From Algebraic Extension to Physical Law: Multiplication, Integration, and the Emergence of Variational Field Reality

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(Dated: January 22, 2026)

Abstract. *We propose a foundational route from elementary mathematical operations to the structural form of physical law. The guiding thesis is that multiplication is the primitive operation that generates geometric extension (e.g., area via bilinear composition), integration is the continuous accumulation of such local extensions into global quantities, and differentiation (or functional variation) is the dual operation that extracts local constraints from global accumulations. From these principles, we show how any consistent description of "physical reality" must be formulated in terms of local densities defined over a continuous geometric support, whose global content is obtained by integration and whose dynamics follows from variational (action) stationarity.*

Within this operational framework, quadratic field terms arise naturally as the simplest scalar invariants built from local degrees of freedom, while source couplings appear as bilinear products between generalized currents and the underlying deformation variables. Furthermore, we show that quantum entanglement is not a dynamical anomaly but a structural inevitability: additive accumulation acting on states represented in a multiplicative (spectral) basis generically produces global correlations that resist local factorization. This reframes Bell-type violations as a failure of structural independence rather than a signal of superluminal causal influence, thereby preserving relativistic causality at the level of dynamical propagation.

Crucially, beyond the contractive modes commonly associated with forces and curvature, the same logic compels expansive degrees of freedom: an entropic sector characterized by an intensive–extensive product structure (temperature-like $\times$ entropy-like) contributing intrinsically to the global action. This viewpoint yields a general blueprint for interpreting electromagnetic, gravitational, and entropic responses as projected modes of a common underlying field structure, and it clarifies why concrete realizations of such a blueprint—including quantum-elastic and gravito-entropic field models—arise as minimal, structurally stable completions rather than independent hypotheses.

"Entia non sunt multiplicanda praeter necessitatem"
— Ockham's Razor

"Padre, Señor del cielo y de la tierra, te doy gracias porque has ocultado todo esto a los sabios y entendidos y se lo has revelado a los que son como niños."
— Matthew 11:25

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CONTENTS

I. INTRODUCTION

Modern theoretical physics is remarkably unified in its formal architecture. Classical field theory, quantum field theory, and general relativity all rely on a small set of shared mathematical ingredients: fields defined over a continuous manifold, local densities, integrals defining global quantities, and differential equations governing dynamics [35, 49, 59]. Despite their empirical success, these features are typically taken as postulates, or justified retrospectively by experimental adequacy rather than derived from deeper mathematical necessity.

This motivates a foundational question that is seldom addressed explicitly: *why does physical law take this form at all?* Why are fundamental quantities expressed as integrals of local densities? Why do interactions repeatedly appear through quadratic or bilinear combinations of fields? And why do the equations governing physical systems arise from variational principles rather than from arbitrary functional relations? These questions echo long-standing reflections on the structural role of mathematics in physical law [4, 62, 63].

In this work, we argue that these features are not contingent modeling choices, but inevitable consequences of the basic operations by which mathematical structures generate *extension*, *accumulation*, and *locality.* Our approach is deliberately pre-physical: instead of starting from spacetime, particles, or forces, we begin with the algebraic and analytical operations that make any notion of "physical quantity" meaningful, in the sense that they permit invariant global constructions from local data [1, 33].

The guiding thesis of the paper is simple but far-reaching:

> Multiplication generates geometric extension; integration accumulates such extensions into global quantities; differentiation (or functional variation) extracts local laws from global quantities.

A key point is that by *extension* we mean an operational property: the ability to construct an additive, coordinate-invariant global quantity from local data via a measure. Multiplication is then not merely arithmetic, but the algebraic mechanism by which dimensional extension—area, volume, norm, orientation-sensitive invariants (e.g., determinants), or action—is produced through bilinear or multilinear composition. Such bilinear constructions underlie the scalar invariants appearing throughout physics [56, 59]. Integration appears as the unique operation that accumulates these local products into global observables, while differentiation or functional variation recovers local structure by probing how such accumulations respond to infinitesimal changes, as formalized in the variational calculus [34, 48].

Together, these operations form a closed logical cycle underlying the architecture of physical theories.

At a deeper structural level, the distinction between conservative multiplicative extension and generative additive accumulation mirrors the orthogonality between multiplication and addition developed in arithmetic and spectral settings in [42, 43, 46].

Once these principles are adopted, several core features of physical law follow with little freedom. First, physically meaningful global quantities must be expressed as *densities* defined over a continuous geometric support, since only densities can be multiplied by an appropriate measure and integrated into coordinate-invariant observables [33, 52]. Second, the primacy of integration implies that *action*—rather than force, energy, or curvature taken in isolation—is the natural universal global quantity, while other conserved quantities arise as derived objects tied to symmetry through Noether-type correspondences [11, 48]. Third, locality of physical law necessitates that dynamics be obtained through differentiation or functional variation of the action, leading inevitably to differential field equations and conservation laws.

A particularly important consequence is the natural emergence of quadratic and bilinear structures in physical Lagrangians. The simplest scalar invariants constructible from local degrees of freedom arise as products of fields with themselves or with conjugate variables, explaining the ubiquity of quadratic kinetic terms and bilinear source couplings across otherwise distinct physical theories [49, 59]. In this sense, these structures are mathematical necessities rather than model-dependent assumptions.

Beyond these well-known features, the present approach leads to a more subtle but crucial conclusion. While multiplication generates geometric extension in a conservative and structure-preserving manner, additive accumulation introduces genuinely new degrees of freedom through mixing and non-factorization. In this sense, the same operational framework that yields contractive modes associated with forces and curvature also compels the existence of expansive, entropic modes associated not with multiplication but with addition. Additive superposition generically destroys factorized structure, producing innovation and irreducible mixing—an effect that, in physical terms, manifests as entropy production [31, 37, 53]. Accordingly, the entropic sector is not an auxiliary statistical overlay, but a structural consequence of additive accumulation itself. It arises whenever local contributions are summed in a way that generates new global configurations not reducible to their constituents, mirroring the role of addition as a generative and entropy-producing operation in arithmetic and spectral frameworks [42, 46].

The purpose of this paper is therefore not to propose a specific physical model, but to establish a mathematical blueprint from which entire classes of physical theories may be understood as realizations. Within this blueprint, electromagnetic, gravitational, and entropic responses can be interpreted as projected modes of a common underlying field structure defined by extension, accumulation, and variation, in a spirit compatible with thermodynamic and informational perspectives on spacetime dynamics [6, 30]. Concrete theories—including quantum-elastic or gravito-entropic field models—then appear not as independent hypotheses, but as minimal implementations of these deeper mathematical principles.

The paper is organized as follows. In Section II, we analyze multiplication as the generator of geometric extension and clarify its relation to dimensionality and invariant construction. Section III develops integration as continuous accumulation and establishes the primacy of action as the fundamental global quantity. In Section IV, we show how locality and dynamical laws arise necessarily through differentiation and variational principles. Section V addresses quantum entanglement from first principles within the present framework, defining it as structural non-factorization under admissible evaluations and identifying additive accumulation as the generic mixing mechanism behind Bell-type correlations without invoking superluminal dynamics [15, 27]. Section VI introduces the entropic sector as an inevitable expansive contribution implied by the additive logic. Finally, in Section X, we discuss implications and outline how specific physical theories may be understood as realizations of the general structure derived here.

II. MULTIPLICATION AS THE GENERATOR OF GEOMETRIC EXTENSION

At the most elementary level, any candidate for a "physical quantity" must possess *extension*: length, area, volume, norm, or accumulated magnitude. Before introducing dynamics or interaction, it is therefore necessary to identify which mathematical operations can generate extension in a consistent and composable manner. In this section, we argue that multiplication plays a distinguished role in this regard, whereas addition alone is structurally insufficient.

A. Addition versus multiplication

Addition combines quantities of the same type by superposition. Given two quantities a and b of identical dimension, their sum $a + b$ preserves dimensionality but does not generate new geometric content. In particular, addition does not create new forms of extension: the sum of two lengths remains a length, and the sum of two energies remains an energy. This reflects the fact that addition is an internal operation on an additive structure, such as a vector space or module.

Multiplication, by contrast, generates new dimensional structures. The simplest geometric example is canonical: the product of two lengths yields an area, and the product of an area with a length yields a volume. More generally, multiplication enables the construction of quantities whose dimensional character reflects a genuine *extension* rather than a mere aggregation. This distinction is fundamental in dimensional analysis and underlies the emergence of geometric measures and scalar invariants in physics [52, 62].

This distinction is not merely pedagogical. It reflects a basic algebraic fact: addition preserves type within a fixed additive structure, while multiplication is the operation by which multilinear and tensorial structures emerge. Only through multiplication can one build objects that encode geometric extension and, in oriented settings, orientation-sensitive invariants such as determinants. In differential geometry, oriented measures arise from alternating multilinear products (e.g. wedge products), whose scalar invariants reduce to determinants in coordinates [1, 33].

This structural split also admits an arithmetic analogue: addition acts as superposition within a fixed ambient structure, while multiplication reorganizes structure by composition—a distinction explored in our orthogonality and spectral characterizations of integer arithmetic [42, 43].[1]

B. Geometric realization of the product

The interpretation of bilinear multiplication as an area provides a canonical realization of this idea. Given two real numbers a and b, their product ab can be represented as the area of a rectangle with sides a and b. This realization is bilinear and scalable, reflecting the defining algebraic properties of the product. Furthermore, in higher-dimensional settings, products appear naturally as inner products, norms, and quadratic forms. Given a local vector $\mathbf{v}$, the scalar $\mathbf{v} \cdot \mathbf{v}$ defines a squared norm, i.e. an intrinsic local measure of extension associated with $\mathbf{v}$. Such constructions are unavoidable whenever one seeks scalar quantities derived

[1] In the arithmetic setting, this distinction is rigorous: multiplication acts diagonally on the prime spectral basis (preserving support), whereas addition acts as a generative, mixing operator that creates new spectral content ("additive innovation"). This serves as a formal prototype for the physical distinction between conservative extension and entropic generation.

from local degrees of freedom and suitable for coordinate-invariant accumulation [59].

Thus, from this viewpoint, multiplication is the minimal algebraic mechanism that transforms local components into extended scalar quantities. It is therefore the primitive operation by which geometric content becomes available for further accumulation.

C. Dimensional extension and density formation

A central consequence is that physically meaningful global quantities are typically expressed as products of a *density* with a geometric measure. For example, an area element dA arises (in local coordinates) from products of infinitesimal lengths, while a volume element dV arises from products of three. In spacetime theories, the measure d^4x is similarly constructed as a product of infinitesimal coordinate differentials.

Local physical quantities—such as energy density or charge density—acquire global meaning only when multiplied by an appropriate measure and accumulated. Densities, by themselves, are not global observables; rather, they are the local factors that, once multiplied by a measure and integrated, generate extended physical magnitudes. The distinction between a scalar field and a density is precisely the Jacobian weight required for $\mathcal{L}\, d^4x$ to be coordinate-invariant under changes of variables [33, 52].

This already hints at the primacy of action. An action element takes the form

$$dS \;=\; \mathcal{L}\, d^4x, \tag{II.1}$$

which is explicitly a product between a local density $\mathcal{L}$ and a geometric measure. Multiplication is essential here: without it, no notion of accumulated global content can be coherently defined [34, 48].

D. Quadratic structures as minimal scalar constructions

Once multiplication is recognized as the generator of extension, the ubiquity of quadratic structures becomes transparent. The simplest scalar quantity constructible from a local field ϕ is ϕ^2 (or $\phi^*\phi$ in complex settings). Likewise, bilinear products between distinct fields naturally encode interaction or coupling terms.

These quadratic and bilinear forms are not arbitrary modeling choices but minimal algebraic constructions compatible with locality, dimensional consistency, and scalar invariance. Any attempt to construct scalar densities from local degrees of freedom without multiplicative composition fails to produce extended quantities suitable for accumulation. Quadratic terms are also the first nontrivial local invariants in a small-field (or small-deformation) expansion, and they typically provide the minimal stable kinetic or elastic response required for a well-posed variational principle [49, 56].

This applies across classical and quantum field theories, as well as gravitational and gauge frameworks [65]. In all cases, the local content of the theory is encoded in products of fields and their derivatives, reflecting the foundational role of multiplication in generating physical extension.

E. Multiplication as a pre-dynamical principle

Importantly, the role of multiplication identified here precedes any notion of time evolution or dynamics. It is a pre-dynamical principle governing how local quantities acquire geometric and physical meaning. Dynamics enters only after such extended quantities are accumulated and subjected to differentiation or variation, which will be analyzed in subsequent sections.

