Newton and Coriolis forces from Einstein Equations

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Abstract

We consider the simple derivation of the Einstein equations. Then, we consider motion of a particle in the non-inertial system and motion of a particle in gravitational field. In case of the constant gravitational field we transform the Einstein equations to the Newtonian form of motion of a particle in gravity and derive potential and the Coriolis force generating by the constant gravity.

1 Introduction

While the electromagnetic field was determined from the motion of charges and currents, the Einstein-Hilbert theory of gravity being the space-time geometry was determined from presence of mass-energy and linear momentum. The corresponding equations - Einstein-Hilbert equations - determine the metric tensor of space-time for a given arrangement of stress-energy in the space-time. The relationship between the metric tensor and the Einstein tensor allows the Einstein-Hilbert equations to be written as a set of non-linear partial differential equations. The solutions of the Einstein-Hilbert equations are the components of the metric tensor. The inertial trajectories of particles and radiation in the resulting geometry are then calculated using the geodesic equations. As well as obeying local energy-momentum conservation, the Einstein-Hilbert equations reduce to Newtons law of gravitation, where the gravitational field is weak and velocities are much less than the speed of light.

We consider here, first, the simple derivation of the Einstein equations by Fock.

2 The Einstein equations derived by Fock

There is the simple derivation of the Einstein-Hilbert equations given by Fock (1964). The similar derivation was performed by Chandrasekhar (1972), Kenyon (1996), Landau et al. (1987), Rindler (2003) and others. Source theory derivation of Einstein equations was performed by Schwinger (1970).

It is well known that the gravity mass M_G of some body is equal to the its inertial mass M_I, where gravity mass is a measure of a massive body to create the gravity field (or, gravity force) and the inertial mass of a massive body is a measure of the ability of the resistance of the body when it is accelerated. At present time we know, that if components of elementary particles have the same gravity and inertial masses, the body composed with such elementary particles has the identical gravity and inertial mass. There is no need to perform experimental verification. So, particle physics brilliantly confirms the identity of the inertial and gravity masses.

According to the Newton theory, the gravity potential is given by the equation

$$U(r) = -\kappa \frac{M}{r}, \tag{1}$$

where r is a distance from the center of mass of a body, κ is the gravitational constant and its numerical value is in SI units $6.67430(15)10^{-11} m^3.kg^{-1}.s^{-2}$ (CODATA, 2018).

The potential U is, as it is well known, the solution of the Poisson equation:

$$\Delta U(r) = -4\pi\kappa\varrho, \tag{2}$$

where ϱ is the density of the distributed masses.

The problem is, what is the geometrical formulation of gravity equation (2) following from the space-time element ds, which has the Minkowski form in case of the special theory of relativity.

Let us postulate that the motion of a body moving in the g-field is determined by the variational principle

$$\delta \int ds = 0. \tag{3}$$

In order to get the Newton equation of motion, we are forced to perform the following identity:

$$g_{00} = c^2 - 2U = -4\pi\kappa\varrho. \tag{4}$$

The second mathematical requirement, which has also the physical meaning is the covariance of the derived equation. It means that the necessary mathematical operation are the following replacing of original symbols:

$$U \to g_{\mu\nu} \tag{5}$$

with

$$\Delta U \to Tensor\ equation \tag{6}$$

and

$$\varrho \to T_{\mu\nu}, \tag{7}$$

where $T_{\mu\nu}$ is the tensor of energy and momentum.

In order to get the tensor generalization of eq. (2) it is necessary to construct new tensor $R_{\mu\nu}$, which is linear combination of the more complicated tensor $R_{\alpha\beta,\mu\nu}$, or

$$R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha,\beta\nu} \tag{8}$$

and the scalar quantity R, which is defined by equation

$$R = g^{\lambda\mu} R_{\lambda\mu} \tag{9}$$

and construct the combination tensor $G_{\lambda\mu}$ of the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{10}$$

which has the mathematical property, that the covariant divergence of this tensor is zero, or,

$$\nabla^{\lambda} G_{\lambda\mu} = 0. \tag{11}$$

.

