   7  13   7   1
 1  9  25  25   9   1
1 11  41  63  41  11  1
```

Figure 8: Tribonacci Triangle
(Photo courtesy of Wikipedia)

is a triangular array of Delannoy numbers named after French army officer and amateur mathematician, Henri-Auguste Delannoy (1833-1915). In this array, the numbers in the first slant are all 1's, the numbers in the second slant are odd numbers, the numbers in the third slant are the centered square numbers, and the numbers in the fourth slant are the centered octahedral numbers. The array resembles the Pascal's triangle, where each number is the sum of the three numbers above it.

6. Lozanic's/Losanitsch's Triangle (1897)

```
            1
          1   1
        1   1   1
      1   2   2   1
    1   2   4   2   1
  1   3   6   6   3   1
1   3   9  10   9   3   1
```

Figure 9: Lozanic's Triangle
(Photo courtesy of Wikipedia)

is named after Serbian chemist Simeon "Sima" Lozanic, who looked into the symmetries exhibited by rows of paraffin (archaic term for alkanes).

7. Bell Triangle (Bell Numbers -1930)

```
            1
         1     2
      2     3     5
   5     7    10    15
15    20    27    37    52
```

Figure 10: Bell Triangle
(Photo courtesy of Wikipedia)

is named such because of its close connection to the Bell numbers which can be found on both sides of the triangle. Named after Eric Temple Bell (1883-1960), a Scottish-born mathematician and science fiction writer, this triangle was discovered independently by Charles Sanders Peirce (1880), Alexander Aitken (1933), and Cohen et al. (1962).

8. Hosoya's/Fibonacci Triangle (1976)

```
        1
      1   1
    2   1   2
  3   2   2   3
5   3   4   3   5
```

Figure 11: Hosoya's Triangle
(Photo courtesy of Wikipedia)

is defined by the recursive relation where the entries in a row have the same *m* and line up according to the value of *n* (*0-m*) from left to right. Haruo Hosoya is a Japanese chemist. The Hosoya index or topological index Z_G used in computational chemistry was named after him.

9. Clark's Triangle (1978)

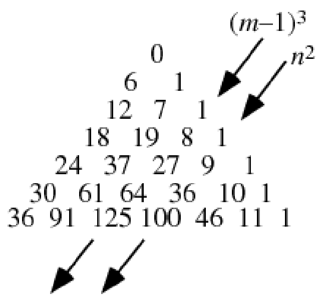


Figure 12: Clark's Triangle
(Photo courtesy of Clark, J.E.)

is a number triangle created by setting the vertex equal to 0, then filling one diagonal with 1's, the other diagonal with multiples of an integer *f*, and filling the remaining entries by summing the elements on either side from one row above [4].

10. Floyd's Triangle (c. 1967)

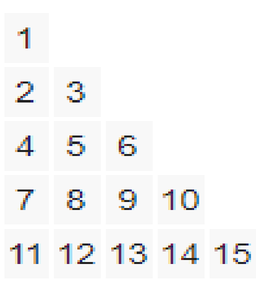


Figure 13: Floyd's Triangle
(Photo courtesy of Wikipedia)

is a right-angled triangular array of natural numbers, used in computer science education. It is named after Robert W (Bob) Floyd (1936-2001), a computer scientist.

11. Dudley Triangle (1987)

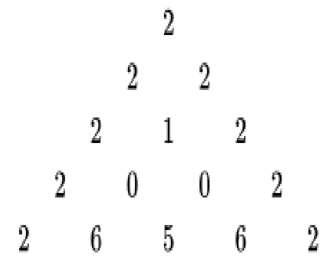


Figure 14: Dudley Triangle
(Photo courtesy of Wikipedia)

is a triangular array of integers that was defined by Underwood Dudley (1987). Dudley exhibited several rows of this triangle, and challenged readers to find the next row; the challenge was met by J.G. Mauldon, who proposed two different solutions.

12. Trinomial Triangle (c. 1987)

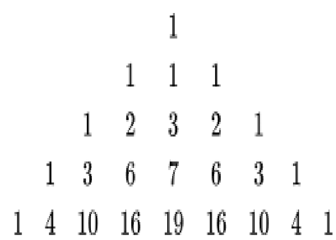


Figure 15: Trinomial Triangle
(Photo courtesy of Wikipedia)

is a variation of Pascal's triangle. The difference between the two is that an entry in the trinomial triangle is the sum of the *three* entries above each number rather than two in Pascal's triangle.

13. Seidel-Entringer-Arnold Triangle

1
0 1
1 1 0
0 1 2 2
5 5 4 2 0
0 5 10 14 16 16

Figure 16: Trinomial Triangle
(Photo courtesy of Wikipedia)

(c. 1991) is a number triangle consisting of the Entringer numbers arranged in “ox-plowing” order.

14. Magog Triangle (c. 1999)

1
1 1
1 1 1
1 1 1 3
1 1 2 4 5.

Figure 17: Magog Triangle
(Photo courtesy of Wikipedia)

is a number triangle of order n with entries 1 to n such that entries are non-decreasing across rows and down columns and all entries in column j are less than or equal to j.

15. Monotone Triangle (c. 1999)

4
2 5
1 4 5
1 3 4 5
1 2 3 4 5

Figure 18: Monotone Triangle
(Photo courtesy of Wikipedia)

is a triangle of order *n* also called a strict Gelfand pattern or a gog triangle. It is a triangle with *n* numbers along each side and the base containing entries between 1 and *n* such that there is strict increase across rows and weak increase diagonally up or down to the right. Each entry in the triangle corresponds to the positions of 1’s in the sum of the first *k* rows of an alternating sign matrix.

16. Triangle of Fibonomial Coefficient (2004)

1
1 1
1 1 1
1 2 2 1
1 3 6 3 1
1 5 15 15 5 1

Figure 19: Fibonomial Triangle
(Photo courtesy of Wikipedia)

is a triangle where its coefficients are similar to binomial coefficients and can be displayed similar to Pascal’s triangle.