In summary, multiplication is the algebraic operation that generates extension, enabling the construction of measures, densities, and scalar invariants suitable for accumulation. Without multiplication, neither geometric extension nor global physical quantities can be coherently defined. In the next section, we show how integration arises as the natural operation that accumulates these locally generated extensions into the global quantity known as action.

III. INTEGRATION AS CONTINUOUS ACCUMULATION AND THE PRIMACY OF ACTION

Having established multiplication as the operation that generates local geometric extension, we now turn to the question of *accumulation*. Physical reality is not composed of isolated local elements; rather, it is characterized by global quantities obtained through the accumulation of local contributions over extended domains. The mathematical operation uniquely suited to perform such accumulation in a consistent, coordinate-invariant, and dimensionally meaningful way is integration [1, 52].

A. From discrete sums to continuous accumulation

At an intuitive level, integration generalizes summation to the continuum. A discrete sum combines finitely many contributions, each treated as an indivisible unit. By contrast, integration arises as the limit of sums over *infinitesimal* elements, each contributing a local product of a density and a measure.

Formally, a definite integral can be expressed as a Riemann limit,

$$\int_\Omega f(x)\, dx \;=\; \lim_{\Delta x \to 0} \sum_i f(x_i)\, \Delta x, \tag{III.1}$$

where each term in the sum is explicitly a product. More generally, what is accumulated is not merely a function against a coordinate increment, but a *density* against a *geometric measure*. In geometric language, global quantities are obtained by integrating an appropriate density ρ against a volume measure μ on a domain Ω:

$$Q[\rho] \;=\; \int_\Omega \rho\, \mu, \tag{III.2}$$

where μ may be understood as a measure or, in differential-geometric terms, a volume form on Ω [33]. In relativistic settings, for instance, the invariant spacetime measure takes the form $\mu = \sqrt{-g}\, d^4x$, making explicit the distinction between coordinate increments and geometric volume [59].

This structure highlights a crucial point: integration introduces no fundamentally new algebra beyond addition and multiplication; it is the organization of an infinite accumulation of locally generated products into a coordinate-invariant global quantity. In this sense, integration is best understood as *continuous accumulation*—the unique operation capable of coherently summing local extensions across a continuous domain while preserving dimensional consistency and geometric meaning [34].

B. Extensive quantities and global meaning

A defining feature of integrated quantities is their *extensivity*. Area scales with the size of the domain, volume scales with spatial extension, and total charge or energy scales with the region over which it is accumulated. This extensivity is not an empirical coincidence but a direct consequence of the integral structure. Only quantities that arise as integrals of densities possess unambiguous global meaning. A density by itself is incomplete: it acquires physical significance only when paired with a measure and accumulated. Put differently, the requirement of coordinate-invariant global observables already constrains admissible physical quantities to be of the form "density × measure" integrated over an extended support [1, 52].

C. Action as the primary physical quantity

Among extensive quantities in physics, action occupies a distinguished position. It is defined as the integral of a local Lagrangian density over spacetime,

$$S[\phi] \;=\; \int \mathcal{L}(\phi, \partial\phi, \ldots)\, d^4x\,, \tag{III.3}$$

or, more invariantly, as the integral of a scalar density over the spacetime volume form.

Within the present framework, the primacy of action is no longer mysterious: it is the most general coordinate-invariant global quantity obtainable by accumulating local extensions generated through multiplication. Crucially, the action is also the natural functional whose infinitesimal variation yields *local* constraints. In this view, variational stationarity is not an additional physical postulate but the unique mathematical mechanism by which a global accumulation gives rise to local laws [34, 48].

Energy, momentum, and other conserved quantities may be derived from the action, but they do not replace it as the foundational object. They typically arise as derived constructs associated with symmetry (via Noether-type correspondences) and with particular decompositions of spacetime, whereas the action remains the universal global scalar functional from which local dynamics are extracted [11].

D. Accumulation, scale, and universality

Because integration accumulates local contributions over a domain, the resulting global quantity is inherently sensitive to scale. This sensitivity underlies the appearance of ultraviolet and infrared regimes, effective descriptions, and the need to compare contributions arising from different resolutions of the same underlying system.

From the present viewpoint, such scale dependence is not an anomaly but a natural consequence of accumulation: an integral necessarily mixes information about local structure (encoded in the integrand) with global extent (encoded in the domain and measure). The action therefore becomes the natural arena in which scale-dependent effects and effective descriptions must appear, as extensively exploited in effective field theory and renormalization frameworks [49].

Furthermore, the universality of action-based formulations across classical mechanics, field theory, and gravitation reflects the universality of integration as an operation. Any theory that aims to describe extended physical reality must specify how local contributions are accumulated into global quantities, and integration provides the canonical and consistent mechanism for doing so [63].

Remark III.1. *Within the operational cycle developed here, temporal ordering appears naturally as the bookkeeping of accumulation: a global quantity is built by successive integration of admissible local contributions. In this sense, chronology is secondary to the algebraic operation of accumulation, while causality is enforced by the finite propagation speed c governing the physical transmission of local deformations in the substrate. Thus, the "flow" associated with time may be interpreted as the sequential updating of the state under accumulation, without reducing spacetime to a mere parameter or denying relativistic covariance.*

E. Integration as the bridge between local and global structure

Integration plays a dual conceptual role. On the one hand, it aggregates local geometric extensions into a single global scalar quantity. On the other hand, it establishes the bridge between local structure and global constraints: an integral is global, yet remains sensitive to infinitesimal variations of its integrand.

This dual role is essential. Without integration, local products would remain isolated and physically unanchored; without sensitivity to variation, global quantities would fail to constrain local behavior. Integration thus prepares the ground for the emergence of dynamics, which arises precisely from probing how the accumulated quantity responds to infinitesimal changes.

In summary, integration is the operation that converts locally generated extensions into global physical quantities. It endows physical theories with extensivity, scale sensitivity, and universality, and it elevates action to the role of primary global quantity. In the next section, we show how locality and dynamics emerge when this accumulated quantity is subjected to differentiation or functional variation.

IV. LOCALITY, DIFFERENTIATION, AND THE EMERGENCE OF DYNAMICS

The previous sections established multiplication as the generator of local extension and integration as the operation that accumulates such extensions into global quantities. We now complete the logical cycle by addressing the question of locality: *how do local physical laws arise from globally accumulated quantities?* The answer lies in differentiation, and more precisely in functional variation.

A. Locality as a structural requirement

Physical laws exhibit locality: the behavior of a system at a given point depends only on the configuration of fields and sources in an infinitesimal neighborhood of that point. This property is not optional. Without locality, predictive physical description would be impossible, as global quantities would fail to constrain local behavior in a controllable manner.

Mathematically, locality requires an operation capable of extracting infinitesimal information from an accumulated quantity. Differentiation is uniquely suited to this task. It probes how a quantity changes under infinitesimal variations, thereby revealing its local structure. No other operation provides a consistent mechanism to relate global quantities to pointwise constraints.

Within the present framework, locality is therefore not postulated but demanded by consistency: if global quantities are defined through integration, then the only way to recover local laws is by differentiating those quantities. This observation already singles out variational calculus as the natural formal language of physical law [1, 34].

B. Variation of accumulated quantities

Let the action be defined as

$$S[\phi] = \int \mathcal{L}(\phi, \partial\phi)\, d^4x, \tag{IV.1}$$

where ϕ denotes a generic field. A variation of the field, $\phi \to \phi + \delta\phi$, induces a variation of the action,

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi}\, \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \delta(\partial_\mu \phi) \right) d^4x. \tag{IV.2}$$

Using $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$ and integrating by parts, one obtains

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi \, d^4x + \text{boundary terms.} \tag{IV.3}$$

Requiring stationarity of the accumulated quantity under arbitrary local variations $\delta\phi$ with fixed boundary data yields the Euler–Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{IV.4}$$

This derivation contains no physical assumptions beyond those already established. Once a global quantity is defined by integration, and once locality is required, differentiation and integration by parts force the Euler–Lagrange structure. The equations of motion are therefore not an independent postulate, but the inevitable mathematical consequence of accumulated extension subjected to infinitesimal variation [22, 34].

C. Dynamics as response to infinitesimal variation

The resulting equations of motion express the response of the accumulated quantity to infinitesimal local deformations. Dynamics is therefore not an independent ingredient, but the manifestation of how extension and accumulation constrain permissible configurations. Importantly, this applies whether the underlying fields represent matter, geometry, or generalized deformation variables. The emergence of dynamics is entirely independent of the specific physical interpretation of the fields; it follows solely from the requirement of local stationarity of a globally accumulated quantity.

Remark IV.1. *From this perspective, time evolution is not fundamental but relational: it describes how configurations must change so as to maintain stationarity of the accumulated extension. This interpretation applies equally to classical mechanics, relativistic field theory, and gravitational dynamics [59].*

D. Conservation laws and symmetries

Once dynamics are expressed in variational form, conservation laws follow automatically. Continuous symmetries of the action correspond to conserved currents, as formalized by Noether's theorem [48]. Within the present framework, this result acquires a particularly transparent interpretation.

Symmetries express redundancy in the description of extension and accumulation. If the accumulated quantity is invariant under a continuous transformation, then differentiation with respect to the corresponding parameter must vanish, yielding a conserved local current. Conservation laws are thus direct consequences of invariance properties of the accumulated extension, rather than independent axioms of motion.

This observation reinforces the central thesis of the paper: fundamental physical structures arise from the interplay between multiplication, integration, and differentiation.

E. Boundary terms and the role of geometry

Differentiation of integrated quantities naturally generates boundary terms. These terms encode the relationship between local dynamics and global geometry, as formalized by generalized versions of Stokes' theorem [9, 55].

In physical terms, boundary contributions describe how local variations propagate to the edges of a domain. Their appearance is unavoidable and highlights the intimate connection between local laws and global structure. In many contexts, boundary terms encode conserved charges and fluxes, further emphasizing the geometric character of physical quantities.

The necessity of boundary terms also underscores the requirement of a continuous geometric substrate. Differentiation and integration presuppose a differentiable structure, which in turn implies the existence of an underlying manifold. As in previous sections, this geometric backdrop is not postulated but inferred as a requirement for the operations that define physical law.

F. Closure of the operational cycle

We may now summarize the operational cycle underlying physical law:

- Multiplication generates local geometric extension.
- Integration accumulates these extensions into global quantities.
- Differentiation or variation extracts local constraints from the accumulated quantity.

This cycle is closed and self-consistent. No additional mathematical operations are required to construct a predictive physical theory. Any framework that respects extension, accumulation, and locality must therefore adopt a variational field-theoretic structure.

G. Algebraic saturation and the necessity of nonlinearity

The previous sections established multiplication as the primitive operation generating extension, integration as the mechanism of accumulation, and variation as the source of local laws. However, these principles alone do not yet explain why physical reality exhibits localized, finite structures rather than unlimited dispersive extension.

A purely quadratic accumulation functional describes a free theory: extension grows without bound, and all excitations propagate indefinitely. Such a structure is mathematically consistent but physically empty, as it cannot support stable localization, finite energy densities, or persistent defects [39, 51]. The requirement that extension be physically meaningful therefore imposes an additional algebraic constraint: extension must be *self-regulating*. This condition implies the existence of higher-order multiplicative contributions that limit or saturate the growth generated by quadratic terms. In algebraic terms, this corresponds to the necessity of nonlinearity.

From this viewpoint, nonlinear terms are not optional embellishments of an otherwise complete theory. They are the minimal algebraic mechanism by which unbounded extension is prevented. Saturation is the mathematical origin of structure: it allows for finite minima, localized configurations, and the stabilization of extended objects [32, 40]. In this sense, the emergence of mass scales, curvature of effective potentials, and localized excitations can be traced back to a single requirement: the algebraic finiteness of extension. Physical structure appears when multiplication is allowed to regulate itself.

Remark IV.2. *In the vicinity of a stable equilibrium configuration, the variational structure of the theory implies that the action admits a quadratic expansion in the deformation modes of the substrate. As is well known, such quadratic variational systems decompose into independent normal modes, each governed by harmonic dynamics. Given the quantization of action introduced earlier, these modes are naturally described as quantum harmonic oscillators. Thus, the linear quantum regime of the theory is structurally equivalent to a collection of quantum harmonic oscillators, without the need for additional postulates.*

The preceding sections have shown how algebraic extension, accumulation, and variational closure lead naturally to global structures in physical law. In the next section, we will address one of the most characteristic and conceptually challenging manifestations of such global structure: quantum entanglement.