With regard to the fact that also the energy-momentum tensor $T_{\mu\nu}$ has the zero divergence, we can identify eq. (10) with the tensor $T_{\mu\nu}$, or

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi\kappa}{c^2} T_{\mu\nu}, \tag{12}$$

where the appeared constant in the last equation is introduce to get the classical limit of the equation.

The approximate solution of the last equation is as follows

$$ds^2 = (c^2 - 2U)dt^2 - \left(1 + \frac{2U}{c^2}(dx^2 + dy^2 + dz^2)\right). \tag{13}$$

The space-time element (13) is able to explain the shift of the frequency of light in gravitational field and the deflection of light in the gravitational field of massive body with mass M.

So, we have seen that the basic mathematical form of the Einstein general relativity is the Riemann manifold specified by the metric with the physical meaning. The crucial principle is the equality of the inertial and gravitational masses.

While the derivation of the Einstein-Hilbert equation is elementary, Feynman wrote that the derivation of Einstein-Hilbert equation by Einstein is difficult to understand. Namely:

Einstein himself, of course, arrived at the same Lagrangian but without the help of a developed field theory, and I must admit that I have no idea how he guessed the final result. We have had troubles enough arriving at the theory - but I feel as though he had done it while swimming underwater, blindfolded, and with his hands tied behind his back! (Feynman et al., 1995).

Let us still remark that the derived Einstein equations (12) can be generalized to form the Einstein equations with the cosmological constant, or, (Einstein, 1917)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi\kappa}{c^2} T_{\lambda\mu}, \tag{14}$$

where Λ is the new cosmological constant introduced formally, with the goal to find new form of the cosmological model and their solutions in the mathematical form. In addition to that the last equation can be still extended in order to involve so called cosmological matrix (Pardy, 2018).

3 Non-inertial system

To understand the Coriolis force, we consider specific characteristics of the mechanical systems in the rotating framework, by means of the of the differential equations describing the mechanical systems in the non-inertial systems. We follow the text of Landau et al. (Landau et al. 1965).

Let be the Lagrange function of a point particle in the inertial system as follows:

$$L_0 = \frac{m\mathbf{v}_0^2}{2} - U \tag{15}$$

with the following equation of motion

$$m\frac{d\mathbf{v}_0}{dt} = -\frac{\partial U}{\partial \mathbf{r}}, \tag{16}$$

where the quantities with index 0 correspond to the inertial system.

The Lagrange equations in the non-inertial system is of the same form as that in the inertial one, or,

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{r}}. \tag{17}$$

However, the Lagrange function in the non-inertial system is not the same as in eq. (15) because it is transformed.

Let us first consider the system K' moving relatively to the system K with the velocity $\mathbf{V}(t)$. If we denote the velocity of a particle with regard to system K' as $\mathbf{v}'$, then evidently

$$\mathbf{v}_0 = \mathbf{v}' + \mathbf{V}(t). \tag{18}$$

After insertion of eq. (18) into eq. (15), we get

$$L_0' = \frac{m\mathbf{v}'^2}{2} + m\mathbf{v}'\mathbf{V} + \frac{m}{2}\mathbf{V}^2 - U. \tag{19}$$

The function $\mathbf{V}^2$ is the function of time only and it can be expressed as the total derivation of time of some new function. It means that the term with the total derivation in the Lagrange function can be removed from the Lagrangian. We also have:

$$m\mathbf{v}'\mathbf{V}(t) = m\mathbf{V}\frac{d\mathbf{r}'}{dt} = \frac{d}{dt}(m\mathbf{r}'\mathbf{V}(t)) - m\mathbf{r}'\frac{d\mathbf{V}}{dt}. \tag{20}$$

After inserting the last formula into the Lagrange function and after removing the total time derivation we get

$$L' = \frac{mv'^2}{2} - m\mathbf{W}(t)\mathbf{r}' - U, \tag{21}$$

where $\mathbf{W} = d\mathbf{V}/dt$ is the acceleration the system K'.

The Lagrange equations following from the Lagrangian (21) are as follows:

$$m\frac{d\mathbf{v}'}{dt} = -\frac{\partial U}{\partial \mathbf{r}'} - m\mathbf{W}(t). \tag{22}$$

We see that after acceleration of the system K' the new force $m\mathbf{W}(t)$ appears. This force is fictitious one because it is not generated by the internal properties of some body.