17. Narayana Number/Triangle (c. 2015)

k =	1	2	3	4	5	6	7	8
n = 1	1							
2	1	1						
3	1	3	1					
4	1	6	6	1				
5	1	10	20	10	1			
6	1	15	50	50	15	1		
7	1	21	105	175	105	21	1	
8	1	28	196	490	490	196	28	1

Figure 20: Narayana Triangle
(Photo courtesy of Wikipedia)

is a triangular array of natural numbers occuring in various counting problems. It is named after Canadian mathematician, T.V. Naranaya (1930-1987).

II. Methods

A. The Problem

Harmonic series has an infinite number of terms [6].

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$$

Unlike arithmetic and geometric series, harmonic series has no formula for finding the sum of the first *n* terms. Because of this, teachers and students alike encountered difficulties in finding the harmonic series of *n* terms especially when *n* gets bigger.

B. An Attempt to Unearth

Harmonic sequence is the reciprocal form of arithmetic sequence [6]. In general, a harmonic sequence may be represented as

$$\frac{1}{a_1}, \frac{1}{a_1 + d}, \frac{1}{a_1 + 2d}, \frac{1}{a_1 + 3d}, \ldots, \frac{1}{a_1 + (n-1)d}.$$

Legend has it that the great mathematician Karl Friedrich Gauss (1777-1855), then a schoolboy, was asked to find the sum of the first 100 positive integers without using paper and pencil.

$$1 + 2 + 3 + \cdots + 98 + 99 + 100.$$

This was how Gauss made his mental calculations:

The series of natural numbers from 1 to 100,

$$1 + 2 + 3 + \cdots + 98 + 99 + 100,$$

and its reverse series,

$100 + 99 + 98 + \cdots + 3 + 2 + 1$ were added.

It followed that,

$$(1 + 100) + (2 + 99) + (3 + 98) + \cdots + (98 + 3) + (99 + 2) + (100 + 1),$$

$$(101) + (101) + (101) + \cdots + (101) + (101) + (101).$$

Since there were one hundred pairs of numbers taken from the series and its reverse order, where each pair was added together, then,

$$\frac{100(101)}{2} = 5{,}050.$$

To the amazement of his math teacher, a supposed to be laborious math problem, was solved in an instant. This was the method used by Gauss, a German mathematician, to derive his formula for the arithmetic series.

Generalizing the pattern and denoting the sum of the first *n* terms of an arithmetic sequence [3] as equation 1 that represents the sum of the first 100 positive integers,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n, \text{ and}$$

rewriting the sum in reverse order as equation 2, that is:

$$S_n = a_n + a_{n-1} + a_{n-2} + \cdots + a_1.$$

Adding equations 1 and 2, one gets

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n).$$

Since there are *n* terms of the form, $a_1 + a_n$, then,

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Using the reasoning of Gauss for the sum of equations 1 and 2 on finding the sum of the first *n*th terms of the harmonic sequence,

For *n* = 1,

$$S_1 = 1$$

Since the number one is a multiplicative identity, 1 is directly taken as it is. Then,

For *n* = 2,

$$S_2 = 1 + \frac{1}{2}$$

$$S_2 = \frac{\frac{1\cdot 2}{1} + \frac{1\cdot 2}{2}}{1\cdot 2}$$

$$S_2 = \frac{\frac{2!}{1} + \frac{2!}{2}}{2!} \qquad \text{(Equation 1)}$$

$$S_2 = \frac{\frac{2!}{2} + \frac{2!}{1}}{2!} \qquad \text{(Equation 2)}$$

$$2S_2 = \frac{\left(\frac{2!}{1} + \frac{2!}{2}\right) + \left(\frac{2!}{2} + \frac{2!}{1}\right)}{2!}$$

$$2S_2 = \frac{2!\left(\frac{1}{1} + \frac{1}{2}\right) + 2!\left(\frac{1}{2} + \frac{1}{1}\right)}{2!}$$

$$S_2 = \frac{2!\left(\frac{3}{2}\right) + 2!\left(\frac{3}{2}\right)}{2(2!)}$$

$$S_2 = \frac{\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)}{2}$$

$$S_2 = \frac{3\left(\frac{1}{2} + \frac{1}{2}\right)}{2}$$

$$S_2 = \frac{\frac{3\left(\frac{2\cdot 2}{2} + \frac{2\cdot 2}{2}\right)}{2\cdot 2}}{2}$$

$$S_2 = \frac{\frac{3}{2}\left(\frac{2\cdot 2}{2} + \frac{2\cdot 2}{2}\right)}{2\cdot 2}$$

For *n* = 3,

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_3 = \frac{\frac{1\cdot 2\cdot 3}{1} + \frac{1\cdot 2\cdot 3}{2} + \frac{1\cdot 2\cdot 3}{3}}{1\cdot 2\cdot 3}$$

$$S_3 = \frac{\frac{3!}{1} + \frac{3!}{2} + \frac{3!}{3}}{3!} \qquad \text{(Equation 1)}$$