V. ENTANGLEMENT AS STRUCTURAL NON-FACTORIZATION AND GLOBAL OBSTRUCTION

A. Structural Separability, Orthogonality, and the Definition of Entanglement

The purpose of this section is to provide a definition of quantum entanglement that does not rely on postulated non-local dynamics or measurement axioms, but instead emerges as a structural property of the underlying algebraic and geometric framework developed in the preceding sections. In particular, we show that entanglement arises naturally as a failure of structural factorization under a complete family of admissible evaluations, and is therefore the generic situation rather than a pathological exception [15, 27].

Structural independence and orthogonality

Recall that in the algebraic framework developed previously, independence between two subsystems is not defined probabilistically, but geometrically. Given a space of configurations $\mathcal{X}$ and a family of admissible evaluations $\mathcal{E} = \{E : \mathcal{X} \to \mathbb{R}\}$, structural independence is characterized by *orthogonality under the full family* $\mathcal{E}$. Concretely, two subsystems A and B are structurally independent if and only if

$$\mathrm{Cov}\left(E_A, E_B\right) = 0 \quad \forall\, E_A \in \mathcal{E}_A,\ E_B \in \mathcal{E}_B, \tag{V.1}$$

where $\mathcal{E}_A$ and $\mathcal{E}_B$ denote the restrictions of admissible evaluations to the respective subsystems.

As emphasized in structural and algebraic approaches to independence, vanishing covariance for a single observable is insufficient; independence requires orthogonality across the entire admissible family [10, 28]. This condition is equivalent to the absence of irreducible cross-terms in any representation compatible with the underlying structure.

Orthogonality implies factorization

A central structural result, already established in arithmetic, geometric, and operator-algebraic settings, is that strong orthogonality enforces factorization. If a composite state admits a representation that is orthogonal with respect to the complete family $\mathcal{E}$, then it decomposes without residual coupling terms:

$$E(S_{AB}) = E_A(S_A)\, E_B(S_B) \quad \forall\, E \in \mathcal{E}. \tag{V.2}$$

In this sense, factorization is not an assumption but a consequence of global orthogonality [10, 46].

1. Definitions of separability and entanglement

A composite state S_{AB} is said to be *structurally separable* if there exists at least one representation of S_{AB}, compatible with all physical constraints (symmetries, finiteness of action, admissibility of evaluations), such that orthogonality holds with respect to the full family $\mathcal{E}$ and the state factorizes accordingly. This notion coincides with standard separability in algebraic and tensor-product formulations, but is expressed here in representation-independent terms [27, 61].

A composite state S_{AB} is said to be *entangled* if no such representation exists. Equivalently, entanglement is defined as the persistent failure of structural factorization: for every admissible representation of S_{AB}, there exists a nonempty subset $\mathcal{E}' \subset \mathcal{E}$ such that

$$\mathrm{Cov}\left(E_A, E_B\right) \neq 0 \quad \text{for some } E_A \in \mathcal{E}_A,\ E_B \in \mathcal{E}_B. \tag{V.3}$$

This formulation aligns with the operator-algebraic characterization of entanglement as the absence of product states compatible with all observables [15, 57], but inverts the usual explanatory burden. Separability is no longer generic, but exceptional: it requires the existence of a globally orthogonal decomposition under all admissible evaluations. Entanglement, by contrast, is the generic situation whenever the structural conditions for complete orthogonality fail.

In particular, no appeal to dynamical non-locality, superluminal influence, or measurement collapse is required at this stage; entanglement is a purely structural obstruction to factorization, defined independently of spacetime considerations. This perspective resonates with algebraic and relational approaches to quantum theory, in which nonseparability is viewed as a kinematic rather than dynamical feature [24, 25].

In the following subsection, we show that this obstruction is not accidental. It arises inevitably from the action of additive (accumulative) operators on states defined over multiplicative or spectral structures, thereby providing a concrete and constructive mechanism for the emergence of non-factorizable correlations.

B. The Mixing Operator: Additive Accumulation and Logical Non-Locality

Having defined entanglement as a structural obstruction to factorization, we now address the origin of this obstruction. The key result of this subsection is that non-factorizable correlations are not accidental, but arise inevitably when additive (accumulative) operators act on states defined over multiplicative or spectral structures. This mechanism is algebraic and logical in nature, rather than dynamical or causal, and therefore does not invoke superluminal propagation [7, 15].

Multiplicative versus additive structure.

In the preceding algebraic development, two fundamentally distinct classes of operations were identified. Multiplicative operations preserve spectral stratification: they act diagonally on the underlying prime or modal decomposition and therefore respect locality in the space of modes. By contrast, additive operations do not act diagonally. They induce global mixing across spectral layers, generating correlations between components that were previously independent.

This distinction is not heuristic but structural. In arithmetic terms, multiplication corresponds to independent shifts along prime directions, whereas addition induces carry propagation across arbitrarily distant scales, producing unavoidable mixing [42, 43]. In spectral terms, multiplicative operators preserve factorization, while additive operators generically destroy it [10].

Accumulation as the fundamental update operation.

Within the present framework, physical interaction and measurement are not modeled as instantaneous creation of new information, but as processes of accumulation. Formally, accumulation corresponds to the integration or summation of local contributions into a global update of the state. This principle was established earlier as a necessary consequence of extension, integration, and variational closure.

Crucially, accumulation is an additive operation. In the arithmetic setting developed previously, the additive shift operator was shown to induce unbounded spectral mixing across prime modes through the propagation of carry chains [46]. The accumulation operator introduced here plays an analogous structural role on the spectral or modal degrees of freedom of the physical substrate.

Non-factorization under additive update.

Let S denote a state represented in a multiplicative (spectral or modal) basis, and let $\mathcal{A}$ denote an additive update operator encoding accumulation. Even if S admits a factorized representation prior to update, the action of $\mathcal{A}$ generically produces a state $\mathcal{A}(S)$ for which no such factorization exists. Symbolically,

$$\mathcal{A}(S_A \otimes S_B) \neq \mathcal{A}(S_A) \otimes \mathcal{A}(S_B), \tag{V.4}$$

except in nongeneric, finely tuned cases. Therefore, *this failure is structural rather than contingent.* Additive operators necessarily mix spectral components, producing irreducible cross-terms that cannot be eliminated by a change of representation consistent with the admissible evaluation family [24, 57].

Logical versus temporal non-locality.

It is essential to emphasize that the action of the additive update operator does not correspond to a signal propagating through spacetime. Rather, it constitutes a reorganization of the global configuration in state space. In this sense, the update is *logically instantaneous* rather than temporally propagating: it represents a change in the informational or contextual determination of the global state, orthogonal to the causal time evolution governed by local field equations. The update is therefore "vertical" in the hierarchy of state determination, not "horizontal" in physical time, in a sense closely related to structural and algebraic notions of non-separability [25].

Remark V.1. *The distinction between causal propagation and logical update can be stated as follows: local disturbances of the deformation field propagate according to hyperbolic (causal) field equations on spacetime, whereas the accumulation-induced update acts on the global state space as a consistency reorganization. The former is constrained by the causal structure (light cone), while the latter is constrained by admissibility and global closure of configurations, without constituting a physical signal.*

Interpretation as structural non-locality

From this perspective, so-called non-local correlations arise whenever the fundamental update rule fails to respect the tensor-product decomposition associated with spatial separation. The non-locality is therefore algebraic rather than dynamical: it expresses the incompatibility between additive accumulation and multiplicative factorization.

This has a profound implication. Standard derivations of Bell inequalities assume statistical independence conditioned on local variables [7]. Within the present framework, this assumption is equivalent to assuming that the global update operator factorizes with respect to spatially separated subsystems. The analysis above shows that such factorization is generically impossible under additive accumulation.

Bell-type violations therefore signal a failure of structural independence, not the presence of superluminal causal mechanisms. They reflect the logical non-locality induced by accumulation, rather than any breakdown of relativistic causality [15, 27].

In the next subsection, we will see how this mechanism acquires a concrete physical realization within Quantum Elastic Geometry (QEG), where additive accumulation acts on a globally constrained elastic substrate whose degrees of freedom are encoded in a symmetric deformation tensor [44, 45].

C. Physical Realization I: The QEG Substrate as a Global Correlator

We now provide a physical realization of the structural non-factorization identified above within the framework of Quantum Elastic Geometry (QEG). The key claim of this subsection is that entanglement emerges naturally once the fundamental degrees of freedom are understood not as localized particles, but as excitations of a globally correlated elastic substrate [45].

The QEG ontology

In QEG, spacetime is a background independent, physical substrate whose state is described by a symmetric rank–2 deformation tensor $\mathcal{G}_{\mu\nu}(x)$. Observable long-range fields and effective metrics arise as coarse-grained manifestations of this underlying structure. Particles are not fundamental point objects, but localized, stable excitations or defects of the substrate [44, 45].

Crucially, this ontology is intrinsically relational. The physical content of the theory is not encoded in isolated local values of $\mathcal{G}_{\mu\nu}$, but in the way deformations of the substrate are correlated across spacetime.

Remark V.2. *A standard concern is whether global correlation would enable faster-than-light signaling. Although the substrate state is globally correlated, such correlations do not provide a controllable channel for superluminal communication. Operationally, local outcome statistics depend only on the reduced (marginal) state of the local subsystem, while remote choices can modify only joint correlations. In this sense, QEG accommodates Bell-type non-factorizability while preserving the no-signaling constraint required by relativistic causality [15, 27].*

Necessity of a correlation structure

Any elastic or oscillatory medium admits a natural notion of correlation between its degrees of freedom. In particular, if the substrate is understood as an effective continuum limit of underlying oscillatory modes, then the physically relevant information is contained not only in mean fields but also in correlations between fluctuations [14].

Accordingly, we introduce, at the effective level, a two-point correlation structure of the form

$$\mathcal{C}_{\mu\nu\rho\sigma}(x, y) \equiv \langle \delta\mathcal{G}_{\mu\nu}(x)\, \delta\mathcal{G}_{\rho\sigma}(y) \rangle, \tag{V.5}$$

where $\delta\mathcal{G}_{\mu\nu} = \mathcal{G}_{\mu\nu} - \langle \mathcal{G}_{\mu\nu} \rangle$ denotes fluctuations about a coarse-grained background configuration. No specific microscopic model is assumed; the correlator is introduced as the minimal object required to encode relational information in a fluctuating elastic medium.

Remark V.3. *The existence of a nontrivial correlator does not imply uniform long-range coherence in generic states. In typical coarse-grained regimes, correlations decay beyond an effective coherence scale set by environmental coupling and preparation constraints. Entangled configurations correspond, in this language, to prepared states whose correlator retains a non-separable structure across spacelike separated regions, even when local mean fields admit a factorized description.*

Global correlations and non-factorization

The existence of a nontrivial correlation structure immediately implies that the physical state of the substrate cannot, in general, be decomposed into independent subsystems associated with spatially separated regions. Even when local expectation values factorize,

$$\langle \mathcal{G}_{\mu\nu}(x) \rangle = \langle \mathcal{G}_{\mu\nu}(x) \rangle_A + \langle \mathcal{G}_{\mu\nu}(x) \rangle_B, \tag{V.6}$$

the correlator $\mathcal{C}_{\mu\nu\rho\sigma}(x, y)$ may remain nonzero for x and y belonging to different regions. In this case, the global state admits no representation compatible with complete structural factorization.

This provides a direct physical instantiation of the abstract definition of entanglement introduced earlier: entanglement corresponds to the persistence of irreducible correlations in the substrate that cannot be eliminated by any admissible change of representation.

Entangled particles as shared substrate excitations

Within this picture, two entangled particles are not independent carriers of information. They are localized manifestations of a single correlated configuration of the substrate. The correlations observed in measurement outcomes do not arise from signals exchanged between particles, but from the fact that both excitations are supported by the same underlying correlated state of $\mathcal{G}_{\mu\nu}$.

In this sense, the substrate itself functions as the carrier of correlation: it is the common cause underlying the observed outcomes, non-local in the Bell sense but not dynamical or signal-mediated [25]. *Measurement acts as a global update of the substrate configuration, selecting a consistent realization of the correlated state.* As emphasized in the previous subsection, this update is logical rather than causal and does not involve propagation through spacetime.

Compatibility with relativistic causality

Because the correlation structure is part of the ontological state of the substrate, no superluminal mechanism is required to account for observed violations of Bell-type inequalities. Local relativistic dynamics governs the propagation of disturbances in $\mathcal{G}_{\mu\nu}$, while entanglement reflects constraints on the admissible global configurations of the substrate. Causality is preserved at the level of dynamical evolution, even though structural independence fails at the level of global state composition [24].

