Bounds for Optimal Golomb Rulers Amendment

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Abstract

In this paper, we improve the lower-bounds for optimal Golomb rulers.

Summary

In my previous paper[1], I presented a new approach on how to determine a good upper-bound (UB) for the OGRs using the previous OGR. I failed to use that same approach for the lower-bound (LB).

OGR Results

The following table shows the OGRs up to order 28 and their lengths and marks.

Table 1 – OGRs up to O(28)

O(n)	L	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	0	0																											
2	1	0	1																										
3	3	0	1	3																									
4	6	0	1	4	6																								
5	11	0	1	4	9	11																							
5	11	0	2	7	8	11																							
6	17	0	1	4	10	12	17																						
6	17	0	1	4	10	15	17																						
6	17	0	1	8	11	13	17																						
6	17	0	1	8	12	14	17																						
7	25	0	1	4	10	18	23	25																					
7	25	0	1	7	11	20	23	25																					
7	25	0	1	11	16	19	23	25																					
7	25	0	2	3	10	16	21	25																					
7	25	0	2	7	13	21	22	25																					
8	34	0	1	4	9	15	22	32	34																				
9	44	0	1	5	12	25	27	35	41	44																			
10	55	0	1	6	10	23	26	34	41	53	55																		
11	72	0	1	4	13	28	33	47	54	64	70	72																	
11	72	0	1	9	19	24	31	52	56	58	69	72																	
12	85	0	2	6	24	29	40	43	55	68	75	76	85																
13	106	0	2	5	25	37	43	59	70	85	89	98	99	106															
14	127	0	4	6	20	35	52	59	77	78	86	89	99	122	127														
15	151	0	4	20	30	57	59	62	76	100	111	123	136	144	145	151													
16	177	0	1	4	11	26	32	56	68	76	115	117	134	150	163	168	177												
17	199	0	5	7	17	52	56	67	80	81	100	122	138	159	165	168	191	199											
18	216	0	2	10	22	53	56	82	83	89	98	130	148	153	167	188	192	205	216										
19	246	0	1	6	25	32	72	100	108	120	130	153	169	187	190	204	231	233	242	246									
20	283	0	1	8	11	68	77	94	116	121	156	158	179	194	208	212	228	240	253	259	283								
21	333	0	2	24	56	77	82	83	95	129	144	179	186	195	255	265	285	293	296	310	329	333							
22	356	0	1	9	14	43	70	106	122	124	128	159	179	204	223	253	263	270	291	330	341	353	356						
23	372	0	3	7	17	61	66	91	99	114	159	171	199	200	226	235	246	277	316	329	348	350	366	372					
24	425	0	9	33	37	38	97	122	129	140	142	152	191	205	208	252	278	286	326	332	353	368	384	403	425				
25	480	0	12	29	39	72	91	146	157	160	161	166	191	207	214	258	290	316	354	372	394	396	431	459	467	480			
26	492	0	1	33	83	104	110	124	163	185	200	203	249	251	258	314	318	343	356	386	430	440	456	464	475	487	492		
27	553	0	3	15	41	66	95	97	106	142	152	220	221	225	242	295	330	338	354	382	388	402	415	486	504	523	546	553	
28	585	0	3	15	41	66	95	97	106	142	152	220	221	225	242	295	330	338	354	382	388	402	415	486	504	523	546	553	585

Lower-Bounds

The previous Trivial LB remains that, while my previous Smart LB is just the Fast one (in keeping with the previous UB summary).

(New and Improved) Smart LB

If we look at the delta distances in each of the OGRs, we can rewrite Table 1 above as follows:

Table 2 – OGRs (Δ) up to O(28)