$$S_3 = \frac{\frac{3!}{3} + \frac{3!}{2} + \frac{3!}{1}}{3!} \qquad \text{(Equation 2)}$$

$$2S_3 = \frac{\left(\frac{3!}{1} + \frac{3!}{3}\right) + \left(\frac{3!}{2} + \frac{3!}{2}\right) + \left(\frac{3!}{3} + \frac{3!}{1}\right)}{3!}$$

$$2S_3 = \frac{3!\left(\frac{1}{1} + \frac{1}{3}\right) + 3!\left(\frac{1}{2} + \frac{1}{2}\right) + 3!\left(\frac{1}{3} + \frac{1}{1}\right)}{3!}$$

$$S_3 = \frac{3!\left(\frac{4}{3}\right) + 3!\left(\frac{4}{4}\right) + 3!\left(\frac{4}{3}\right)}{2(3!)}$$

$$S_3 = \frac{\left(\frac{4}{3}\right) + \left(\frac{4}{4}\right) + \left(\frac{4}{3}\right)}{2}$$

$$S_3 = \frac{4\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{3}\right)}{2}$$

$$S_3 = \frac{\frac{4\left(\frac{3\cdot 4\cdot 3}{3} + \frac{3\cdot 4\cdot 3}{4} + \frac{3\cdot 4\cdot 3}{3}\right)}{3\cdot 4\cdot 3}}{2}$$

$$S_3 = \frac{\frac{4}{2}\left(\frac{3\cdot 4\cdot 3}{3} + \frac{3\cdot 4\cdot 3}{4} + \frac{3\cdot 4\cdot 3}{3}\right)}{3\cdot 4\cdot 3}$$

For *n* = 4,

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$S_4 = \frac{\frac{1\cdot 2\cdot 3\cdot 4}{1} + \frac{1\cdot 2\cdot 3\cdot 4}{2} + \frac{1\cdot 2\cdot 3\cdot 4}{3} + \frac{1\cdot 2\cdot 3\cdot 4}{4}}{1\cdot 2\cdot 3\cdot 4}$$

$$S_4 = \frac{\frac{4!}{1} + \frac{4!}{2} + \frac{4!}{3} + \frac{4!}{4}}{4!} \qquad \text{(Equation 1)}$$

$$S_4 = \frac{\frac{4!}{4} + \frac{4!}{3} + \frac{4!}{2} + \frac{4!}{1}}{4!} \qquad \text{(Equation 2)}$$

$$2S_4 = \frac{\left(\frac{4!}{1}+\frac{4!}{4}\right)+\left(\frac{4!}{2}+\frac{4!}{3}\right)+\left(\frac{4!}{3}+\frac{4!}{2}\right)+\left(\frac{4!}{4}+\frac{4!}{1}\right)}{4!}$$

$$2S_4 = \frac{4!\left(\frac{1}{1}+\frac{1}{4}\right)+4!\left(\frac{1}{2}+\frac{1}{3}\right)+4!\left(\frac{1}{3}+\frac{1}{2}\right)+4!\left(\frac{1}{4}+\frac{1}{1}\right)}{4!}$$

$$S_4 = \frac{4!\left(\frac{5}{4}\right)+4!\left(\frac{5}{6}\right)+4!\left(\frac{5}{6}\right)+4!\left(\frac{5}{4}\right)}{2(4!)}$$

$$S_4 = \frac{\left(\frac{5}{4}\right)+\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)+\left(\frac{5}{4}\right)}{2}$$

$$S_4 = \frac{5\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\frac{1}{4}\right)}{2}$$

$$S_4 = \frac{\frac{5\left(\frac{4\cdot 6\cdot 6\cdot 4}{4}+\frac{4\cdot 6\cdot 6\cdot 4}{6}+\frac{4\cdot 6\cdot 6\cdot 4}{6}+\frac{4\cdot 6\cdot 6\cdot 4}{4}\right)}{4\cdot 6\cdot 6\cdot 4}}{2}$$

$$S_4 = \frac{\frac{5}{2}\left(\frac{4\cdot 6\cdot 6\cdot 4}{4}+\frac{4\cdot 6\cdot 6\cdot 4}{6}+\frac{4\cdot 6\cdot 6\cdot 4}{6}+\frac{4\cdot 6\cdot 6\cdot 4}{4}\right)}{4\cdot 6\cdot 6\cdot 4}$$

Instead of using the Least Common Denominator (LCD) rule, a bit of thinking outside-the-box has been the method of adding fractions by taking the product of all denominators, where

$$1\cdot 2\cdot 3\cdot\cdots\cdot n = n!$$

It is to be noted that,

$$S_n = \frac{\left(\frac{n!}{1}+\frac{n!}{2}+\frac{n!}{3}+\cdots+\frac{n!}{n}\right)}{n!} \qquad \text{(Equation 1)}$$

$$S_n = \frac{\left(\frac{n!}{n}+\frac{n!}{n-1}+\frac{n!}{n-2}+\cdots+\frac{n!}{1}\right)}{n!} \qquad \text{(Equation 2)}$$

And that the sum of the equations 1 and 2 are as follows:

$$2S_n = \frac{\left(\frac{n!}{1}+\frac{n!}{n}\right)+\left(\frac{n!}{2}+\frac{n!}{n-1}\right)+\left(\frac{n!}{3}+\frac{n!}{n-2}\right)+\cdots+\left(\frac{n!}{4}+\frac{n!}{1}\right)}{4!}$$

$$2S_n = \frac{n!\left(\frac{1}{1}+\frac{1}{n}\right)+n!\left(\frac{1}{2}+\frac{1}{n-1}\right)+n!\left(\frac{1}{3}+\frac{1}{n-2}\right)+\cdots+n!\left(\frac{1}{n}+\frac{1}{1}\right)}{n!}$$

$$S_n = \frac{\left(\frac{n+1}{n}\right)+\left(\frac{n-1+2}{2(n-1)}\right)+\left(\frac{n-2+3}{3(n-2)}\right)+\cdots+\left(\frac{1+n}{n}\right)}{2}$$

$$S_n = \frac{\left(\frac{n+1}{n}\right)+\left(\frac{n+1}{2n-2}\right)+\left(\frac{n+1}{3n-6}\right)+\cdots+\left(\frac{n+1}{n}\right)}{2}$$

$$S_n = \frac{(n+1)[\left(\frac{1}{n}\right)+\left(\frac{1}{2n-2}\right)+\left(\frac{1}{3n-6}\right)+\cdots+\left(\frac{1}{n}\right)]}{2}$$

Further generalizing the equation in Figure 21.