D. Physical Realization II: Entanglement as Shared Topological Obstruction

We now consider a complementary realization of structural non-factorization in the nonlinear regime of Quantum Elastic Geometry. While the previous subsection emphasized global correlations in the linear or weakly fluctuating substrate, the present discussion shows that entanglement can also be understood geometrically, as a shared topological obstruction associated with finite-action configurations [44].

Particles as nonlinear defects of the substrate

In the nonlinear extension of QEG, particles are not merely localized fluctuations but stable, finite-energy configurations of the deformation field. Such configurations are naturally interpreted as solitons or topological defects of the elastic substrate. Their stability is ensured by a combination of energetic minimization and topological protection, subject to the requirement of finite action [39, 51].

Importantly, the admissibility of a configuration is a global condition. A defect that is locally well-defined may nevertheless fail to extend to a globally finite-action configuration unless additional constraints are satisfied.

Global extensibility and obstruction

Let $\mathcal{S}$ denote the space of admissible substrate configurations modulo gauge and redundancy relations. A localized excitation corresponds to a class $[\mathcal{G}] \in \mathcal{S}$. In general, not every local assignment of topological or geometric charges defines an element of $\mathcal{S}$; obstructions may arise that prevent global extension.

Such obstructions are familiar in many areas of physics, including vortex confinement, flux tubes, and defect pairing in condensed matter systems [32, 40]. In QEG, the same principle applies: finite-action realizability imposes global compatibility conditions on the distribution of topological data.

Entanglement as a shared obstruction

Within this framework, an entangled pair can be interpreted as a composite configuration in which each subsystem, taken in isolation, carries an obstruction to global extension. The combined configuration, however, is admissible because the obstructions cancel when considered together.

Formally, let S_A and S_B denote two localized excitations. Individually, the classes $[S_A]$ and $[S_B]$ do not belong to $\mathcal{S}$ due to incompatible boundary or topological conditions. The composite configuration S_{AB}, by contrast, defines a valid element of $\mathcal{S}$ because the global constraints are satisfied only at the level of the pair.

This provides a geometric realization of entanglement: the physical state exists only as a whole, not as a tensor product of independently admissible parts.

Remark V.4. *The topological picture is fully consistent with the algebraic definition introduced earlier. The impossibility of assigning independent, physically admissible states to the subsystems is precisely the failure of structural factorization. No choice of local representation yields a separable description, because separability would violate the global admissibility conditions imposed by finite action. Thus, topological entanglement and algebraic non-factorization are two manifestations of the same underlying structural constraint.*

The present construction bears a formal resemblance to proposals relating entanglement to geometric connections [38]. However, no additional spacetime bridges or exotic geometrical objects are postulated. In the present framework, the "bridge" is not a new structure but the constraint itself: the necessity of a globally consistent configuration satisfying variational and topological admissibility.

The connection between entangled subsystems is therefore encoded in the global constraint equations, not in a traversable geometric channel. In this sense, entanglement reflects a requirement of global closure rather than a mechanism of communication.

Physical interpretation

From this viewpoint, measurement-induced decoherence corresponds to a transition in which the nonlinear configuration can no longer maintain its shared obstruction, typically due to coupling to an environment. Once the global constraint is relaxed, the system admits a factorized description and entanglement is lost.

This geometric realization completes the physical interpretation of entanglement within QEG. In the concluding subsection, we synthesize the algebraic, dynamical, and geometric perspectives and clarify the implications for Bell-type inequalities and the interpretation of quantum nonlocality.

E. Synthesis and Consequences: Bell Inequalities and Logical Non-Locality

We conclude by synthesizing the algebraic, dynamical, and geometric perspectives developed above and by clarifying their implications for Bell-type inequalities and the interpretation of quantum non-locality.

Structural assumptions behind Bell inequalities and reinterpretation of Bell violations

Bell's theorem demonstrates that no theory simultaneously satisfying locality, realism, and statistical independence can reproduce all quantum correlations [7].

Crucially, the notion of statistical independence employed in Bell-type arguments presupposes that the joint state of spatially separated subsystems admits a factorized description conditioned on local variables [20]. In the present framework, this assumption corresponds precisely to the existence of a representation in which structural orthogonality holds under the full family of admissible evaluations. As shown in the preceding subsections, such factorization is generically incompatible with additive update rules acting on multiplicative or spectral state spaces.

Violations of Bell inequalities are therefore not interpreted here as evidence for superluminal influences or failures of relativistic causality. Rather, they signal the breakdown of structural independence: the global state of the substrate does not admit a factorized representation compatible with the admissible evaluation structure and finite-action constraints.

In this sense, Bell inequalities fail because one of their implicit premises—complete local factorizability of the global state—is mathematically untenable in the presence of additive accumulation [15, 27].

A key distinction must be emphasized. The non-local correlations discussed here arise from logical constraints on the global configuration space, not from causal propagation through spacetime. The update of the state under accumulation is a reorganization of the spectral or topological structure of the configuration, orthogonal to the causal time evolution governed by local field equations.

Thus, the apparent instantaneous character of entanglement reflects logical non locality rather than physical superluminality. Relativistic causality remains intact at the level of dynamical response and signal propagation [24].

Ontological clarification

Within Quantum Elastic Geometry, the fundamental ontological entity is the correlated state of the substrate [45]. Particles are localized manifestations of this state, not independent carriers of information. Measurement does not create correlations but selects a consistent realization of an already correlated global configuration. This perspective avoids both naive local determinism and ad hoc non-local dynamics. It is realist, but not locally factorizable.

Scope, limitations, and conclusions

The present construction does not attempt to reproduce the full formalism of quantum mechanics from first principles, nor does it propose a hidden-variable completion in the traditional sense. Its purpose is structural: to explain why non-factorizable correlations are generic in a universe governed by additive accumulation acting on multiplicative or spectral substrates.

In this respect, entanglement is not an anomaly to be explained away, but a natural consequence of the algebraic and geometric architecture underlying physical law. It emerges as a manifestation of structural non-factorization and global admissibility constraints. Its apparent non-locality reflects logical properties of state composition rather than violations of causal order.

By grounding this phenomenon in algebraic structure and geometric realizability, the framework provides a coherent interpretation of quantum correlations consistent with both relativistic causality and a realist ontology.

F. F. Outlook: The Born Rule as Structural Concordance

The interpretation of entanglement as structural non-factorization suggests a reformulation of the measurement problem and the origin of the Born rule ($P \propto |\psi|^2$).

In standard quantum mechanics, the probabilistic character of measurement is postulated. Within the present framework, it admits a structural interpretation grounded in the interaction between global modes and local accumulation in a fluctuating substrate [8].

A measurement event corresponds to the attempt of a localized defect (detector or probe) to consistently couple to a global configuration of the substrate. Since the global state does not physically factorize, the interaction cannot be interpreted as a purely local scattering process, but rather as a global consistency or stability test: *can the substrate support a localized, dynamically coherent history at point x?*

If the substrate admits a wave-mechanical description, the local availability of deformation modes must be characterized by a scalar density constructed from the global mode $\psi(x)$. Under the minimal requirements of positivity, locality, and invariance under global phase transformations, the only admissible scalar density is proportional to $|\psi(x)|^2$.

This mirrors a well-known structural fact: in wave-bearing and elastic media, quadratic quantities represent energy, intensity, or probability densities [13, 36]. Here, $|\psi|^2$ is interpreted more generally as the density of structurally available micro-configurations.

It is therefore structurally natural to associate the likelihood of a localized outcome at x with the degree of structural concordance between the global mode and the local probe. In this view, the probability $P(x)$ is not an intrinsic randomness attached to the particle, but an emergent statistical weight measuring how many consistent local realizations are supported by the global configuration:

$$P(x) \propto |\psi(x)|^2. \tag{V.7}$$

From this perspective, the Born rule is not introduced as an independent axiom, but arises as the unique quadratic local invariant governing the rate of successful localization events in a fluctuating substrate [21]. A full derivation of this correspondence from the microscopic dynamics of the substrate is left as a natural direction for future work.

VI. THE NECESSITY OF AN ENTROPIC (EXPANSIVE) SECTOR

The operational cycle established in the previous sections—multiplication generating extension, integration accumulating it, and variation extracting local dynamics—is mathematically complete. However, its physical realizations are not unique. In this section, we argue that restricting the action *a priori* to purely contractive (force-like) contributions constitutes an additional hidden constraint. Once the full class of local, integrable contributions compatible with the operational cycle is admitted, an expansive sector becomes a natural and, in an important sense, unavoidable completion: it is the generic way for accumulation to include a non-vanishing homogeneous component [13, 36].

A. Contractive versus expansive contributions

Most familiar field-theoretic contributions to the action are contractive in a variational sense. Kinetic terms, curvature norms, gauge-field strengths, and elastic energies typically penalize deviations from local uniformity by increasing the action when fields develop gradients or depart from equilibrium. Their leading effect is therefore restoring: they suppress inhomogeneities and encode resistance or stiffness [41, 49, 59].

However, the multiplicative mechanism that generates extension does not, by itself, privilege contractive over expansive effects. Both are admissible at the algebraic level: multiplication can build positive-definite gradient norms, but it can also build extensive densities that persist in the homogeneous limit. Integration then accumulates all admissible contributions without discrimination [34].

A theory built exclusively from contractive terms is therefore a restricted subclass rather than the generic case. If the operational cycle is taken seriously as a foundational blueprint, the question is not "why include an expansive sector?", but rather "what justifies excluding all extensive contributions that survive coarse-graining?"[2]

B. Intensive–extensive product structure

An expansive contribution to the accumulated quantity must satisfy two conditions. First, it must be expressible as a local product, consistent with the primacy of multiplication. Second, it must scale extensively under integration, contributing intrinsically to the global accumulation.

The simplest algebraic pattern meeting these requirements is an intensive–extensive product. In thermodynamics, this structure appears in the differential form

$$\delta Q \;=\; T\, dS, \tag{VI.1}$$

where T is intensive and S is extensive [13]. Here we do *not* import thermodynamic equilibrium postulates; we only adopt the structural fact that an intensive factor multiplying an extensive increment yields an extensive contribution upon accumulation [36].

Within the present framework, the intensive factor controls the local strength of the contribution, while the extensive factor encodes the accumulated extension. This is precisely the minimal algebraic form of an expansive sector compatible with the operational cycle.

C. Entropic contribution to the action

Guided by the intensive–extensive product structure, we introduce an entropic sector as a genuine contribution to the action functional. By construction, this contribution must be expressed as an integral of a local density over spacetime, so that its dimensional character matches that of action:

$$S_{\rm ent} \;=\; \int_\Omega \mathcal{L}_{\rm ent}(x)\, d^4x. \tag{VI.2}$$

[2] This necessity is further supported by the arithmetic prototype: while multiplication preserves information (is conservative), addition is intrinsically entropy-producing, generating quantifiable "additive innovation" across prime scales. Excluding the expansive sector would amount to enforcing a physically unnatural conservation of information in the aggregation process [42, 46].

We take the entropic Lagrangian density in the schematic (minimal) factorized form

$$\mathcal{L}_{\rm ent}(x) \;\equiv\; \mathcal{T}(x)\, \sigma(x), \tag{VI.3}$$

where $\mathcal{T}(x)$ is an intensive, temperature-like field and $\sigma(x)$ is an extensive, entropy-like density (per unit spatial volume, or more generally per unit spacetime measure depending on convention) [13].

Dimensional bridge

In standard thermodynamics, the product $T\,dS$ carries units of energy. In a variational setting, the correct action units arise because the entropic contribution is integrated over spacetime. Writing $d^4x = dt\, d^3x$ (or equivalently $x^0 = ct$), σ may be interpreted as an entropy density per unit spatial volume, so that $\mathcal{T}\sigma$ has units of energy density. Consequently,

$$S_{\rm ent} \;\sim\; \int (\text{energy density})\, d^3x\, dt, \tag{VI.4}$$

which has the correct units of action [34].

D. Expansive sector and cosmological constant-like contributions

The terms *contractive* and *expansive* are often used informally and may be confused with the attractive character of gravity or with cosmological expansion in specific coordinates. In the present framework, the distinction is variational and therefore unambiguous [59].

