TRIVOLUTIONS FROM CUBIC PENCILS: A SYNTHETIC VIEWPOINT

YERKEBULAN BOLAT

Abstract. A pencil of plane conics induces an involution on any transversal line (Desargues' Involution Theorem). For cubics, the analogous construction yields a natural degree–3 correspondence on a line, which we call a *trivolution*.

Although the underlying mechanism is classical (Cayley–Bacharach and genus–one geometry), we give a synthetic treatment focused on the projective picture and (when relevant) circular cubics. We describe the induced degree–3 covering of a line, its monodromy via discriminants, and how this explains the rigidity of involution methods in degree 2. As concrete consequences we include a projective butterfly theorem for 2–torsion points and a cubic–method collinearity theorem of Sakhipov.

1. Introduction

Desargues' Involution Theorem (DIT) states that a pencil of conics through four base points induces a projective involution on any transversal line. This phenomenon explains many classical collinearity and concurrency results: the "hidden symmetry" is the involution.

A pencil of plane cubics behaves differently. Cubics intersect a line in three points, so there is no canonical involution on the line in general position. Instead, one naturally obtains an *unordered triple* in each fiber and hence a triple–valued correspondence.

The goal of this note is to describe this correspondence cleanly, to explain its monodromy and degenerations, and to record a few elementary consequences. The viewpoint aligns with classical "cubic method" arguments: many surprising linear conclusions are forced by the Cayley–Bacharach principle.

What is and is not claimed. The degree–3 correspondence induced by a cubic pencil is not a single-valued map in general, and one should not expect a canonical order–3 projective automorphism of the line. The correct general object is a correspondence with generically full S_3 monodromy. Only in special situations does the monodromy reduce to A_3, producing a canonical 3–cycle.

2. Cayley–Bacharach and cubic pencils

Lemma 1 (Cayley–Bacharach for cubics)**.** *Let C_1, C_2 be plane cubics intersecting in nine points $B_1, \dots, B_9$ (counted with multiplicity). Any cubic passing through $B_1, \dots, B_8$ also passes through B_9.*

Proof. See [1, 2]. □

Thus, a pencil of cubics through eight general points automatically has a ninth base point. We will denote the base locus by

$$\mathcal{B} = \{B_1, \dots, B_9\} \subset \mathbb{P}^2.$$

3. Trivolutions on a transversal line

Let $\mathcal{P} = \{C_t\}_{t \in \mathbb{P}^1}$ be a pencil of plane cubics with base locus $\mathcal{B}$, and let $\ell \subset \mathbb{P}^2$ be a line avoiding $\mathcal{B}$.

3.1. The incidence covering $\ell \to \mathbb{P}^1$.

For each point $X \in \ell$, the condition "$X \in C_t$" is linear in the pencil parameter t. Hence there is a uniquely determined member of the pencil passing through X. Equivalently, we obtain a rational map

$$\phi_\ell : \ell \simeq \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad X \longmapsto t(X)$$

sending X to the parameter of the unique cubic $C_{t(X)}$ containing X. For a given t, the fiber $\phi_\ell^{-1}(t)$ consists precisely of the three intersection points $\ell \cap C_t$. Thus ϕ_ℓ has degree 3.

3.2. The trivolution correspondence.

Definition 1 (Trivolution correspondence)**.** Define the set of intersection triples

$$T_\ell := \left\{ (X, Y, Z) \in \ell^3 \;\middle|\; \{X, Y, Z\} = \ell \cap C_t \text{ for some } t \in \mathbb{P}^1 \right\}.$$

We call T_ℓ the *trivolution correspondence* induced by the pencil $\mathcal{P}$ on ℓ.

The object T_ℓ is naturally *unordered*: the pencil determines the set $\{X, Y, Z\}$ but (in general) no preferred cyclic order. This is the first structural difference from conics.

3.3. Monodromy via the discriminant.

Choose an affine coordinate x on $\ell \simeq \mathbb{P}^1$. Restricting the pencil to ℓ gives a pencil of binary cubics, hence a degree–3 equation

$$F(x) + t\,G(x) = 0,$$

whose three roots are the points of $\ell \cap C_t$. Branch points occur exactly when two or three roots collide, i.e. when the discriminant $\Delta(t)$ vanishes. The monodromy group of the covering ϕ_ℓ is the Galois group of this cubic over the function field; generically it is S_3, and it drops to A_3 precisely when $\Delta(t)$ is a square (the standard cubic criterion; see [3]).