In case that the system K' rotates with the angle velocity $\mathbf{\Omega}$ with regard to the system K, vectors $\mathbf{v}$ and $\mathbf{v}'$ are related as (Landau et al., 1965)

$$\mathbf{v}' = \mathbf{v} + \mathbf{\Omega} \times \mathbf{r}. \tag{23}$$

The Lagrange function for this situation is (Landau et al., 1965)

$$L = \frac{mv^2}{2} - m\mathbf{W}(t)\mathbf{r} - U + m\mathbf{v} \cdot (\mathbf{\Omega} \times \mathbf{r}) + \frac{m}{2}(\mathbf{\Omega} \times \mathbf{r})^2. \qquad 24)$$

The corresponding Lagrange equations for the last Lagrange function are as follows (Landau et al., 1965):

$$m\frac{d\mathbf{v}}{dt} = -\frac{\partial U}{\partial \mathbf{r}} - m\mathbf{W} + m\mathbf{r} \times \dot{\mathbf{\Omega}} + 2m\mathbf{v} \times \mathbf{\Omega} + m\mathbf{\Omega} \times (\mathbf{r} \times \mathbf{\Omega}) \,. \tag{25}$$

We observe in eq. (25) three so called inertial forces. The force $m\mathbf{r} \times \dot{\mathbf{\Omega}}$ is connected with the nonuniform rotation of the system K' and the forces $2m\mathbf{v} \times \mathbf{\Omega}$ and $m\mathbf{\Omega} \times \mathbf{r} \times \mathbf{\Omega}$ correspond to the uniform rotation. The force $2m\mathbf{v} \times \mathbf{\Omega}$ is so called the Coriolis force and it depends on the velocity of a particle. The force $m\mathbf{\Omega} \times \mathbf{r} \times \mathbf{\Omega}$ is called the centrifugal force. It is perpendicular to the rotation axes and the magnitude of it is $m\varrho\omega^2$, where ϱ is the distance of the particle from the rotation axis.

Equation (25) can be applied to many special cases. For instance to the case of the mathematical pendulum swinging in the gravitational field of the rotating Earth, so called Foucault pendulum.

Foucault pendulum was studied by Léon Foucault (1819 - 1868) as the big mathematical pendulum with big mass m swinging in the gravitational field of the Earth. He used a 67 m long pendulum in the Panthéon in Paris and showed the astonished public that the direction of its swing changed over time rotating slowly. The experiment proved that the earth rotates. If the earth would not rotate, the swing would always continue in the same direction.[1]

4 Motion of particle in gravitational field

The motion of a particle in a gravitational field is determined by the principle of least action in the following form (Landau et al., 1987),

$$\delta S = -mc\delta \int ds = 0, \tag{26}$$

since the gravitational field is nothing but a change in the metric of space-time, manifesting itself only in a change in the expression for ds in terms of the dx^i. Thus, in a gravitational field the particle moves so that its world point moves along an extremal or, as it is called, a geodesic line in the four-space however, since in the presence of the gravitational field space-time is not Galilean, this line is not a "straight line", and the real spatial motion of the particle is neither uniform nor rectilinear.

Instead of starting once again directly from the principle of least action, it is simpler to obtain the equations of motion of a particle in a gravitational field by an appropriate generalization of the differential equations for the free motion of a particle in the special theory of relativity, i.e. in a Galilean four-dimensional coordinate system. These equations are $du^i/ds = 0$, or, $du^i = 0$, where $u^i = dx^i/ds$ is the four velocity.