$$S_n = \frac{\frac{(n+1)\{[\frac{(n)(2n-2)(3n-6)\cdots(n)}{n}]+[\frac{(n)(2n-2)(3n-6)\cdots(n)}{2n-2}]+[\frac{(n)(2n-2)(3n-6)\cdots(n)}{3n-6}]+\cdots+[\frac{(n)(2n-2)(3n-6)\cdots(n)}{n}]\}}{(n)(2n-2)(3n-6)\cdots(n)}}{2}$$

$$S_n = \frac{[\frac{(n+1)}{2}]\{[\frac{(n)(2n-2)(3n-6)\cdots(n)}{n}]+[\frac{(n)(2n-2)(3n-6)\cdots(n)}{2n-2}]+[\frac{(n)(2n-2)(3n-6)\cdots(n)}{3n-6}]+\cdots+[\frac{(n)(2n-2)(3n-6)\cdots(n)}{n}]\}}{(n)(2n-2)(3n-6)\cdots(n)}$$

Figure 21: Showing the product of denominators from an attempt to the harmonic series.

Now, the product of denominators can be expressed and deduced into:

$$(n)(2n-2)(3n-6)\cdots(n)$$
$$=[1(n-0)][2(n-1)][3(n-2)]\cdots[1(n-0)]$$
$$=n!^2$$

C. The Accidental Discovery

The product of the denominators has created a symmetrical pattern, artistically awe-inspiring in its presence.

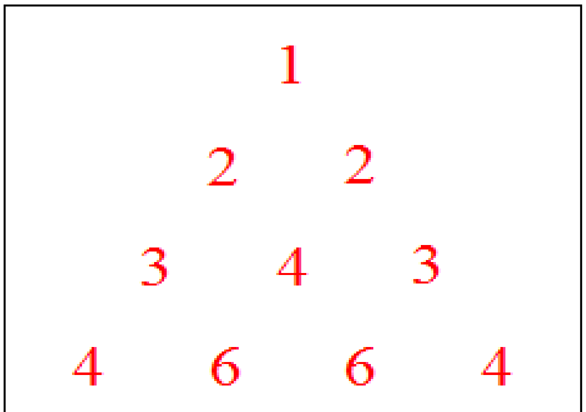


Figure 22: The product of denominators in a triangular array.

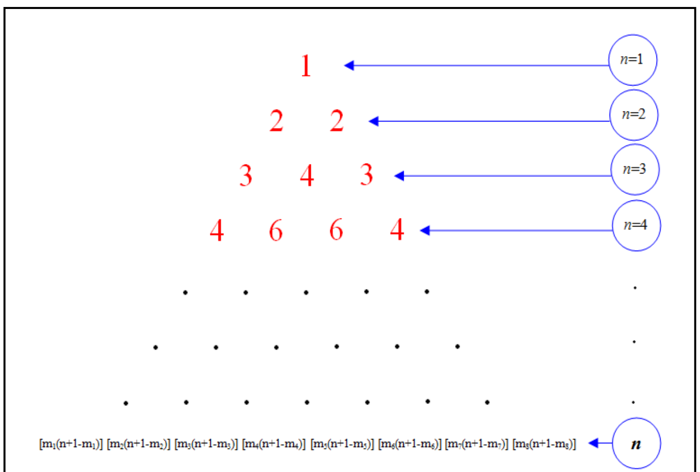


Figure 23: The infinite progression of the product of denominators in triangular array with notations for each element.

III. Results

A. The Birth of a New Number Triangle

By listing these products of denominators in a triangular array, the newly-found triangle resembles a Pascal's triangle.

Each entry is represented by a closed form:

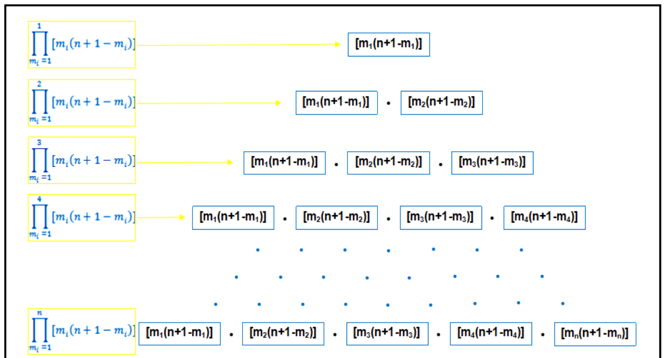


Figure 24: Triangular array of the product of denominators in closed form.

B. The Fortaliza Number

Comparing this triangle to those existed before, it can be considered new and unique in its structure and essence. To claim this as an original discovery is enough to name the pattern

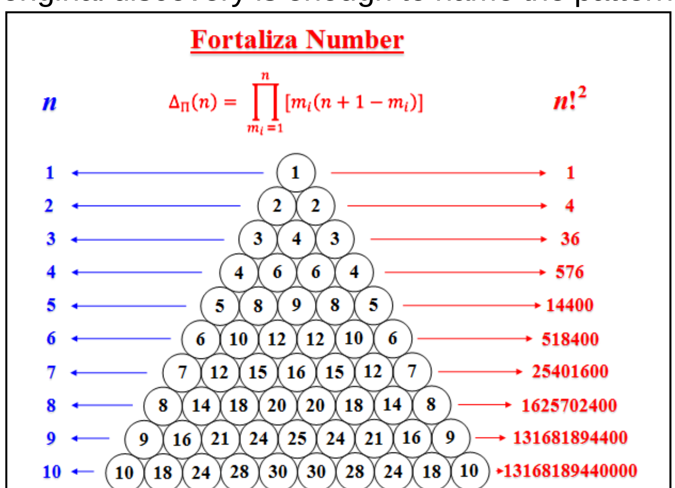


Figure 25: The Fortaliza Number as a product of the factors in each row in a triangular array of numbers.

