Some questions related to the omega constant

Edgar Valdebenito

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Abstract: This note is about a specific value of Lambert's W function

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1. Omega Constant

The Omega constant $\Omega = W(1) = 0.567143290409\ldots$ is the unique real solution of the transcendental equation:

$$\Omega\, e^{\Omega} = 1 \tag{1}$$

where $W(x)$ is the Lambert W-function: $W(x)\, e^{W(x)} = x$.

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2. Integral Representation

$$\frac{1}{1+\Omega} = \int_{-\infty}^{+\infty} \frac{1}{(e^x - x)^2 + \pi^2}\, dx \tag{2}$$

Remark: (2) is due to V. Adamchik ([5],[8]).

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3. Some Relations

$$\frac{1}{1+\Omega} = \int_0^{\infty} \left(\frac{1}{(e^x - x)^2 + \pi^2} + \frac{1}{(e^{-x} + x)^2 + \pi^2} \right) dx \tag{3}$$

$$\frac{1}{1+\Omega} = \int_{-s}^{s} \frac{1}{(e^x - x)^2 + \pi^2}\, dx + \int_s^{\infty} \left(\frac{1}{(e^x - x)^2 + \pi^2} + \frac{1}{(e^{-x} + x)^2 + \pi^2} \right) dx \quad , \ s > 0 \tag{4}$$

For $a > 0.91021\ldots$ we have

$$\int_0^{\infty} \frac{1}{(e^x - x)^2 + \pi^2}\, dx = \int_0^{a} \frac{1}{(e^x - x)^2 + \pi^2}\, dx + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} \frac{(-1)^k\, 2^{n-k}\, \pi^{2k-2m}\, \Gamma(n-k+2m+1,\, (n+k+2)\, a)}{(n+k+2)^{n-k+2m+1}} \tag{5}$$

For $a > \pi$ we have

$$\int_0^\infty \frac{1}{(e^{-x}+x)^2+\pi^2}\,dx = \int_0^a \frac{1}{(e^{-x}+x)^2+\pi^2}\,dx + \frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{a}\right) + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k+1}\binom{n+1}{k}\binom{n-k+1}{m}(-1)^{n+1}\pi^{2k}\,2^{n-k-m+1}\,(n-k+m+1)^{n+k+m+2}\,\Gamma(-n-k-m-2,(n-k+m+1)\,a) \tag{6}$$

Remark: In (5) and (6) $\Gamma(x, y)$ is the incomplete Gamma function.

$$\frac{1}{1+\Omega} = \int_0^{e^{-1}} \frac{1}{x\left((\ln(x))^2+\pi^2\right)}\left(\frac{1}{1+W(0,-x)} - \frac{1}{1+W(-1,-x)}\right)dx \tag{7}$$

Remark: $W(0, x)$ and $W(-1, x)$ are the two real branches of $W(x)$.

$$\frac{1}{1+\Omega} = \int_1^\infty \frac{1}{x\left((x-\ln(x))^2+\pi^2\right)}\,dx + \int_1^\infty \frac{x}{\left((1+x\ln(x))^2+x^2\pi^2\right)}\,dx \tag{8}$$

$$\frac{1}{1+\Omega} = \int_0^\infty \frac{x}{\left((1+x\ln(x))^2+x^2\pi^2\right)}\,dx \tag{9}$$

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4. A Fixed Point

$$F(s) = \int_s^\infty\left(\frac{1}{(e^x-x)^2+\pi^2} + \frac{1}{(e^{-x}+x)^2+\pi^2}\right)dx \ , \ s>0 \tag{10}$$

$$s^* = F(s^*) \Longrightarrow s^* = 0.539720\ldots \tag{11}$$

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5. Iterations

Iteration 1:

$$s_1 = \frac{1}{2} \ , \ s_{n+1} = F(s_n) \Longrightarrow s_n \longrightarrow s^* = 0.539720\ldots \tag{12}$$

Iteration 2:

$$s_1 = \frac{1}{2} \ , \ s_{n+1} = \frac{G(s_n)}{1+G(s_n)}\,s_n + \frac{F(s_n)}{1+G(s_n)} \Longrightarrow s_n \longrightarrow s^* = 0.539720\ldots \tag{13}$$

where

$$G(x) = \frac{1}{(e^x-x)^2+\pi^2} + \frac{1}{(e^{-x}+x)^2+\pi^2} \tag{14}$$

6. A Series

$$\frac{1}{1+\Omega} = s^* + \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^{2n+2}} \left(\frac{2 (s^*)^{2n+1}}{2n+1} - \sum_{k=1}^{2n} \binom{2n}{k} \frac{\Gamma(2n-k+1, k s^*) - \Gamma(2n-k+1, -k s^*)}{k^{2n-k+1}} \right) \tag{15}$$

where $\Gamma(x, y)$ is the incomplete Gamma function.