A contribution to the action is *contractive* when its leading effect penalizes local inhomogeneities, typically through positive-definite quadratic forms in derivatives (gradient energies, curvature-like norms). Such terms favor locally uniform configurations and generate restoring tendencies [41, 59]. Other hand, contribution is *expansive* when it contains an intrinsic extensive density whose leading effect does *not* vanish in the homogeneous limit. In that case, the dominant influence is not a restoring force but a background component that persists under coarse-graining and accumulates with spacetime volume.

Homogeneous limit and vacuum-like behavior

Consider $\mathcal{L}_{\rm ent} = \mathcal{T}\sigma$. In large-scale equilibrium or coarse-grained regimes, $\mathcal{T}(x)$ and $\sigma(x)$ may admit slowly varying components over large domains. Denoting the approximately homogeneous contribution by $\rho_{\rm vac}$,

$$\mathcal{L}_{\rm ent}(x) \;=\; \rho_{\rm vac} \;+\; \delta\mathcal{L}_{\rm ent}(x), \tag{VI.5}$$

where $\delta\mathcal{L}_{\rm ent}$ captures fluctuations and gradient-dependent parts. The spacetime integral of the homogeneous component yields

$$S_{\rm vac} \;=\; \int_\Omega \rho_{\rm vac}\, d^4x, \tag{VI.6}$$

which is intrinsically extensive and survives in the homogeneous limit.

Covariant form and cosmological constant-like term

In a generally covariant formulation, the invariant volume element is $\sqrt{-g}\,d^4x$ [41, 59], and the corresponding contribution takes the form

$$S_{\text{vac}} = -\int_{\Omega} \rho_{\text{vac}} \sqrt{-g}\, d^4x. \quad \text{(VI.7)}$$

Up to conventional sign choices, this is formally equivalent to a cosmological constant term: varying (VI.7) with respect to the metric yields an effective stress-energy contribution proportional to $g_{\mu\nu}$,

$$T^{\text{vac}}_{\mu\nu} \propto \rho_{\text{vac}}\, g_{\mu\nu} \quad \text{(VI.8)}$$

(cf. standard derivations in relativistic field theory) [41, 59]. It is worth emphasizing that the spacetime measure $\sqrt{-g}\,d^4x$ in (VI.7) is itself a manifestation of the primitive operation of multiplication. Geometrically, $\sqrt{-g}$ represents the volume of the parallelepiped spanned by a tangent basis and can be expressed in terms of determinants, i.e. multiplicative invariants of the metric [33, 52]. In this sense, even the measure defining vacuum accumulation arises from the same multiplicative principle underlying local extension and coupling.

Thus, the expansive sector admits a direct cosmological reading: its homogeneous component behaves as a vacuum-like contribution that naturally accumulates with spacetime volume, while its inhomogeneous component contributes through fluctuations and gradients, thereby generating force-like responses through derived contractive modes. This separation clarifies the origin of attractive, restoring behavior (gradient-driven contractive responses) versus vacuum-like accumulation (homogeneous expansive contribution), avoiding common ambiguities in the use of "contractive" and "expansive."

Extremization versus maximization (thermodynamic analogy)

The intensive–extensive structure $\mathcal{T}\sigma$ is algebraically analogous to the thermodynamic product $T\,s$, but its variational role is different. In thermodynamics, equilibrium principles are often phrased in terms of entropy maximization under constraints [13, 36]. Here, by contrast, $\mathcal{L}_{\text{ent}} = \mathcal{T}\sigma$ enters an action functional that is extremized [34].

The physical effect of the entropic sector is therefore controlled by its sign and by how it contributes to stress-energy in the homogeneous limit. In particular, a vacuum-like contribution $-\rho_{\text{vac}}\sqrt{-g}$ yields an effective equation of state consistent with negative pressure, while gradient-dependent parts generate restoring, force-like responses through derived contractive modes. The thermodynamic analogy therefore refers to the algebraic structure of local products, not to an identity of variational principles.

E. Entropic degrees of freedom as structural, not statistical

In conventional treatments, entropy is often introduced as a statistical or emergent quantity associated with coarse-graining. In the present framework, this interpretation is neither required nor primary. Here, entropic degrees of freedom arise as a direct consequence of the algebraic and variational structure of physical law. They are admitted by the same principles that give rise to fields, actions, and equations of motion. Entropy, in this sense, is a geometric and variational quantity encoding an expansive aspect of accumulated extension [13].

This viewpoint does not deny statistical interpretations of entropy, but places them downstream of a more fundamental structural role: statistical entropy becomes a particular realization of a more general entropic mode inherent in the architecture of physical law [31].

F. Completeness of the variational framework

With the inclusion of an entropic sector, the variational framework becomes structurally complete: the action incorporates both contractive and expansive contributions, reflecting the full range of algebraically admissible extensions compatible with the operational cycle.

The resulting picture is one in which physical reality is governed not solely by restoring forces and constraints, but also by intrinsic extensive tendencies encoded at the same fundamental level. This balance is not imposed by hand; it follows from allowing the full expressive power of multiplication, integration, and variation.

Remark VI.1. *It is worth noting that the effective dynamics of substrate modes naturally separates into two regimes. Purely entropic modes obey diffusion-type equations, analogous to the heat equation, while contractive and transport modes obey wave-type dynamics [36]. In the non-relativistic limit, the oscillatory propagation of these modes recovers the structure of a Schrödinger-type equation [5]. From this perspective, the imaginary unit reflects the conservative, phase-preserving character of the oscillatory sector, in contrast with the real dissipative dynamics of the entropic sector.*

G. Reconciliation with Statistical Mechanics: The Field as the Substrate of Statistics

A common objection to treating entropy as a fundamental field is the success of statistical mechanics, which derives thermodynamic behavior from the counting of microstates ($S = k_B \ln \Omega$) [13]. The present framework does not contradict this view; rather, it provides the structural substrate on which such counting becomes physically meaningful [31].

a. Variational Necessity of the Expansive Sector. As established in Sec. VI.A, a complete variational theory built on multiplication and integration requires both contractive (restoring) and expansive (accumulative) sectors. Excluding the expansive sector leads to a degenerate theory capable only of describing localized defects in a vacuum, but incapable of representing the vacuum's own extensive capacity.

b. The Field is the Object Being Counted. In the QEG realization, this expansive sector is identified with the isotropic trace (dilatational) mode of the substrate, $\Theta \propto \text{tr}(\mathcal{G}_{ij})$, which provides the physical degree of freedom carrying extensivity and thermal response [45]. Standard statistical mechanics is then recovered as an effective description of the fluctuations of this fundamental mode. Just as the electromagnetic field admits photon modes whose statistics yield Planckian laws, the thermo-entropic sector admits excitations of Θ whose coarse-grained (overdamped) dynamics are diffusive, and whose combinatorial multiplicity yields the Boltzmann entropy in the thermodynamic limit [13, 36]. QEG identifies this sector with the trace/dilatational mode and its telegrapher-type effective dynamics [44].

c. Conclusion: Structure Supports Statistics. From this perspective, entropy is not merely a measure of ignorance about a microstate, but a measure of excitation of a real expansive mode of the substrate. Statistical thermodynamics describes *how* this mode populates its accessible configurations; the field itself appears as a necessary structural component of the variational action. The macroscopic arrow of time is then associated with the specific dissipative/telegrapher dynamics governing this sector in the effective (coarse-grained) regime [36].

Remark VI.2. *The effective "dissipation" associated with viscosity in the substrate does not represent a loss of energy into nothingness, but a transfer from coherent transport modes (vector/tensor sectors) to the incoherent entropic sector (scalar-like degrees of freedom). At the level of the full variational system, conservation laws follow from symmetry principles: in particular, time-translation invariance yields a conserved total energy functional (up to exchange* *with boundary and environment) [36, 48]. Thus, an emergent thermodynamic arrow of time can coexist with fundamental conservation, through a mechanism of spectral transfer and recycling within the unified field structure.*

In the following section, we show how conventional electromagnetic and gravitational modes can be understood as derived or scaled responses within this expanded framework. In particular, we outline how force-like fields may arise as projected, differentiated, or rescaled manifestations of the entropic sector, rather than as independent fundamental ingredients.

VII. DERIVED CONTRACTIVE MODES: FROM EXPANSIVE ACCUMULATION TO LOCAL GRADIENTS

A. From expansive accumulation to local gradients

The introduction of an expansive (entropic) sector in Section VI enlarges the variational framework beyond purely contractive contributions. We now address a central implication: once an extensive contribution to the action exists, the requirement of locality forces the appearance of *gradient-driven* responses. In other words, force-like (contractive) modes emerge naturally as local manifestations of spatial or spacetime variations in the underlying expansive sector [22, 34].

Entropic sector as a scalar accumulated contribution

For definiteness, consider an entropic contribution written as an integral over a spacetime domain Ω,

$$S_{\text{ent}}[\Sigma, \mathcal{T}] \;=\; \int_{\Omega} \mathcal{F}(\mathcal{T}(x), \Sigma(x))\; d^4x, \tag{VII.1}$$

where $\mathcal{T}$ is an intensive, temperature-like field and Σ is an extensive, entropy-like field or density. The function $\mathcal{F}$ encodes the local intensive–extensive product structure discussed previously.

The key point is that S_{ent} is a *scalar accumulated* quantity: it is defined by integration of a local density, as required by covariance and dimensional consistency [34, 59].

A purely uniform entropic sector, with $\partial_\mu \mathcal{T} = 0$ and $\partial_\mu \Sigma = 0$, contributes only as a constant offset to the accumulated quantity and cannot induce local dynamics. Any local influence must therefore arise from *inhomogeneities*, i.e. from spacetime dependence of $\mathcal{T}$ and/or Σ.

In a variational setting, the local influence of inhomogeneities is encoded through derivatives of the fields. This is not a model assumption but a structural fact: if the action is to constrain local behavior, the Euler–Lagrange equations must involve derivatives that detect changes across infinitesimal neighborhoods. Consequently, any framework in which $\mathcal{T}(x)$ or $\Sigma(x)$ varies generically implies the emergence of gradient-sensitive terms and corresponding local responses [17, 22].

Potentials and forces as a mathematical duality

The passage from an accumulated scalar quantity to a local driving field is mathematically analogous to the relationship between potentials and forces. A scalar potential Φ does not act through its absolute value but through its gradient $\nabla\Phi$. The same logic applies in the present context: the entropic sector contributes a scalar density to the action, but local responses arise from how this density *changes* [35].

To make this explicit, suppose that an effective scalar potential $\Phi_{\text{ent}}(x)$ can be associated with the entropic sector, for instance by a suitable identification of Φ_{ent} with a functional of $\mathcal{T}$ and Σ (or with a reduced description thereof). Then the simplest local driving field compatible with locality and isotropy is the gradient

$$\mathbf{G}_{\text{ent}} \;\propto\; -\nabla\Phi_{\text{ent}}. \tag{VII.2}$$

This expression is not yet a physical law; it is the minimal mathematical form of a local response extracted from a scalar accumulated contribution. It follows from the fact that the only coordinate-covariant, first-order local object one can build from a scalar is its derivative.

Why force-like modes are generically contractive

If Φ_{ent} is interpreted as encoding an expansive contribution to the action, then its gradients encode *spatial differentials of expansion*. Local systems respond to differentials by redistributing or equilibrating, which mathematically appears as a contractive tendency: gradients drive flows that reduce inhomogeneity.

In the variational language, this corresponds to the emergence of terms that penalize gradients, typically of the form $(\partial\Phi_{\text{ent}})^2$ (or analogous quadratic expressions), yielding Euler–Lagrange equations that smooth out variations [34, 36]. Hence, even if the fundamental sector is expansive at the level of accumulated contribution, its local manifestations are naturally expressed through gradient-driven, typically contractive modes. These modes are the mathematical progenitors of force-like fields.

First-order derivatives as universal derived fields

A robust conclusion follows: whenever an entropic sector is present and is allowed to vary across spacetime, the variational requirement of locality forces the existence of derived first-order fields built from derivatives of entropic potentials (or entropic state variables). These derived fields are universal in form:

$$\mathcal{A}_\mu \;\sim\; \partial_\mu \Psi_{\text{ent}}, \tag{VII.3}$$

where Ψ_{ent} denotes an effective scalar descriptor of the expansive sector (not necessarily unique). The specific physical interpretation of $\mathcal{A}_\mu$ depends on how the underlying degrees of freedom are organized, but its *mathematical origin* is fixed: it is the first-order local object extractable from an accumulated scalar contribution.

Summary

The key result of this subsection is structural: an expansive entropic sector defined through accumulated scalar contributions generically implies the emergence of gradient-driven, force-like responses when inhomogeneities are present. These responses arise because locality demands that only derivatives can transmit local influence from an integrated scalar density.