[1] The big Foucault pendulum is inside of the rotunda of the Flower garden in Kroměříž (Moravia, Czech Republic)

Clearly, in curvilinear coordinates this equation is generalized to the equation

$$Du^i = 0. \tag{27}$$

From the expression

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k\right)dx^l \tag{28}$$

for the covariant differential of a vector A^i, we have

$$du^i + \Gamma^i_{kl}u^k dx^l = 0. \tag{29}$$

Dividing this equation by ds, we have

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0 \tag{30}$$

This is the required equation of motion. We see that the motion of a particle in gravitational field is determined by the quantities Γ^{kl}_i . The derivative $\frac{d^2x^i}{ds^2}$ is the four-acceleration of the particle. Therefore we may call the quantity $-m\Gamma^i_{kl}u^ku^l$ the "four-force", acting on the particle in the gravitational field. Here, the tensor g_{ik} plays the role of the "potential" of the gravitational field. Its derivatives determine the field "intensity" Γ^i_{kl}.

Following Landau et al., (1987) it was shown that by a suitable choice of the coordinate system one can always make all the Γ^i_{kl} zero at an arbitrary point of space-time. We now see that the choice of such a locally-inertial system of reference means the elimination of the gravitational field in the given infinitesimal element of space-time, and the possibility of making such a choice is an expression of the principle of equivalence in the relativistic theory of gravitation.

5 Potential and Coriolis force of constant gravitational field

A gravitational field is constant if there is a system of reference in which all the components of the metric tensor are independent of the time coordinate called the world time. The choice of a world time is not completely unique. Thus, if we add to time coordinate an arbitrary function of the space coordinates, the metric tensor will still not contain time.

In addition, of course, the world time can be multiplied by an arbitrary constant, i.e. the units for measuring it are arbitrary. Strictly speaking, only the field produced by a single body can be constant.

If the body producing the field is fixed (in the reference system in which the metric tensor do not depend on time, then both directions of time are equivalent.

However, for the field produced by a body to be constant, it is not necessary for the body to be at rest. Thus the field of an axially symmetric body rotating uniformly about its axis will also be constant. (Landau, et al., 1987)

Let us determine the force acting on a particle in a constant gravitational field. We follow Landau et al. (1987). For the components of Γ^i_{kl} which we need, we find the following expressions

$$\Gamma^\alpha_{00} = \frac{1}{2}h^{;\alpha} \tag{31}$$

$$\Gamma^{\alpha}_{0\beta} = \frac{h}{2}(g^{\alpha}_{;\beta} - g^{;\alpha}_{\beta}) - \frac{1}{2}g_{\beta}h^{;\alpha} \tag{32}$$

$$\Gamma^{\alpha}_{\beta\gamma} = \lambda^{\alpha}_{\beta\gamma} + \frac{h}{2}[g_{\beta}(g^{;\alpha}_{\gamma} - g^{\alpha}_{;\gamma}) + g_{\gamma}(g^{;\alpha}_{\beta} - g^{\alpha}_{;\beta})] - \frac{1}{2}g_{\beta}g_{\gamma}h^{;\alpha}. \tag{33}$$

In these expressions all the tensor operations (covariant differentiation, raising and lowering of indices) are carried out in the three-dimensional space with metric $\gamma_{\alpha\beta}$, on the three-dimensional vector g^{α} and the three-dimensional scalar h (Landau et al.; 1987) $\lambda^{\alpha}_{\beta\gamma}$ is the three-dimensional Christoffel symbol, constructed from the components of the tensor yae in just the same way as Γ^{li}_{k} , is constructed from the components of g_{ik} ; in the computations we use (84.9-12).

$$\gamma^{\alpha\beta} = -g^{\alpha\beta}, \tag{34}$$

$$-g = g_{00}\gamma, \tag{35}$$

$$g_{\alpha} = -g_{0\alpha}/g_{00}, \tag{36}$$

$$g^{\alpha} = -\gamma^{\alpha\beta}g_{\beta} = -g^{0\alpha}, \tag{37}$$

Substituting (30) in the equation of motion

$$\frac{du^{\alpha}}{ds} = \Gamma^{\alpha}_{00}(u^0)^2 - 2\Gamma^{\alpha}_{0\beta}u^0u^{\beta} - 2\Gamma^{\alpha}_{\beta\gamma}u^{\beta}u^{\gamma} \tag{38}$$

and using the expression

$$u^{\alpha} = \frac{v^{\alpha}}{c\sqrt{1-\frac{v^2}{c^2}}};\ u^0 = \frac{1}{\sqrt{h}\sqrt{1-\frac{v^2}{c^2}}} + \frac{g_{\alpha}v^{\alpha}}{c\sqrt{1-\frac{v^2}{c^2}}}. \tag{39}$$