of this triangle after its author, the Fortaliza number, i.e., the numbers, 1, 4, 36, 576, 14400, 518400, 25401600, …, $n!^2$ generated by the function,

$$\Delta_{\Pi}(n) = \prod_{m_i=1}^{n} [m_i(n+1-m_i)] = n!^2$$

where,

$$\prod_{m_i=1}^{n} [m_i(n+1-m_i)]$$
$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] \cdot [m_3(n+1-m_3)] \cdot \ldots \cdot [m_i(n+1-m_i)]$$

C. Mathematical Induction of Fortaliza Number

The first explicit statement of the principle of mathematical induction was made by Blaise Pascal in his *Treatise on the Arithmetical Triangle*, written in the late 1950s [5].To establish the validity of statements that are given in terms of natural numbers, mathematical induction is a powerful method of proof that is frequently used [1]. Following the induction process of Raji [19], to show that, $\forall\, n \in \mathbb{N}$, note that,

$$\Delta_{\Pi}(1) = \prod_{m_i=1}^{n=1} [1(1+1-1)] = 1!^2$$

$$\Delta_{\Pi}(1) = 1 = 1$$

and thus, the statement is true for n = 1. For the remaining inductive step, suppose that the formula holds for n, that is,

$$\Delta_{\Pi}(n) = \prod_{m_i=1}^{n} [m_i(n+1-m_i)] = n!^2$$

It can be shown that,

$$\Delta_{\Pi}(n+1) = \prod_{m_i=1}^{n+1} [m_i(n+2-m_i)] = (n+1)!^2$$

to complete the proof by induction. Indeed,

$$\Delta_{\Pi}(n+1) = \prod_{m_i=1}^{n+1} [m_i((n+1)+1-m_i)] = (n+1)!^2$$

$$\Delta_{\Pi}(n+1) = \prod_{m_i=1}^{n+1} [m_i(n+2-m_i)] = (n+1)!^2$$

$$\Delta_{\Pi}(n+1) = \prod_{m_i=1}^{n+1} [m_i(n+2-m_i)]$$
$$= [m_1(n+2-m_1)] \cdot [m_2(n+2-m_2)] \cdot [m_3(n+2-m_3)] \cdot \ldots \cdot [m_i(n+2-m_i)] = (n+1)!^2$$

And the result follows the proof by showing the validity of this mathematical statement using the principle of mathematical induction.■

To show some instances on the validity of the proof of the Fortaliza number, the following examples are illustrated:

1. If n = 2,

$$\Delta_{\Pi}(n) = \prod_{m_i=1}^{n} [m_i(n+1-m_i)]$$
$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] \cdot [m_3(n+1-m_3)] \cdot \ldots \cdot [m_i(n+1-m_i)] = n!^2$$

$$\Delta_{\Pi}(2) = \prod_{m_i=1}^{n=2} [m_i(n+1-m_i)]$$
$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] = 2!^2$$

$$\Delta_{\Pi}(2) = \prod_{m_i=1}^{n=2} [m_i(n+1-m_i)]$$
$$= [(2+1-1)] \cdot [2(2+1-2)] = 2!^2$$

$$\Delta_{\Pi}(2) = \prod_{m_i=1}^{n=2} [m_i(n+1-m_i)] = 2 \cdot 2 = 4.$$

2. If n = 3,

$$\Delta_{\Pi}(n) = \prod_{m_i=1}^{n} [m_i(n+1-m_i)]$$
$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] \cdot [m_3(n+1-m_3)] \cdot \ldots \cdot [m_i(n+1-m_i)] = n!^2$$

$$\Delta_{\Pi}(3) = \prod_{m_i=1}^{n=3} [m_i(n+1-m_i)]$$

$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] \cdot [m_3(n+1-m_3)] = 3!^2$$

$$\Delta_{\Pi}(3) = \prod_{m_i=1}^{n=3} [m_i(n+1-m_i)]$$

$$= [1(3+1-1)] \cdot [2(3+1-2)] \cdot [3(3+1-3)] = 3!^2$$

$$\Delta_{\Pi}(3) = \prod_{m_i=1}^{n=3} [m_i(n+1-m_i)] = 3 \cdot 4 \cdot 3 = 36.$$

3. If $n = 4$,

$$\Delta_{\Pi}(n) = \prod_{m_i=1}^{n} [m_i(n+1-m_i)]$$

$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] \cdot [m_3(n+1-m_3)] \cdot \ldots \cdot [m_i(n+1-m_i)] = n!^2$$

$$\Delta_{\Pi}(4) = \prod_{m_i=1}^{n=4} [m_i(n+1-m_i)]$$

$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] \cdot [m_3(n+1-m_3)] \cdot [m_4(n+1-m_4)] = 4!^2$$

$$\Delta_{\Pi}(4) = \prod_{m_i=1}^{n=4} [m_i(n+1-m_i)]$$

$$= [1(4+1-1)] \cdot [2(4+1-2)] \cdot [3(4+1-3)] \cdot [4(4+1-4)] = 4!^2$$