In subsequent subsections we refine this statement by analyzing scaling and dimensional structures, showing how electromagnetic- and gravitational-like modes can be understood as distinct projections and rescalings of these derived gradient fields.

B. Scaling and dimensional reduction of the entropic sector

In Section VII A we argued that an expansive scalar contribution to the action generically implies derivative-driven local responses whenever inhomogeneities are present. We now refine this result by analyzing how *scaling* and *geometric projection* of the same underlying entropic sector give rise to distinct classes of derived (typically contractive) modes.

Intensive and extensive quantities as scaling primitives

A fundamental distinction in the present framework is that between intensive and extensive quantities. Extensive quantities scale with the size of the domain over which they are accumulated, while intensive quantities remain invariant under such scaling. This distinction is not thermodynamic in origin; it is a structural property of accumulation by integration [13, 36].

Let $\sigma(x)$ denote an entropy-like *density* (local, but extensive upon accumulation) and $\mathcal{T}(x)$ an intensive field. Their product defines a local contribution whose integral scales extensively,

$$S_{\text{ent}} = \int \mathcal{T}(x)\, \sigma(x)\, d^4x. \tag{VII.4}$$

Derived local fields inherit their scaling behavior from this structure. In particular, differentiation introduces inverse-length factors and converts scalar descriptors into higher-rank local objects, thereby increasing locality while reducing the effective scaling dimension of the response [17, 34].

Dimensional reduction as scaling and form-degree change

Differentiation induces a controlled "dimensional reduction" in the sense of scaling: it converts a scalar (0-form) into a covector (1-form), and more generally increases tensorial complexity while introducing inverse-length factors. Schematically, if Ψ_{ent} is an effective scalar descriptor of the expansive sector, then

$$\Psi_{\text{ent}} \longrightarrow A_\mu \equiv \partial_\mu \Psi_{\text{ent}} \longrightarrow F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{VII.5}$$

where A_μ is the minimal first-order derived field and $F_{\mu\nu}$ is its antisymmetric curvature (the exterior derivative of the 1-form A). Each step increases locality and introduces additional inverse-length scaling, which is precisely what allows a globally accumulated scalar sector to generate localized, force-like responses [9, 55].

This hierarchy provides a natural classification of derived modes:

- scalar (accumulated, expansive) sector: Ψ_{ent},
- vector-like gradient sector: $A_\mu \sim \partial_\mu \Psi_{\text{ent}}$,
- antisymmetric tensor sector associated with circulation/curvature: $F_{\mu\nu} \sim \partial_{[\mu} A_{\nu]}$.

1. *Radial and azimuthal projections*

In a spatially isotropic setting, local derived fields naturally decompose into radial and azimuthal (circulation-like) components. The radial component is associated with variations along $\hat{\mathbf{r}}$, while the azimuthal component captures circulation around that direction.

For a spherically symmetric scalar descriptor $\Psi_{\text{ent}}(r)$, one obtains

$$\nabla \Psi_{\text{ent}}(r) = \frac{d\Psi_{\text{ent}}}{dr}\, \hat{\mathbf{r}}, \tag{VII.6}$$

which is purely radial. By contrast, circulation arises when considering antisymmetric derivatives of a vector-like derived field (e.g. the curl of $\mathbf{A}$ or, covariantly, $F_{\mu\nu}$). Thus, from a single scalar expansive sector, two qualitatively distinct derived responses appear:

- radial, gradient-driven modes,
- azimuthal, circulation/curvature-driven modes.

This dichotomy anticipates gravitational-like and electromagnetic-like structures, respectively, without presupposing their physical interpretation [29, 59].

Emergence of inverse-distance and inverse-square laws

A key consequence of the derivative hierarchy is the appearance of characteristic distance dependences. In three spatial dimensions, outside sources (or in regions where the effective descriptor satisfies a Laplace-type equation), isotropic solutions yield scalar potentials of the form

$$\Psi_{\text{ent}}(r) \sim \frac{1}{r}. \tag{VII.7}$$

Differentiation then produces radial fields scaling as

$$\nabla \Psi_{\text{ent}}(r) \sim \frac{1}{r^2}. \tag{VII.8}$$

These scalings are therefore not empirical inputs but mathematical consequences of isotropy, spatial dimensionality, and the requirement that local responses arise from derivatives of an underlying scalar descriptor [17, 29].

Scaling constants as projection factors

The transition from the expansive scalar sector to derived contractive modes may involve fixed numerical or dimensional scaling factors. Such factors serve to normalize different projections of the same underlying structure and to translate between the units appropriate to each derived mode.

Importantly, these scaling constants do not define independent interactions. They encode how a single expansive contribution is redistributed across different tensorial projections and scaling dimensions. Their universality reflects the universality of the projection mechanism rather than the introduction of separate physical sectors [63].

Summary

Once admitted as an expansive scalar contribution to the action, the entropic sector contains within it the seeds of multiple derived modes. Differentiation induces a hierarchy of increasingly local fields (scalar → 1-form → antisymmetric curvature), while symmetry enforces radial and azimuthal decompositions. The resulting gradient- and circulation-based responses exhibit characteristic inverse-distance scaling laws and typically contractive behavior through gradient-sensitive invariants.

In the next subsection, we exploit these structural results to show how electromagnetic and gravitational modes can be identified as distinct projections of the same underlying entropic sector, differing by symmetry, scaling, and geometric interpretation. In particular, the expansive sector admits both a homogeneous vacuum-like component (cosmological-constant-like) and an inhomogeneous component whose derivatives generate derived force-like modes.

C. Electromagnetic and gravitational modes as geometric projections

In the previous subsection we showed that once an expansive scalar sector is admitted, differentiation and symmetry considerations inevitably generate distinct classes of derived local modes. We now demonstrate that these classes can be naturally identified with gravitational-like and electromagnetic-like responses, not as independent fundamental interactions, but as geometric projections of a common underlying structure [22, 34].

Remark on the nature of the underlying scalar sector

Throughout the preceding sections, the expansive sector has been described as scalar in the sense that it defines an accumulated invariant contribution to the action. It is important to clarify that this does not require the underlying field to be a globally trivial real scalar. In particular, the scalar sector may possess an internal phase structure without altering its scalar character at the level of accumulated quantities.

Scalar origin and projection principle

We therefore describe the expansive sector by a complex scalar field,

$$\Phi(x) = \sqrt{\rho(x)}\, e^{i\theta(x)}, \tag{VII.9}$$

where $\rho(x)$ is a positive real scalar encoding the extensive magnitude of the sector, and $\theta(x)$ is a dimensionless phase variable. The accumulated contribution to the action depends only on scalar combinations of Φ and its derivatives, ensuring invariance, while the phase structure introduces additional geometric degrees of freedom at the local level.

The projection principle may then be stated as follows:

> All contractive field modes arise as geometric projections of derivatives of an underlying expansive scalar sector.

Different projections correspond to different ways of extracting local structure from $\rho(x)$ and $\theta(x)$, constrained by symmetry, dimensionality, and covariance.

Algebraic closure and the role of complex numbers

The introduction of a complex scalar field is not merely a convenient way to encode phase information. From an algebraic perspective, if multiplication is taken as the primitive operation generating physical extension, it is natural to require that the underlying number system be algebraically closed. The complex numbers constitute the minimal algebraic closure of the real numbers under multiplication. In this sense, the appearance of a complex structure reflects the requirement of algebraic completeness of the extension-generating operation, rather than an ad hoc choice motivated by gauge considerations [58, 60, 63].

Radial projection and gravitational-like response

In an isotropic setting, the most direct projection of the expansive sector arises from gradients of the scalar magnitude $\rho(x)$. This yields a vector field of the form

$$\mathbf{g}(r) \;\propto\; -\nabla\rho(r). \tag{VII.10}$$

Such a field is:

- radial,
- central,
- source-driven,
- decaying as $1/r^2$ in three spatial dimensions.

These properties coincide with the defining structural features of gravitational fields. Importantly, no appeal to mass, curvature, or spacetime geometry is required at this stage. The gravitational-like response appears purely as the radial projection of the expansive sector under the requirements of isotropy and locality.

The contractive nature of this mode follows from its origin: gradients of expansion encode differential tendencies toward equilibration, which manifest as attractive, force-like behavior [17, 59].

Azimuthal projection and electromagnetic-like response

In addition to radial gradients, the phase field $\theta(x)$ gives rise to a distinct class of local structures. The derivative of the phase defines a one-form,

$$A_\mu \;\equiv\; \partial_\mu\theta, \tag{VII.11}$$

which locally resembles a gradient but does not, in general, correspond to a globally exact form.

In the presence of nontrivial topology, phase singularities, or multivalued configurations of θ, the associated field strength

$$F_{\mu\nu} \;\equiv\; \partial_\mu A_\nu - \partial_\nu A_\mu \tag{VII.12}$$

is non-vanishing. This field strength measures the curvature associated with phase transport and captures circulation-like, azimuthal responses [2, 18, 47].

Electromagnetic-like modes therefore emerge not from the gradient of a real scalar, but from the curvature of a phase connection associated with the expansive sector. Such modes are characterized by:

- azimuthal structure,
- coupling to current-like sources associated with transport of the expansive quantity,
- inverse-distance scaling in stationary configurations.

The electromagnetic-like response thus arises as a circulation-oriented projection of the same underlying expansive structure that generates the gravitational-like response. This construction does not assume a pre-existing gauge symmetry; rather, the gauge-like structure emerges from the geometric properties of the phase field.

Topological quantization (remark)

A further consequence of the phase-based construction is the natural appearance of quantization conditions [18, 47]. Since the phase must be single-valued up to integer winding, one has

$$\oint \partial_\mu\theta\, dx^\mu \;=\; 2\pi n, \qquad n \in \mathbb{Z}. \tag{VII.13}$$

In gauge-theoretic language, this implies that the holonomy of the emergent connection A_μ is quantized whenever nontrivial winding or defects are present [2]. This provides a structural route to charge- or flux-quantization without introducing it as an independent postulate.

1. Sources: mass and current as dual projections

The distinction between mass-like and current-like sources arises naturally within the projection framework. Scalar accumulations of the expansive sector generate radial gradients and hence mass-like sourcing. By contrast, transport or phase winding of the expansive quantity generates circulation and hence current-like sourcing.

In this sense, mass and electric current are not independent primitives. They represent different modes of coupling between localized systems and the same underlying expansive sector:

- mass couples to radial projections of $\rho(x)$,
- current couples to azimuthal projections associated with $\theta(x)$.

This duality mirrors the distinction between gravitational attraction and electromagnetic circulation.

Minimal bilinear coupling as the source–field interaction

The projection framework identifies derived local fields (e.g. a gradient-driven radial mode and a phase-connection mode). To complete the variational architecture, one must specify how localized sources enter the action. This is not an additional physical assumption but a question of minimal algebraic construction.

Given a one-form field A_μ (arising from phase transport) and a conserved current J^μ encoding transport of the underlying expansive quantity, the simplest local scalar density that couples source and field is the bilinear contraction

$$\mathcal{L}_{\rm int} \;=\; -\,J^\mu A_\mu. \tag{VII.14}$$

This term is singled out by three structural requirements: locality (no nonlocal functionals), covariance (scalar contraction), and minimality (lowest order in fields and derivatives) [48, 49].

Analogously, the radial sector admits a minimal scalar coupling between a scalar source density ρ_m and an effective scalar potential $\Phi_{\rm ent}$ extracted from the expansive magnitude,

$$\mathcal{L}^{(g)}_{\rm int} \;=\; -\,\rho_m\,\Phi_{\rm ent}. \tag{VII.15}$$

In both cases, the interaction terms arise as the minimal bilinear constructions that connect sources to the corresponding projected fields. Together with the quadratic invariant terms discussed earlier, these bilinears close the algebraic loop anticipated in the Introduction: the dynamics is governed by the simplest scalar densities constructible from local degrees of freedom.

Unified geometric picture

The emergence of gravitational and electromagnetic modes can therefore be summarized as follows:

- a single expansive scalar sector defines the accumulated background,
- differentiation produces local sensitivity to inhomogeneities,
- symmetry enforces radial and azimuthal projections,
- these projections yield force-like fields with characteristic scaling laws.

What distinguishes the resulting modes is not their origin but their geometric character. Gravitational and electromagnetic fields appear as complementary responses to the same underlying structure, shaped by how extension is distributed and transported.