O(n)	L	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	SUN
1	0	0																											1
2	1	1																											2
3	3	1	2																										3
4	6	1	3	2																									4
5	11	1	3	5	2																								4
5	11	2	5	1	3																								4
6	17	1	3	6	2	5																							4
6	17	1	3	6	5	2																							4
6	17	1	7	3	2	4																							5
6	17	1	7	4	2	3																							5
7	25	1	3	6	8	5	2																						4
7	25	1	6	4	9	3	2																						5
7	25	1	10	5	3	4	2																						6
7	25	2	1	7	6	5	4																						3
7	25	2	5	6	8	1	3																						4
8	34	1	3	5	6	7	10	2																					4
9	44	1	4	7	13	2	8	6	3																				5
10	55	1	5	4	13	3	8	7	12	2																			6
11	72	1	3	9	15	5	14	7	10	6	2																		4
11	72	1	8	10	5	7	21	4	2	11	3																		6
12	85	2	4	18	5	11	3	12	13	7	1	9																	6
13	106	2	3	20	12	6	16	11	15	4	9	1	7																5
14	127	4	2	14	15	17	7	18	1	8	3	10	23	5															6
15	151	4	16	10	27	2	3	14	24	11	12	13	8	1	6														4
16	177	1	3	7	15	6	24	12	8	39	2	17	16	13	5	9													4
17	199	5	2	10	35	4	11	13	1	19	22	16	21	6	3	23	8												7
18	216	2	8	12	31	3	26	1	6	9	32	18	5	14	21	4	13	11											7
19	246	1	5	19	7	40	28	8	12	10	23	16	18	3	14	27	2	9	4										6
20	283	1	7	3	57	9	17	22	5	35	2	21	15	14	4	16	12	13	6	24									8
21	333	2	22	32	21	5	1	12	34	15	35	7	9	60	10	20	8	3	14	19	4								6
22	356	1	8	5	29	27	36	16	2	4	31	20	25	19	30	10	7	21	39	11	12	3							6
23	372	3	4	10	44	5	25	8	15	45	12	28	1	26	9	11	31	39	13	19	2	16	6						7
24	425	9	24	4	1	59	25	7	11	2	10	39	14	3	44	26	8	40	6	21	15	16	19	22					5
25	480	12	17	10	33	19	55	11	3	1	5	25	16	7	44	32	26	38	18	22	2	35	28	8	13				4
26	492	1	32	50	21	6	14	39	22	15	3	46	2	7	56	4	25	13	30	44	10	16	8	11	12	5			9
27	553	3	12	26	25	29	2	9	36	10	68	1	4	17	53	35	8	16	28	6	14	13	71	18	19	23	7		5
28	585	3	12	26	25	29	2	9	36	10	68	1	4	17	53	35	8	16	28	6	14	13	71	18	19	23	7	32	5

Similarly, as done for the Smart UB, we can check for the smallest unused number (SUN) not in the delta list. If we select the smallest number to add at the end, the next OGR must be greater than or equal to that combined length.

Proof

Assume the $L_{n+1} < L_n + k$, where k = min(x) s.t. {x | x > 0 and x ∉ O(n)}. Thus, the closed integer set {1, … k, … k*}, where k* ≥ k, gives us {1, …, k*} ∈ O(n) ∪ k. Then, O(n+1) can be rotated until the distance k* or less is at an end, and that end can be removed to create a smaller O(n) length. Thus, a contradiction. In addition, since O(n) is optimal, a smaller length than that would give two deltas that are equal. Also, a contradiction.

Using O(5) as an example, the delta length 4 is not in O(4). Thus, $L_5 \geq L_4$ + SUN(4) = 6 + 4 = 10.

For unique, sets or sets with the same lowest missing number, the minimum LB is obvious.

Looking at n = 6, 7, and 11, we have multiple SUNs for each group of n.

Table 3 – Non-Unique OGRs (Δ)

O(n)	L	0	1	2	3	4	5	6	7	8	9	SUN
5	11	1	3	5	2							4
5	11	2	5	1	3							4
6	17	1	3	6	2	5						4
6	17	1	3	6	5	2						4
6	17	1	7	3	2	4						5
6	17	1	7	4	2	3						5
7	25	1	3	6	8	5	2					4
7	25	1	6	4	9	3	2					5
7	25	1	10	5	3	4	2					6
7	25	2	1	7	6	5	4					3
7	25	2	5	6	8	1	3					4
11	72	1	3	9	15	5	14	7	10	6	2	4
11	72	1	8	10	5	7	21	4	2	11	3	6

However, by the same logic, L_{n+1} must be greater than L_n with the largest SUN otherwise we also arrive at a contradiction.

Thus, the table from my previous paper can be updated as follows:

Table 4 – Bounds for OGRs up to O(29)

O(n)	LB	L	UB
1	0	0	0
2	1	1	1
3	3	3	3
4	6	6	7
5	10	11	13
6	15	17	23
7	22	25	31
8	31	34	45
9	38	44	58
10	49	55	72
11	61	72	91
12	78	85	98
13	91	106	133
14	111	127	177
15	133	151	183
16	155	177	249
17	181	199	303
18	206	216	288
19	223	246	307
20	252	283	296
21	291	333	381
22	339	356	418
23	362	372	496
24	379	425	491
25	430	480	553
26	484	492	570
27	501	553	651
28	558	585	585
29	590	TBD	757

The two highlights in yellow are the only two for this list where the upper-bound is closer to the OGR length than the lower-bound.

References

[1] Abdul-Baki, Bassam S. "Bounds for Optimal Golomb Rulers" (PDF).