7. Two Fixed Points

$$u = \int_u^{\infty} \frac{1}{(e^x - x)^2 + \pi^2} \, dx \Longrightarrow u = 0.140603 \ldots \tag{16}$$

$$v = \int_v^{\infty} \frac{1}{(e^{-x} + x)^2 + \pi^2} \, dx \Longrightarrow v = 0.443954 \ldots \tag{17}$$

$$\frac{1}{1+\Omega} = u + v + \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^{2n+2}} \left(\frac{u^{2n+1} + v^{2n+1}}{2n+1} + \sum_{k=1}^{2n} \binom{2n}{k} \frac{\Gamma(2n-k+1, -k u) - \Gamma(2n-k+1, k v)}{k^{2n-k+1}} \right) \tag{18}$$

where $\Gamma(x, y)$ is the incomplete Gamma function.

8. Two Identities

For $a > 0$, $b > 0$ we have

$$\begin{aligned} \frac{1}{1+\Omega} = {} & \frac{a}{(e^a - a)^2 + \pi^2} + \int_a^{\infty} \frac{1}{(e^x - x)^2 + \pi^2} \, dx + \int_{((e^a - a)^2 + \pi^2)^{-1}}^{(1+\pi^2)^{-1}} \left(-\sqrt{\frac{1}{x} - \pi^2} - W\left(-1, -e^{-\sqrt{\frac{1}{x} - \pi^2}}\right) \right) dx + \\ & \frac{b}{\left(e^{-b} + b\right)^2 + \pi^2} + \int_b^{\infty} \frac{1}{(e^{-x} + x)^2 + \pi^2} \, dx + \int_{\left((e^{-b} + b)^2 + \pi^2\right)^{-1}}^{(1+\pi^2)^{-1}} \left(\sqrt{\frac{1}{x} - \pi^2} + W\left(0, -e^{-\sqrt{\frac{1}{x} - \pi^2}}\right) \right) dx \end{aligned} \tag{19}$$

For $a > 0$, $b > 0$, $e^a - a = e^{-b} + b$, $c = \frac{1}{(e^a - a)^2 + \pi^2}$ we have

$$\begin{aligned} \frac{1}{1+\Omega} = {} & (a + b)\, c + \int_a^{\infty} \frac{1}{(e^x - x)^2 + \pi^2} \, dx + \\ & \int_b^{\infty} \frac{1}{(e^{-x} + x)^2 + \pi^2} \, dx + \int_c^{(1+\pi^2)^{-1}} \left(W\left(0, -e^{-\sqrt{\frac{1}{x} - \pi^2}}\right) - W\left(-1, -e^{-\sqrt{\frac{1}{x} - \pi^2}}\right) \right) dx \end{aligned} \tag{20}$$

Remark: $W(0, x)$ and $W(-1, x)$ are the two real branches of $W(x)$.

9. Final Formulas

$$\frac{1}{1+\Omega} = r + \int_0^{r} \frac{1}{x\left((x - \ln(x))^2 + \pi^2\right)} \, dx \tag{21}$$

where $r = 0.269121\ldots$

$$r_1 = \frac{1}{4}\ ,\ r_{n+1} = H(r_n) \Longrightarrow r_n \longrightarrow r \tag{22}$$

$$H(y) = \int_y^\infty \frac{1}{x\left((x - \ln(x))^2 + \pi^2\right)}\, dx \tag{23}$$

For $a > 0$, $a \longrightarrow \infty$ we have

$$\frac{1}{1+\Omega} \sim \int_{-a}^{a} \frac{1}{(e^x - x)^2 + \pi^2}\, dx + \frac{e^{-2a}}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{\pi}{a}\right) \tag{24}$$

$$\frac{1}{1+\Omega} \sim \int_{-a}^{a} \frac{1}{(e^x - x)^2 + \pi^2}\, dx + \frac{1}{2\pi^2} \ln\left(1 + \pi^2 e^{-2a}\right) + \frac{1}{\pi} \tan^{-1}\left(\frac{\pi}{a}\right) \tag{25}$$

10. References

1. Corless, R.M., Gonnet, G.H., Hare, D.E.G., and Jeffrey, D.J.: Lambert's W function in Maple, The Maple Technical Newsletter, 9, 12-22, 1993.
2. Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J, Knuth, D.E.: On the Lambert W- functions. Adv. Comput. Math. 5, 329-359, 1996.
3. Lambert, J.H.: Observationes variae in mathesin puram, Acta Helvetica, physico-mathematico-anatomico-botanico-medica, 3, Basel, 1758, 128-168.
4. Lambert, J.H.: Observations Analytiques, in Nouveaux mémoires de l'Académie royale des sciences et belles-lettres, Berlin, 1772, vol. 1, for 1770.
5. Moll, V.H.: Some Questions in the Evaluation of Definite Integrals. MAA Short Course, San Antonio, TX. Jan. 2006.
6. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST Handbook of Mathematical Functions. Cambridge University Press, 2010.
7. Sloane, N.J.A.: Sequences A030178 and A115287. in "The On-Line Encyclopedia of Integer Sequences".
8. Weisstein, E.W.: Omega Constant. From MathWorld-A Wolfram Web Resource.