$$\Delta_{\Pi}(4) = \prod_{m_i=1}^{n=4} [m_i(n+1-m_i)] = 4 \cdot 6 \cdot 6 \cdot 4 = 576.$$

4. If $n = 10$,

$$\Delta_{\Pi}(n) = \prod_{m_i=1}^{n} [m_i(n+1-m_i)]$$

$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] \cdot [m_3(n+1-m_3)] \cdot \ldots \cdot [m_i(n+1-m_i)] = n!^2$$

$$\Delta_{\Pi}(10) = \prod_{m_i=1}^{n=10} [m_i(n+1-m_i)]$$

$$= [m_1(n+1-m_1)] \cdot [m_2(n+1-m_2)] \cdot [m_3(n+1-m_3)] \cdot \ldots \cdot [m_{10}(n+1-m_{10})] = 10!^2$$

$$\Delta_{\Pi}(10) = \prod_{m_i=1}^{n=10} [m_i(n+1-m_i)]$$

$$= [1(10+1-1)] \cdot [2(10+1-2)] \cdot [3(10+1-3)] \cdot \ldots \cdot [10(10+1-10)] = 10!^2$$

$$\Delta_{\Pi}(10) = \prod_{m_i=1}^{n=10} [m_i(n+1-m_i)]$$

$$= 10 \cdot 18 \cdot 24 \cdot 28 \cdot 30 \cdot 30 \cdot 28 \cdot 24 \cdot 18 \cdot 10 = 13{,}168{,}189{,}440{,}000$$

D. *A Multiplication Table in a Different Perspective*

One interesting property of this triangle is the multiplication table in a different perspective that even the school children appreciate is its intersection-product. Suppose one takes the

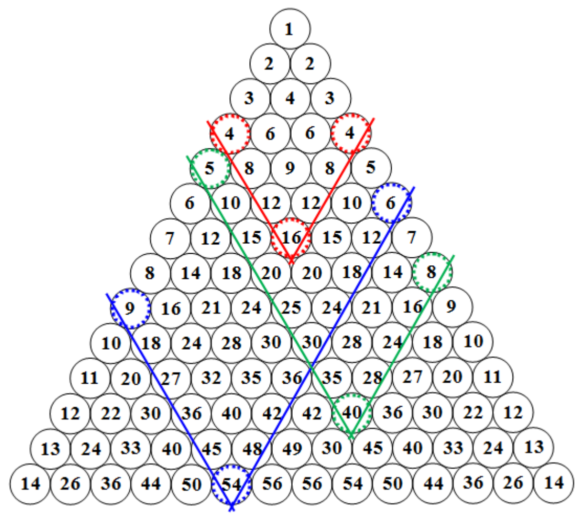


Figure 26: The triangular array of the product of denominators as a "multiplication table".

product of (4)(4)=16 as shown on the red-colored intersecting lines, the factors are situated at the outer sides of the triangle (4 at the left side and 4 at the right side marked in perforated red circle), while the product is at the point of intersection, also in perforated red circle is equal to 16. The green-marked intersecting lines has the equation (5)(8)=40 in it. And also, the blue-marked intersecting lines has (9)(6)=54. The triangle can be extended indefinitely up to the *n*th row and freely explore its multiplication properties.

E. *The Tetrahedral Number in the New Number Triangle*

The triangle that shows the Fortaliza number also shows the Tetrahedral numbers: 1, 4, 10, 20, 35,… as generated by the function (the closed form was designed by the author of this newly-found triangle),

$$\Delta_{\Sigma}(n) = \sum_{m_i=1}^{n} [m_i(n+1-m_i)] = T_{e_n} = T_n$$

where,

$$\sum_{m_i=1}^{n} [m_i(n+1-m_i)] = [m_1(n+1-m_1)] + [m_2(n+1-m_2)] + [m_3(n+1-m_3)] + \cdots + [m_i(n+1-m_i)]$$

where in, T_{e_n} and T_n are tetrahedral numbers as illustrated by Wolfram MathWorld and Wikipedia [13][18].

$$T_{e_n} = \sum_{k=1}^{n} T_k = \frac{1}{6}n(n+1)(n+2) = \binom{n+2}{3}$$

$$T_n = \frac{n(n+1)(n+2)}{6} = \frac{n^{\overline{3}}}{3!} = \binom{n+2}{3}$$

A tetrahedral number or triangular pyramidal number is a figurate number that represents a pyramid with a triangular base and three sides, called a tetrahedron. The *n*th tetrahedral number is the sum of the first *n* triangular numbers [13].

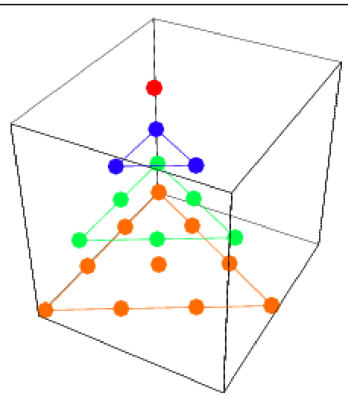


Figure 27: Illustration of the 4th tetrahedral number (20) in 4 layers of triangular numbers.
(Photo courtesy of MathWorld)

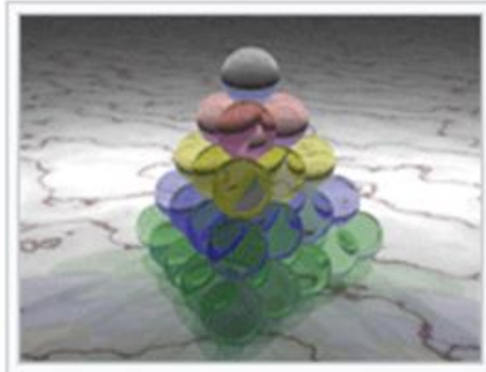


Figure 28: A pyramid of marbles in 5 layers contains 35 spheres. Each layer represents one of the first five triangular numbers.
(Photo courtesy of Wikipedia)

The tetrahedral (triangular pyramidal) number as illustrated in the triangle where the Fortaliza number was first illustrated.

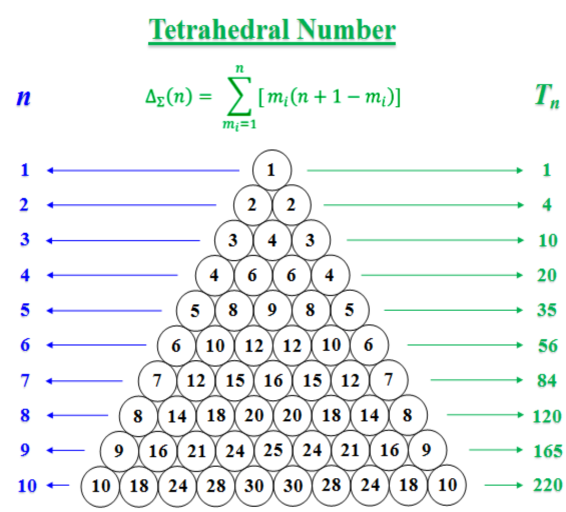


Figure 29: The sum of the terms in each row of the triangle illustrates the Tetrahedral number.