for the components of the four-velocity, we find after some simple transformations:

$$\frac{d}{ds}\frac{v^{\alpha}}{c\sqrt{1-\frac{v^2}{c^2}}} = -\frac{h^{;\alpha}}{2h\left(1-\frac{v^2}{c^2}\right)} - \frac{\sqrt{h}(g^{\alpha}_{;\beta} - g^{;\alpha}_{\beta})v^{\beta}}{c\left(1-\frac{v^2}{c^2}\right)} - \frac{\lambda^{\alpha}_{\beta\gamma}v^{\beta}v^{\alpha}}{c^2\left(1-\frac{v^2}{c^2}\right)} \tag{40}$$

The force f acting on the particle is the derivative of its momentum p with respect to the (synchronized) proper time, as defined by the three-dimensional covariant differential:

$$f^{\alpha} = c\sqrt{1-\frac{v^2}{c^2}}\frac{Df^{\alpha}}{ds} = c\sqrt{1-\frac{v^2}{c^2}}\frac{d}{ds}\frac{mv^{\alpha}}{c\sqrt{1-\frac{v^2}{c^2}}} + \lambda^{\alpha}_{\beta\gamma}\frac{mv^{\beta\gamma}}{c\sqrt{1-\frac{v^2}{c^2}}}. \tag{41}$$

From (41) we therefore have (for convenience we lower the index a):

$$f_{\alpha} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\left\{-\frac{\partial}{\partial x^{\alpha}}\ln\sqrt{h} + \sqrt{h}\left(\frac{\partial g_{\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha}}{\partial x^{\beta}}\right)\frac{v^{\beta}}{c}\right\}. \tag{42}$$

Now the question arises, what is the the force acting on the point moving in the homogenous gravitational field. It was calculated in the 3-form as follows (Landau, et al., 1987):

$$\mathbf{f} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\left\{\mathbf{grad}\ \ln\sqrt{h} + \sqrt{h}\left[\frac{\mathbf{v}}{c} rot\ \mathbf{g}\right]\right\}, \tag{43}$$

with (Landau, et al., 1987).

$$h = 1 + \frac{2\varphi}{c^2}, \tag{44}$$

where φ s gravitational potential generating the acceleration $\mathbf{g}$. So, we see that it is not in the simple Newton form.

We note that if the body is at rest, then the force acting on it - the first term in eq. (43) - has a potential. For low velocities of motion the second term in eq. (43) has the form $mc\sqrt{h}\ \mathbf{v} \times rot\ \mathbf{g}$ analogous to the Coriolis force which would appear (in the absence of the field) in a coordinate system rotating with angular velocity

$$\Omega = \frac{c}{2}\sqrt{h}\ rot\ \mathbf{g}. \tag{45}$$

Let us remark that formula (43) is valid only for the specific form of gravity. In other words, only for the constant gravitational field. It is evident that there is possibility, in general, to transform the Einstein gravity equation to the Newtonian language with the Galileo-Newton-Hook gravity force (Inwood, 2002; Frova et al., 2006;). This mathematical problem was still not solved. It represents the new deal of theory of gravity and mathematical physics.

6 Discussion

We have considered here the derivation of the Einstein equations. Then, we have considered motion of a particle in the non-inertial system and motion of a particle in gravitational field. In case of the constant gravitational field we have transformed the Einstein equations to the Newtonian form of motion of a particle in gravity and derived the potential and the Coriolis force generating by the constant gravity (43).

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The transformation of Einstein gravity to the Newton form does not involve the principle of equivalence. The principle of equivalence states that it is impossible to distinguish between the action of gravity and non-gravity acceleration (Lyle, 2008).

The controversy between different opinions on the principle of equivalence can be easily solved by the physical definition of gravity and inertia. Namely: gravity is the specific form of matter, or, form of vacuum. And inertia is the interaction of the massive body with vacuum which is the physical medium. So, Gravity is form of matter and inertia is form of interaction.

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