Proposition 1 (Discriminant/monodromy classification)**.** *Let $\Delta(t)$ be the discriminant of the restricted cubic pencil on ℓ.*

(1) *If $\Delta(t)$ has only simple zeros, the local branching is simple and the monodromy is generically S_3.*
(2) *If $\Delta(t)$ is a square in the relevant function field, then the monodromy lies in A_3 and the correspondence admits a canonical 3–cycle after choosing a basepoint.*
(3) *A double zero of Δ corresponds to a nodal degeneration along ℓ (two intersection points collide).*
(4) *A triple zero corresponds to a cuspidal degeneration along ℓ (three intersection points collide).*

Remark 1 (When do you get an actual order–3 map on ℓ?)*.* If the covering ϕ_ℓ is Galois with deck group $\mathbb{Z}/3\mathbb{Z}$ (equivalently, monodromy A_3 and no further obstruction), then there exists a projective automorphism $\tau \in \mathrm{PGL}_2$ of order 3 on ℓ that cyclically permutes the three points in each fiber. This is the closest degree–3 analogue of the "fixed involution" phenomenon in DIT.

4. Involutions on a cubic and torsion-generated 3–cycles

The previous section produces a degree–3 correspondence *on a line*. There is a different, classical source of actual order–3 dynamics: composing chord involutions *on the cubic itself*. We record this because it often appears implicitly in the integrable–maps literature and provides a clean "trivolution" on the curve.

4.1. **Chord involutions.** Let $E \subset \mathbb{P}^2$ be a smooth plane cubic equipped with an elliptic curve group law (choose any origin $O \in E$). For $P \in E$, define a birational involution

$$s_P : E \dashrightarrow E,$$

by letting $s_P(X)$ be the third point of intersection of the line PX with E (interpreting PX as the tangent when $X = P$). Equivalently, in the group law one has $s_P(X) = -(P + X)$.

Proposition 2 (Composition equals translation)**.** *For any $A, B \in E$ and any $X \in E$ in the domain of definition,*

$$(s_A \circ s_B)(X) = X + (B - A).$$

In particular, $s_A \circ s_B$ has order 3 if and only if $B - A$ is a nontrivial 3–torsion point.

Proof. Using $s_P(X) = -(P + X)$,

$$(s_A \circ s_B)(X) = -(A + (-(B + X))) = X + (B - A).$$

The order statement is immediate from iterating a translation. □

Remark 2. The general principle "composition of two involutions on a genus–one curve is a translation" is standard; see for example [4, §1]. Related constructions on cubic pencils appear in the theory of Manin transformations and birational maps preserving cubic pencils [5, 6].

5. A projective butterfly theorem (2–torsion)

The classical butterfly theorem is metric. On elliptic curves the natural analogue uses 2–torsion.

Theorem 1 (2–torsion butterfly)**.** *Let E be a smooth cubic with identity O and let $M \in E$ be a point of order 2. Let chords AA' and BB' pass through M. Define X as the third intersection of AB' with E and Y as the third intersection of $A'B$ with E. Then*

$$X + Y = O,$$

i.e. X, Y, O are collinear. For a circular cubic with identity chosen at an infinite flex, the chord XY is parallel to the asymptote direction.

Proof. Collinearity gives $A + A' + M = O$ and $B + B' + M = O$. Since $M = -M$, we obtain $A + A' = B + B' = M$. By definition of addition, $X = -(A + B')$ and $Y = -(A' + B)$. Therefore

$$X + Y = -(A + B') - (A' + B) = -(A + A' + B + B') = -(2M) = O.$$

□

5.1. An olympiad-style application on the triangle cubic $K(141)$.

The 2–torsion butterfly identity is a minimal example of a broader phenomenon: once a configuration is encoded on a suitable cubic, simple torsion bookkeeping often forces classical-looking Euclidean conclusions. We record one such application (from an AoPS discussion) in which the relevant cubic is the isocubic $K(141)$ in Kimberling's encyclopedia. One convenient characterization is the following: let $*$ denote isotomic conjugation in $\triangle ABC$, let K be the symmedian point, and set $Q = K^*$. Then $K(141)$ is the locus of points P such that P, P^*, Q are collinear [8, 9].

Theorem 2 (AoPS)**.** *In triangle ABC, let K be the symmedian point, E the Exeter point, and let X be the symmedian point of the anticomplementary triangle. Let A' satisfy $ABA'C$ is a parallelogram, and define*

$$F = XA' \cap BC, \qquad J = KF \cap AX, \qquad I = JA' \cap EF.$$

Then $AI \perp BC$.