Mathematical consistency

The emergence of nonzero $F_{\mu\nu}$ does not contradict the identity $\nabla \times \nabla\psi = 0$ for real scalar fields. The present construction relies on the distinction between locally exact and globally non-exact forms, a standard feature of gauge theories and topologically nontrivial configurations. The electromagnetic-like sector thus arises from geometric properties of the phase field rather than from a violation of basic differential identities.

Summary

Gravitational-like and electromagnetic-like fields arise inevitably as geometric projections of derivatives of an expansive scalar sector. Their defining features—central attraction, circulation, inverse-distance scaling, and source duality—follow from symmetry, dimensionality, and topology rather than from independent postulates. This projection-based understanding prepares the ground for analyzing coupling constants and scale factors as artifacts of dimensional translation, a task undertaken in the next subsection.

D. Universality of geometric invariants and scaling of coupling constants

Having established that gravitational-like and electromagnetic-like modes arise as geometric projections of an underlying expansive scalar sector, we now address the role of coupling constants. In conventional formulations, coupling constants are often introduced as independent empirical parameters characterizing distinct interactions. In the present framework, however, their origin admits a different and more constrained interpretation.

Invariants as the only admissible building blocks of the action

The action is a scalar quantity obtained by integrating a local density over a geometric domain. Consequently, only scalar quantities constructed from the available fields and their derivatives can contribute. Moreover, these scalars must be invariant under coordinate transformations in order for the variational principle to be well-defined.

This requirement severely restricts the admissible structures. Once an underlying scalar sector $\Psi_{\rm ent}$ is assumed, the only nontrivial local objects that can be constructed are its derivatives. The simplest invariant scalar density then takes the generic quadratic form

$$\mathcal{I} \;\sim\; (\partial_\mu \Psi_{\rm ent})(\partial^\mu \Psi_{\rm ent}), \tag{VII.16}$$

or, more generally, quadratic contractions of tensors derived from $\Psi_{\rm ent}$ through projection and differentiation.

The appearance of such quadratic invariants is therefore not a modeling choice but a mathematical necessity. Any consistent variational theory based on an expansive scalar sector must ultimately reduce to combinations of this type [34, 48, 59]. [3]

[3] This reliance on quadratic invariants mirrors the structural logic of arithmetic, where the interactions between integers are governed by a positive-definite kernel $\mathcal{P}(m, n) = \sum_p v_p(m)v_p(n)/p$ that quantifies structural overlap. In physics, the action plays the analogous role of the positive-definite invariant defining the theory's geometry.

Projection-dependent interpretations of the same invariant

Although the invariant $\mathcal{I}$ is unique at the geometric level, its physical interpretation depends on how it is projected and decomposed. Radial projections emphasize gradient magnitudes and lead naturally to gravitational-like interpretations, while azimuthal projections emphasize circulation and give rise to electromagnetic-like interpretations.

Crucially, these interpretations do not correspond to different invariants. They correspond to different geometric decompositions of the same underlying scalar quantity. The invariant remains unchanged; only its expression in terms of projected variables differs.

This observation implies that what are traditionally regarded as distinct interaction sectors share a common geometric origin. Their apparent independence arises from the way the invariant is parametrized and measured in different physical regimes [26, 59].

Scaling factors as dimensional translators

When the same geometric invariant is expressed in different projected forms, dimensional and numerical scaling factors necessarily appear. These factors serve to translate between:

- different units associated with distinct projections,
- different dimensional reductions induced by differentiation,
- different normalization conventions for the derived fields.

In this sense, coupling constants function as *dimensional translators*. They do not introduce new dynamical content but rescale the same invariant structure so that it may be consistently interpreted within different physical contexts.

The universality of coupling constants across a given sector reflects the universality of the underlying invariant, not the existence of independent fundamental interactions [49, 60].

Regimes and effective couplings

Because integration accumulates local contributions over a domain, the effective influence of an invariant may depend on scale. Different physical regimes—local versus global, microscopic versus macroscopic—correspond to different effective weightings of the same invariant contribution.

From this viewpoint, variations in effective coupling strength need not signal new physics. They may instead reflect changes in how the underlying invariant is sampled, accumulated, or projected across scales. This observation provides a natural conceptual basis for scale-dependent couplings without introducing additional degrees of freedom.

Implications for unification

The geometric interpretation of coupling constants has important unifying implications. If gravitational-like and electromagnetic-like modes arise from the same invariant structure, and if their couplings differ only by projection-dependent scaling factors, then unification does not require the introduction of new fundamental interactions. It requires only a recognition of the shared geometric origin of seemingly distinct sectors.

This perspective shifts the focus of unification away from force-based taxonomies and toward invariant-based structures. What is unified is not a collection of forces, but the geometric and variational principles that govern how extension, accumulation, and locality are expressed mathematically.

Dimensionless constants as geometric ratios

While dimensional coupling constants naturally arise as scale-setting factors associated with different projections and regimes, the present framework also constrains the possible origin of *dimensionless* constants. Since no additional scales can be introduced at the invariant level, such constants cannot encode new dynamics. Rather, they must reflect intrinsic geometric relations already present in the underlying structure.

If radial and azimuthal modes emerge as distinct projections of a common invariant sector, then the relative strength of these modes need not be arbitrary. In this perspective, a dimensionless constant such as the fine-structure constant α may be interpreted as encoding a ratio between geometric efficiencies of different projection channels.

Importantly, the framework restricts the admissible sources of such ratios. Dimensionless constants can only arise from elementary geometric invariants that are themselves scale-free and universal. Typical examples include:

- normalization factors associated with Green-kernel solutions of Laplace-type operators under isotropy and locality,
- dimensionless coefficients fixed by self-consistency or stability conditions of homogeneous modes,
- purely geometric factors determined by dimensionality, symmetry, or boundary structure.

These quantities are not introduced as free parameters, but are fixed once the mathematical structure of extension, accumulation, and projection is specified. No explicit derivation of α is attempted here; the purpose of this discussion is only to delimit the class of admissible origins for dimensionless couplings. Within this class, specific realizations may identify particular geometric invariants as physically relevant, while the general framework remains independent of such choices [54, 63].

Summary

Coupling constants emerge naturally as scaling factors associated with different projections and dimensional interpretations of a universal geometric invariant. They do not define independent interactions but encode how the same underlying expansive structure manifests across distinct regimes. This understanding prepares the ground for explicit realizations of the framework in concrete physical theories, which may be viewed as particular parameterizations of a common invariant-based architecture.

E. Regimes, scale separation, and effective couplings

In the previous subsection we argued that coupling constants may be interpreted as scaling and normalization factors translating a universal invariant structure into different projected descriptions. We now make this statement more precise by introducing the notion of *regimes* and showing that effective couplings arise generically as scale-dependent consequences of accumulation, projection, and coarse sampling of the same underlying invariant.

Why regimes are inevitable in an integral-based reality

A defining feature of the present framework is that physically meaningful quantities are accumulated by integration. Any integral over a domain Ω implicitly involves scale: the domain may be small or large, the integrand may vary rapidly or slowly, and the measure may weight contributions differently across regions.

As a result, even if the underlying invariant density is universal, its accumulated impact need not be uniform across scales. The existence of regimes is therefore unavoidable: different scales probe different effective portions of the same invariant content. [4] This provides a structural explanation for why physics admits distinct effective descriptions (local versus cosmological, microscopic versus macroscopic) without requiring different foundational principles in each regime [17, 34].

Local versus global accumulation

Let $\mathcal{I}(x)$ denote a universal invariant density constructed from the underlying expansive sector and its derived fields. Consider its accumulation over two domains, a local domain $\Omega_{\rm loc}$ and a global domain $\Omega_{\rm glob}$,

$$S_{\rm loc} = \int_{\Omega_{\rm loc}} \mathcal{I}(x)\, d^4x, \qquad S_{\rm glob} = \int_{\Omega_{\rm glob}} \mathcal{I}(x)\, d^4x. \tag{VII.17}$$

Even if $\mathcal{I}(x)$ has the same functional form in both cases, the effective influence of the accumulated contribution may differ because:

- $\mathcal{I}(x)$ can contain slowly varying background components and rapidly varying local components,
- the effective measure and boundary conditions weight these components differently across scales,
- global constraints and admissible variations differ between domains.

Hence, local and global regimes correspond to different effective reductions of the same invariant accumulation problem [41, 59].

Effective couplings as coarse-sampling coefficients

To formalize this idea, it is useful to invoke a separation-of-scales ansatz and write

$$\mathcal{I}(x) \;=\; \mathcal{I}_{\rm slow}(x) \;+\; \mathcal{I}_{\rm fast}(x), \tag{VII.18}$$

where $\mathcal{I}_{\rm slow}$ varies weakly over large domains and $\mathcal{I}_{\rm fast}$ captures localized structure. The precise split is not unique; it represents a regime-dependent coarse-graining choice.

In local regimes, admissible perturbations are supported on small neighborhoods, and the stationarity problem is dominated by $\mathcal{I}_{\rm fast}$. In global regimes, coherent accumulation of $\mathcal{I}_{\rm slow}$ over very large domains can dominate, even if its pointwise magnitude is small.

This naturally leads to *effective* invariant densities obtained by coarse sampling (averaging, integrating out fast structure, or restricting the class of admissible variations). Denoting the resulting regime-dependent effective density by $\mathcal{I}^{(R)}_{\rm eff}$, one has schematically

$$S^{(R)}_{\rm eff} \;=\; \int_{\Omega_R} \mathcal{I}^{(R)}_{\rm eff}(x)\, d^4x, \tag{VII.19}$$

and in many situations $\mathcal{I}^{(R)}_{\rm eff}$ admits an approximate representation of the form

$$\mathcal{I}^{(R)}_{\rm eff}(x) \;\approx\; \alpha_R\, \mathcal{I}(x), \tag{VII.20}$$

where α_R is an effective weighting/normalization factor induced by domain size, boundary constraints, and the relative dominance of slow versus fast components. Importantly, α_R does not represent a new fundamental interaction; it summarizes how a universal invariant is sampled and normalized in a given regime [60, 64]. Its value is fixed once the regime, admissible variations, and normalization conventions are specified.

Scale separation and apparent parameter multiplicity

A striking consequence is that a single universal invariant may appear to generate multiple "constants" when expressed in different effective theories. This is a natural outcome of scale separation: different regimes yield different effective normalizations and hence different apparent couplings, even though the underlying invariant structure is common.

Thus, multiplicity of parameters in effective physics may be interpreted not as evidence for independent fundamental sectors, but as an artifact of projecting, coarse-graining, and canonically normalizing the same invariant content under different accumulation conditions [60, 63].

Locality, boundary conditions, and regime-dependent stationarity

The stationarity condition $\delta S = 0$ depends not only on the integrand but also on the class of allowed variations. In local regimes, variations are typically restricted by local boundary conditions (or by the requirement that they vanish sufficiently fast at the boundary of $\Omega_{\rm loc}$). In global regimes, admissible variations may be constrained by global boundary terms, topology, or large-scale symmetry requirements.

Therefore, even with the same underlying invariant density $\mathcal{I}(x)$, the resulting effective equations of motion can differ in their apparent coupling strength because the variational problem is solved under different constraint classes. This is a purely variational mechanism by which regimes emerge [34, 48].

Analogy with spontaneous symmetry breaking

The regime separation above admits an interpretive analogy with spontaneous symmetry breaking. A decomposition into a homogeneous background component and local excitations—as in $\mathcal{L}_{\rm ent} = \rho_{\rm vac} + \delta\mathcal{L}_{\rm ent}$—mirrors the standard split between an order-parameter condensate and fluctuations. In this analogy, the phase variable introduced in Section VII C plays a role reminiscent of a Goldstone-like degree of freedom, while the homogeneous contribution behaves as a vacuum energy density. This remark is not required for the structural argument, but connects the invariant-based regime picture to familiar mechanisms in field theory.

Consequently, distinct physical regimes may be understood as different dominance patterns within the same invariant structure. The linear regime, in which quadratic terms dominate, corresponds to long-range propagating

[4] A sharp arithmetic analogue of this projection-based viewpoint is the existence of two *incompatible* spectral decompositions for the integers: a prime-exponent (multiplicative) basis in which multiplication is diagonal/translation-like, and a digit-expansion (additive, b-adic) basis in which addition is (nearly) local while multiplication becomes strongly mixing. Crucially, there is *no* representation in which both operations are simultaneously local or diagonal. This intrinsic obstruction provides a clean structural template for why a single invariant substrate, when probed through distinct operational channels (accumulation vs. extension), necessarily manifests as complementary effective modes rather than a single simultaneously-simple description.

modes such as gravitational and electromagnetic interactions. By contrast, the nonlinear regime, where self-multiplicative terms dominate, leads to saturation, localization, and finite-range behavior. The transition from infinite-range fields to confined or massive structures is therefore not a change of fundamental law, but a shift of algebraic regime within a unified variational framework [23, 60].