F. *Mathematical Induction of Tetrahedral Number*

Following the induction process of Raji [19], to show that, $\forall\, n \in \mathbb{N}$.

Note that,

$$\Delta_\Sigma(1) = \sum_{m_i=1}^{n=1} [1(1+1-1)] = T_n = \frac{1}{6}(1)(1+1)(1+2)$$

$$\Delta_\Sigma(1) = 1 = T_n = 1$$

and thus the statement is true for $n = 1$. For the remaining inductive step, suppose that the formula holds for n, that is

$$\Delta_\Sigma(n) = \sum_{m_i=1}^{n} [m_i(n+1-m_i)] = T_n = \frac{1}{6}(n)(n+1)(n+2)$$

It is shown that,

$$\Delta_\Sigma(n+1) = \sum_{m_i=1}^{n+1} [m_i(n+2-m_i)] = T_n = \frac{1}{6}(n+1)(n+2)(n+3)$$

to complete the proof by induction. Indeed,

$$\Delta_\Sigma(n+1) = \sum_{m_i=1}^{n+1} [m_i(n+1+1-m_i)] = T_n = \frac{1}{6}(n+1)(n+1+1)(n+2+1)$$

$$\Delta_\Sigma(n+1) = \sum_{m_i=1}^{n+1} [m_i(n+2-m_i)] = T_n = \frac{1}{6}(n+1)(n+2)(n+3)$$

$$\Delta_\Sigma(n+1) = \sum_{m_i=1}^{n+1} [m_i(n+2-m_i)]$$
$$= [m_1(n+2-m_1)] + [m_2(n+2-m_2)] + [m_3(n+2-m_3)] + \cdots + [m_i(n+2-m_i)]$$

$$\Delta_\Sigma(n+1) = \sum_{m_i=1}^{n+1} [m_i(n+2-m_i)] = T_n = \frac{1}{6}(n+1)(n+2)(n+3)$$

And the result follows the proof by showing the validity of this mathematical statement using the principle of mathematical induction. ■

To verify the validity of the proof of the tetrahedral number, some examples are shown:

1. If $n = 2$,

$$\Delta_\Sigma(n) = \sum_{m_i=1}^{n} [m_i(n+1-m_i)]$$
$$= [m_1(n+1-m_1)] + [m_2(n+1-m_2)] + [m_3(n+1-m_3)] + \cdots + [m_i(n+1-m_i)] = T_n = \frac{1}{6}(n)(n+1)(n+2)$$

$$\Delta_\Sigma(2) = \sum_{m_i=1}^{n=2} [m_i(n+1-m_i)] = [m_1(2+1-m_1)] + [m_2(2+1-m_2)]$$
$$= T_2 = \frac{1}{6}(2)(2+1)(2+2)$$

$$\Delta_\Sigma(2) = \sum_{m_i=1}^{n=2} [m_i(n+1-m_i)] = [1(2+1-1)] + [2(2+1-2)] = T_2$$
$$= \frac{1}{6}(2)(3)(4)$$

$$\Delta_\Sigma(2) = \sum_{m_i=1}^{n=2} [m_i(n+1-m_i)] = 2 + 2 = T_2 = 4.$$

2. If $n = 3$,

$$\Delta_\Sigma(n) = \sum_{m_i=1}^{n} [m_i(n+1-m_i)]$$
$$= [m_1(n+1-m_1)] + [m_2(n+1-m_2)] + [m_3(n+1-m_3)] + \cdots + [m_i(n+1-m_i)] = T_n = \frac{1}{6}(n)(n+1)(n+2)$$

$$\Delta_\Sigma(3) = \sum_{m_i=1}^{n=3} [m_i(n+1-m_i)]$$
$$= [m_1(3+1-m_1)] + [m_2(3+1-m_2)] + [m_3(3+1-m_3)] = T_3 = \frac{1}{6}(3)(3+1)(3+2)$$

$$\Delta_\Sigma(3) = \sum_{m_i=1}^{n=3} [m_i(n+1-m_i)]$$
$$= [1(3+1-1)] + [2(3+1-2)] + [3(3+1-3)] = T_3 = \frac{1}{6}(3)(4)(5)$$

$$\Delta_\Sigma(3) = \sum_{m_i=1}^{n=3} [m_i(n+1-m_i)] = 3 + 4 + 3 = T_2 = 10.$$

3. If $n = 4$,

$$\Delta_\Sigma(n) = \sum_{m_i=1}^{n} [m_i(n+1-m_i)]$$
$$= [m_1(n+1-m_1)] + [m_2(n+1-m_2)] + [m_3(n+1-m_3)] + \cdots + [m_i(n+1-m_i)] = T_n = \frac{1}{6}(n)(n+1)(n+2)$$

$$\Delta_\Sigma(4) = \sum_{m_i=1}^{n=4}[m_i(n+1-m_i)]$$

$$= [m_1(4+1-m_1)] + [m_2(4+1-m_2)] + [m_3(4+1-m_3)] + [m_4(4+1-m_4)] = T_4 = \frac{1}{6}(4)(4+1)(4+2)$$

$$\Delta_\Sigma(4) = \sum_{m_i=1}^{n=4}[m_i(n+1-m_i)]$$

$$= [1(4+1-1)] + [2(4+1-2)] + [3(4+1-3)] + [4(4+1-4)] = T_4 = \frac{1}{6}(4)(5)(6)$$

$$\Delta_\Sigma(4) = \sum_{m_i=1}^{n=4}[m_i(n+1-m_i)] = 4+6+6+4 = T_2 = 20.$$

4. If n = 10,

$$\Delta_\Sigma(n) = \sum_{m_i=1}^{n}[m_i(n+1-m_i)]$$

$$= [m_1(n+1-m_1)] + [m_2(n+1-m_2)] + [m_3(n+1-m_3)] + \cdots + [m_i(n+1-m_i)] = T_n = \frac{1}{6}(n)(n+1)(n+2)$$

$$\Delta_\Sigma(10) = \sum_{m_i=1}^{n=10}[m_i(n+1-m_i)]$$

$$= [m_1(10+1-m_1)] + [m_2(10+1-m_2)] + [m_3(10+1-m_3)] + \cdots + [m_{10}(10+1-m_{10})] = T_{10} = \frac{1}{6}(10)(10+1)(10+2)$$

$$\Delta_\Sigma(10) = \sum_{m_i=1}^{n=10}[m_i(n+1-m_i)]$$

$$= [1(10+1-1)] + [2(10+1-2)] + [3(10+1-3)] + \cdots + [10(10+1-10)] = T_{10} = \frac{1}{6}(10)(11)(12)$$

$$\Delta_\Sigma(10) = \sum_{m_i=1}^{n=10}[m_i(n+1-m_i)]$$

$$= 10+18+24+28+30+30+28+24+18+10 = T_{10} = 240.$$

G. Connection to the Pascal's Triangle

The link of this new found triangle to Pascal's triangle can be seen on the fourth row of left-justified triangle of Pascal's, the tetrahedral number.