Proof sketch (Due to AoPS user numbersandnumbers). Consider the cubic $\mathcal{C} = K(141)$ with the group law chosen so that collinear triples sum to 0. It is known (and can be verified in barycentric coordinates) that

$$A, B, C, G, A', K, Q, H, E \in \mathcal{C},$$

where G and H are the centroid and orthocenter, and $Q = K^*$. Moreover, tangency relations on $\mathcal{C}$ imply the "doubling" identities

$$2G = 2A' = -Q, \qquad 2A = 2B = 2C = 2Q = -K$$

(the tangent at a point meets $\mathcal{C}$ again at the negative of the double of that point).

Writing points of $\mathcal{C}$ as "integer multiples of G plus 2–torsion", one obtains

$$K = 4G, \quad X = -5G, \quad H = 7G, \quad E = -8G,$$

and there exist distinct nontrivial 2–torsion points $t_A, t_B, t_C \in \mathcal{C}[2]$ such that

$$A = -2G + t_A, \quad B = -2G + t_B, \quad C = -2G + t_C, \quad A' = G + t_A.$$

Since $\mathcal{C}[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$, the three nonzero 2–torsion points satisfy $t_A + t_B + t_C = 0$. Define $F' := 4G + t_A$. Then the collinearity relations

$$A' + X + F' = 0, \qquad B + C + F' = 0$$

show that F' lies on both $A'X$ and BC, hence $F' = F$.

Finally, one applies Pascal's theorem to a suitable degenerate hexagon on $\mathcal{C}$ (built from the points above) to deduce that the three points

$$A', \quad J = AX \cap KF, \quad AH \cap EF$$

are collinear. Therefore $I = JA' \cap EF$ coincides with $AH \cap EF$, so A, I, H are collinear. Since $AH \perp BC$, we obtain $AI \perp BC$. ☐

Remark 3 (Bonus angle). The same $K(141)$ bookkeeping identifies F as the unique point of BC lying on XA' with prescribed divisor class, and one can additionally show $\angle EFC = \angle XFB$ (see the AoPS thread for details).

6. Degenerate fibers and local behavior

The discriminant description in Proposition 1 is also a practical degeneration dictionary. We record the geometric meaning of the main cases.

6.1. **Nodal fibers.** If some member C_{t_0} has a node, then along a transversal line one typically sees a collision of two intersection points (a double root). The local monodromy around t_0 is a transposition.

6.2. **Cuspidal fibers.** If some member C_{t_0} has a cusp, then along a transversal line the triple may collapse completely (a triple root). The local monodromy is trivial.

6.3. **Reducible fibers.** If $C_{t_0} = Q \cup L$ is reducible (conic plus line), then $\ell \cap C_{t_0}$ splits into two points from Q and one point from L, exhibiting an "involution + fixed point" shadow of the conic case.

7. A cubic–method collinearity theorem of Sakhipov

The next theorem illustrates a typical cubic–method pattern: a nontrivial collinearity is proved by constructing two cubics and applying Cayley–Bacharach.

Theorem 3 (Sakhipov)**.** *Let $ABCD$ be a convex quadrilateral and set $E = AD \cap BC$, $F = AB \cap CD$. Define X by requiring that AX is isogonal to AC with respect to $\angle BAD$ and CX is isogonal to CA with respect to $\angle BCD$. Define Y analogously from the diagonal BD using the angles at B and D. Define Z using the line EF with isogonality with respect to $\angle AFD$ and $\angle CED$. Then X, Y, Z are collinear.*