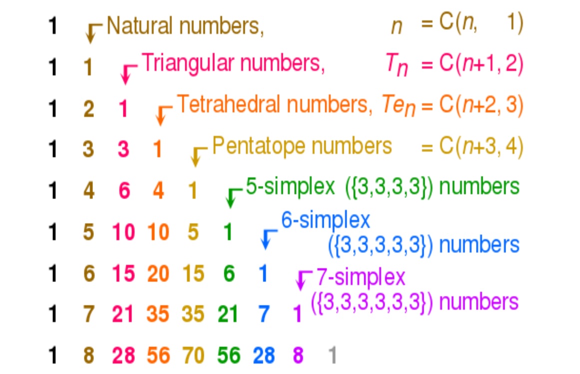


Figure 30: Derivation of simplex numbers from a left-justified Pascal's triangle.
(Photo courtesy of Wikipedia)

IV. Discussion

The Fortaliza-Tetrahedral Triangle

Since the two numbers, Fortaliza number and Tetrahedral number, are both contained in the same triangle, it is but fitting and proper to name this triangle as Fortaliza-Tetrahedral Triangle.

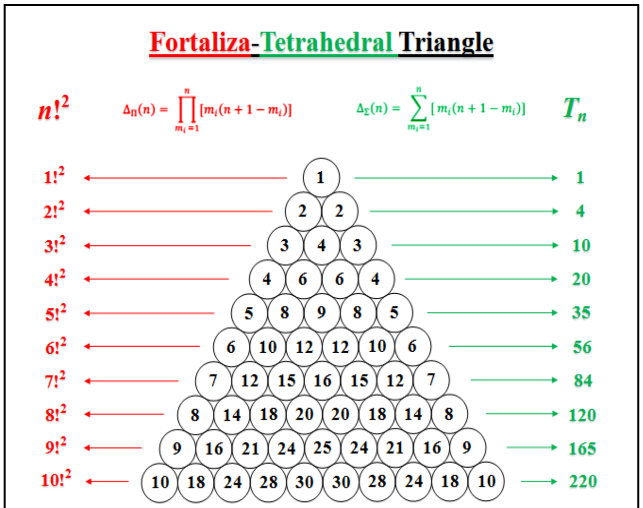


Figure 31: The triangle sharing both numbers, Fortaliza number and Tetrahedral number.

In essence, the Fortaliza number was originally directed in finding the formula for the sum of first n terms of the harmonic sequence. While the study was still underway, it was by accident that this newly-found triangle was discovered while digging on the formula for harmonic number. While the tetrahedral number has been in the mathematical world many years before its newly-found connection with the Fortaliza number thru the Pascal's triangle.

V. Conclusion

The discovery of Fortaliza-Tetrahedral Triangle had been in arduous process of curiosity, intuition, and ingenuity like other mathematicians of the past who travelled the same road [2]. In pure math, this triangle is an art masterpiece in itself, and due to its connection to the harmonic series and tetrahedral number, this triangle will go a long way in unearthing up many discoveries yet to be known. The harmonic sequence is not a distant relative of the Riemann Zeta Function [10]. The Riemann hypothesis, considered one of the greatest unsolved problems in mathematics [10], where the solution is key in explaining the powerful properties of prime numbers that in turn explains many things in mathematics and the sciences. The Tetrahedral numbers have practical applications in sphere packing [14] and its tetrahedron have applications in various fields like numerical analysis, chemistry, electricity and electronics, games, color space, contemporary art, popular culture, geology, structural engineering, and aviation [15].

VI. Recommendation

Math enthusiasts and lovers of art and symmetry may find the Fortaliza-Tetrahedral Triangle interesting. In the surface, it resembles a multiplication table in a different perspective. In its sublime context, number theorists and mathematical researchers may find this Fortaliza-Tetrahedral Triangle worthy of research as it connects with the harmonic sequence and other related concepts. This triangle also links with the Pascal's triangle through the tetrahedral numbers.

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Greatest gratitude is for the Divine Power who has given me the gift of seeing the beauty and grandeur of this newly-discovered number triangle;

My heartfelt gratitude to my professors, past and present, who gave me sparks of mathematical curiosity;

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To persons and entities who in one way or another had helped in the realization of this research, my sincere gratitude; and

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