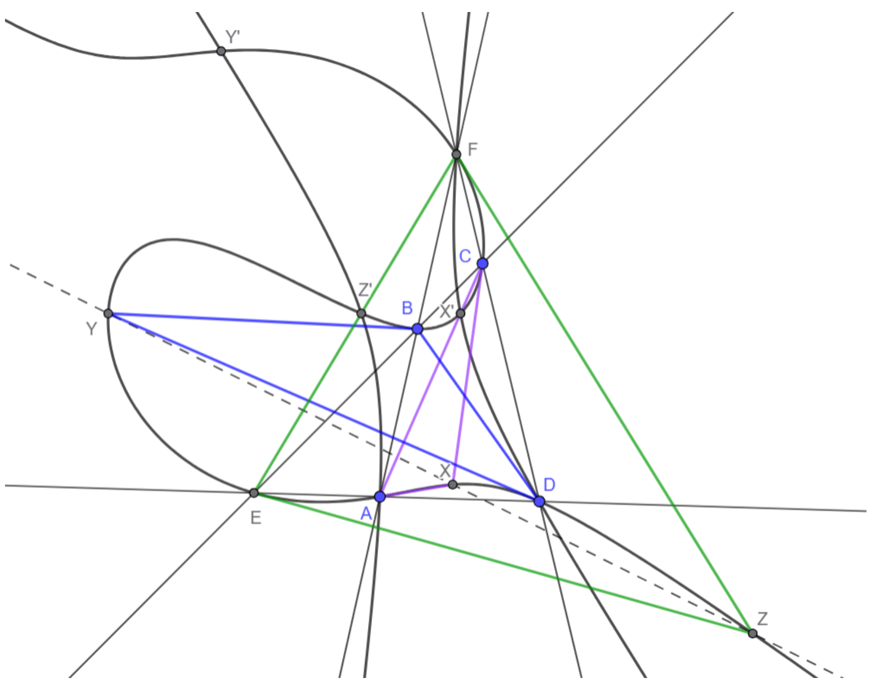
Proof (A. Sakhipov). Let $\mathcal{P}$ be the isogonal cubic of the quadrilateral $ABCD$, i.e. the locus of points admitting an isogonal conjugate in $ABCD$ (see [7]). Let $X' = AC \cap \mathcal{P} \setminus \{A, C\}$ and define Y', Z' analogously on BD and EF. By construction, X is the isogonal conjugate of X' (and similarly $Y \leftrightarrow Y'$, $Z \leftrightarrow Z'$).

Let $\mathcal{Q} = AC \cup BD \cup EF$. Then $\mathcal{P} \cap \mathcal{Q}$ consists of the nine points

$$A, C, X', \quad B, D, Y', \quad E, F, Z'.$$

Let ω be the conic through A, D, F, X', Y' and set $\mathcal{S} = BC \cup \omega$. Then $\mathcal{S}$ passes through the eight points A, B, C, D, E, F, X', Y'. By Lemma 1, $\mathcal{S}$ also passes through Z', hence $Z' \in \omega$. Thus A, D, F, X', Y', Z' lie on the same circumconic of $\triangle ADF$.

Isogonal conjugation with respect to $\triangle ADF$ maps circumconics to lines, so the isogonal conjugates X, Y, Z are collinear.



□

Special thanks to Sultan Kurmankulov for providing this result.

8. Outlook: why degree 2 is special

DIT and its dual owe their power to a rigidity phenomenon specific to degree 2: a conic cuts a transversal line in two points, and a two–point fiber admits a canonical symmetry (swapping). This symmetry is precisely the involution underlying Desargues' theorem.

In contrast, a cubic cuts a line in three points, and without extra structure there is no canonical ordering of each triple. Accordingly, the natural object associated to a pencil of cubics is not a self–map of the line but a triple–valued correspondence, which we have called a *trivolution*. From the covering–theoretic viewpoint, the induced degree–3 covering is typically non–Galois and has full S_3 monodromy.

This perspective explains why involution methods are ubiquitous and effective in classical geometry but do not automatically extend to higher degree. Meaningful extensions usually require auxiliary choices (marked base points, tangency constraints, or group–law data) or restriction to special families (e.g. circular cubics or pencils with extra symmetry). Even in these cases, the correct replacement for an involution is generally a multi–valued correspondence rather than a genuine self–map.

References

[1] D. Eisenbud, M. Green, J. Harris, *Cayley–Bacharach theorems and conjectures*, Bull. Amer. Math. Soc. **33** (1996), no. 3, 295–324.

[2] Q. Ren, J. Richter-Gebert, B. Sturmfels, *Cayley–Bacharach formulas*, arXiv:1405.6438.

[3] K. Conrad, *Galois groups of cubics and quartics (not in characteristic* 2*)*, Expository notes available online.

[4] G. Felder, *The Poncelet property of the Boltzmann system*, arXiv:2008.03480.

[5] P. H. van der Kamp, *Generalised Manin transformations and QRT maps*, preprint available online.

[6] P. H. van der Kamp, *Manin transformations with involution curves*, SIGMA **17** (2021), 067.

[7] D. Hu, *Constructions in the locus of isogonal conjugates in a quadrilateral*, arXiv:1912.08296.

[8] C. Kimberling, *Encyclopedia of Triangle Centers*, online resource, entry $X(141)$ and the associated triangle cubic $K(141)$.

[9] B. Gibert, *Examples and notes on the isocubic* $K(141)$, online notes, available on the author's webpage.