Summary

Regimes are inevitable in an integral-based description of physical reality. Because physical quantities arise through accumulation, scale separation and boundary constraints naturally produce different effective coarse samplings of a universal invariant density. Effective couplings therefore arise as regime-dependent normalization factors rather than as independent fundamental parameters. This regime-based viewpoint provides a bridge between local force-like physics and global expansive behavior, and it sets the stage for discussing explicit realizations in which local and global couplings become identifiable as distinct projections of a common underlying structure.

VIII. DISCRETE ACCUMULATION AND THE NECESSITY OF QUANTIZATION

The framework developed in the previous sections relies on multiplication to generate local extension and on integration to accumulate such extensions into global quantities. While this construction is naturally written in terms of continuous fields and integrals, it raises a deeper structural question: under what conditions does an accumulation process represent a *physically individuated* quantity rather than a purely formal analytic limit?

In this section, we argue that physically meaningful accumulation requires a notion of *elementary contribution* that cannot be resolved indefinitely. Quantization is therefore not introduced as an additional dynamical hypothesis, but emerges as a structural requirement for accumulation to possess objective content.

A. Minimal resolution and the individuation of accumulation

By definition, an integral is the limit of a discrete sum,

$$\int_\Omega \mathcal{L}(x)\, d^4x \;=\; \lim_{\Delta V\to 0} \sum_i \mathcal{L}(x_i)\, \Delta V_i. \tag{VIII.1}$$

Each term is an elementary contribution built as a product between a local density and an elementary measure element. *This statement is purely mathematical.* The structural issue is that, in physics, an "accumulated quantity" is not merely a convergent limit: it is meant to count or encode *something that is present* in a configuration. For accumulation to have such individuation, its elementary contributions must admit a minimal resolution [18, 34].

If elementary contributions can be subdivided without end, accumulation becomes purely formal: there is no intrinsic scale at which a contribution becomes a distinguishable "unit" of content. To avoid this, we introduce a minimal *resolution* of accumulated extension, denoted by S_0, such that changes in the accumulated quantity below this scale are not physically distinguishable:

$$\delta S \;\sim\; S_0. \tag{VIII.2}$$

Equivalently, one may state that physically individuated accumulation proceeds in discrete steps,

$$S \;=\; N\, S_0 \;+\; \mathcal{O}(S_0)\,, \qquad N \in \mathbb{Z}, \tag{VIII.3}$$

where the remainder represents coarse-grained uncertainty below the resolution scale. The constant S_0 is not introduced here as a coupling or a model parameter; it is the minimal unit required for accumulation to represent individuated physical content rather than an infinitely divisible abstraction [18, 50].

The existence of a minimal resolution does not contradict the use of continuous fields or integrals. Rather, continuum calculus provides an effective description valid when many elementary contributions are accumulated and $S \gg S_0$. In that regime, discrete steps become unresolvable and the integral approximation becomes accurate.

In this sense, *continuity is not fundamental but emergent: it arises as the coarse-grained limit of discrete accumulation.* The variational calculus developed in previous sections operates at this effective level, while the underlying discreteness ensures that accumulation has physical individuation [31, 36]. [5]

B. Consequences: Multi-scale measures, ordering, composition, and non-commutativity

Once accumulation is recognized as fundamentally discrete, it is natural to allow that the effective measure governing accumulation may be scale-dependent. In particular, the "volume element" μ need not behave as a smooth four-dimensional measure at all scales; it may exhibit multi-scale behavior in which the effective dimensionality depends on resolution.

At the structural level, this can be represented by allowing a scale-dependent measure μ_ℓ (or an effective Hausdorff/Minkowski scaling) such that accumulated quantities interpolate between different effective dimensions as the resolution scale ℓ varies. The variational framework remains intact at macroscopic scales, while the discrete accumulation picture naturally accommodates scenarios in which spacetime exhibits fractal-like or multi-scale geometric properties at sufficiently small scales. This provides a natural setting in which modified power laws and scale-dependent effective couplings arise as consequences of accumulation on a nontrivial measure, without altering the operational cycle itself [3, 12].

Additionally, once accumulation is understood as composition of discrete elementary contributions, ordering can become physically relevant whenever contributions encode geometric transport or phase information. In particular, if elementary steps involve local phase transport (as in the connection-like structures of Section VII C), then composing steps around different loops can lead to nontrivial holonomy.

In such cases, the composition law for elementary contributions is naturally non-commutative: different orderings correspond to different accumulated geometric configurations. Non-commutativity thus appears as a structural possibility of discrete accumulation with internal phase/connection content, rather than as an independent postulate. [6] [16, 47].

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[5] Integer arithmetic provides a rigorous model for this emergence: the prime spectrum constitutes an intrinsically discrete "space" where continuity is absent at the fundamental level, yet statistical regularities (like the Erdős–Kac theorem) emerge effectively at large scales.

[6] In the arithmetic setting, discreteness and ordering acquire an explicit dynamical mechanism: in a digit (additive) basis, addition is local except for *carry propagation*, whose chain length is governed by p-adic valuations and exhibits geometric-tail statistics (e.g. for binary carries). Dually, in a prime (multiplicative) basis, the additive shift acts as a globally mixing operator producing large valuation jumps. This provides a concrete model of how discrete accumulation can generate rare but arbitrarily long-range updates, naturally suggesting multi-scale (and potentially fractal-like) effective structure in any continuum limit built from such elementary steps

C. Preparation for a quantum of action

The minimal resolution scale S_0 characterizes the granularity of accumulation. Its numerical value and detailed physical interpretation are not fixed at this stage. What matters structurally is its existence as the scale that individuates accumulation. In subsequent developments, S_0 may be identified with a fundamental quantum of action, thereby connecting the accumulation principle to quantum formulations. At the present level, however, the role of S_0 is to ensure that extension, accumulation, and the variational extraction of dynamics are grounded in discrete elementary contributions rather than in purely formal limits.

Remark VIII.1. *The integer winding numbers n associated with phase transport (Section VII C) and the discrete accumulation steps implied by* (VIII.3) *point to the unified structural origin for quantum discreteness. In standard formulations, the phase of a complex amplitude is proportional to an action measured in units of a universal constant [19, 50]. Here the correspondence can be read in reverse: if winding is intrinsically quantized and accumulation is intrinsically discrete, then a universal conversion factor relating geometric phase transport to accumulated action is naturally expected to emerge as the bridge between the two discretizations.*

IX. A CONCRETE REALIZATION: THE ARITHMETIC–GEOMETRIC SUBSTRATE

The previous sections established that physically meaningful accumulation requires discreteness, multi-scale structure, and topological individuation. We now show that these requirements admit a concrete realization in terms of a background-free arithmetic geometric substrate.

A. Background-Free Geometry and Arithmetic Constraints

1. Ontological Motivation and Background Independence

Any fundamental physical theory aiming at ultimate parsimony must confront the question of background independence. If a physical substrate is to be truly fundamental, it cannot presuppose an external stage on which dynamics unfolds. Otherwise, the assumed background would itself demand explanation. By a standard ontological minimality argument, the ultimate substrate must therefore be *background-free.*

In such a setting, what remains as the minimal physical content is not a field *on* space, but rather space—or more precisely, relational extension itself—as the primary entity. Without extension, differentiation, or relational structure, no physical observables can be defined. Consequently, the fundamental substrate must be intrinsically geometric in nature, not as a fixed metric manifold, but as the very condition of physical existence.

2. From Geometry to Topology in the Fundamental Regime

At the most fundamental level, geometric structure cannot be assumed to be smooth or metric. Metrics encode notions of length, scale, and units, all of which are emergent concepts requiring prior structure. In the absence of such assumptions, the appropriate language is topological rather than metrical.

Moreover, the fundamental regime cannot be linear. Linearity is a hallmark of effective descriptions, valid only after suitable coarse-graining. The underlying substrate must therefore be described by non-linear, relational, and topological structures, characterized by connectivity, equivalence classes, and global invariants rather than local distances.

3. Elliptic Curves, Fractals, and Non-Linear Geometry

Within non-linear topology and geometry, certain mathematical structures arise naturally as minimal yet non-trivial carriers of global structure. Elliptic curves play a distinguished conceptual role in this context, not as literal spacetime geometries, but as archetypal minimal non-linear structures supporting global consistency, iteration, and protected modes. Topologically simple (genus one), yet algebraically rich, they admit internal composition laws, iteration, and uniformization. As such, they represent minimal non-linear geometric objects capable of supporting stable global modes.

Fractal geometries, on the other hand, embody scale-free structure. They are characterized not by smoothness, but by self-similarity under iteration and the absence of a preferred length scale. In a background-free setting, where no fundamental scale is given *a priori*, fractal-like structures are not pathological but rather natural candidates for the microscopic geometry of the substrate.

Importantly, smooth manifolds should then be understood as effective descriptions emerging from suitable coarse-graining of an underlying non-smooth, possibly fractal, geometric substrate.

4. Arithmetic Constraints as Geometric Selection Principles

If physical reality emerges from a topological and non-linear geometric substrate, not all conceivable geometries can correspond to physically realizable configurations. Only those structures that are globally stable under composition and iteration can persist as effective degrees of freedom.

Here, deep arithmetic constraints acquire physical relevance. Principles such as the *abc* conjecture express fundamental limitations on the coexistence of additive cancellation and multiplicative simplicity. In essence, they prohibit arbitrarily fine additive cancellations without an accompanying increase in global multiplicative complexity.

Interpreted geometrically, such constraints act as global selection rules. They restrict the persistence of highly fine-tuned, self-similar configurations under repeated composition. In this sense, *abc*-type bounds function as anti-fine-tuning principles, limiting the class of admissible geometric patterns that can survive across scales.

5. Emergent Physics and Mode Selection

Within this framework, physical entities such as waves, particles, and masses correspond to topologically stable excitations or modes of the underlying substrate. These modes are not imposed externally but arise endogenously as robust features under coarse-graining.

While the microscopic geometry may be fractal or highly irregular, symmetry requirements such as isotropy and homogeneity enforce universality at large scales. Under such conditions, effective operators—notably the Laplacian—emerge generically as equilibrium descriptions, yielding familiar inverse-distance laws and continuum field equations.

6. Summary

Summary. If the ultimate physical substrate is background-free, it must be intrinsically geometric and relational. At the fundamental level, this geometry is necessarily topological and non-linear, admitting fractal-like microstructure and minimal non-trivial configurations such as elliptic curves. Physical phenomena then emerge as topologically stable modes of this substrate under coarse-graining. Deep arithmetic constraints, exemplified by *abc*-type anti-cancellation principles, act as global selection rules restricting which geometric configurations are physically realizable. In this view, the laws of physics reflect not arbitrary dynamics, but the subset of geometric structures that remain stable under both topological and arithmetic constraints.

B. A Hybrid Toy Model: Arithmetic–Geometric Emergence via a Discrete Action

1. Arithmetic substrate as a weighted graph

Let $G = (V, E)$ be a connected, locally finite, weighted graph. We interpret G as a discrete, background-free relational substrate: geometry is encoded by adjacency and edge weights rather than by an external metric.

We take vertices to be positive integers $V = \{1, 2, \ldots, N\}$ (with N large) and define the edge set as the union

$$E \;=\; E_{\times} \,\cup\, E_{+}.$$

The *multiplicative* edges encode structure-preserving scaling by a finite set of primes $\mathcal{P}$:

$$(i,j) \in E_{\times} \quad\Longleftrightarrow\quad \exists\, p \in \mathcal{P} :\; j = pi \text{ or } i = pj.$$

The *additive* edges encode mixing via primitive additive relations. Concretely, we include a finite set of primitive triples $\mathcal{T}$ with $a + b = c$ and $\gcd(a, b, c) = 1$, and connect their endpoints by edges, e.g.

$$(a,b), (a,c), (b,c) \in E_{+} \quad \text{for each } (a,b,c) \in \mathcal{T}.$$

Assign positive symmetric weights $w_{ij} = w_{ji} > 0$ to each edge $(i, j) \in E$, allowing different weight schemes on $E_{\times}$ and E_{+}.

2. Discrete action and Euler–Lagrange equation (graph Laplacian)

Let $\phi : V \to \mathbb{R}$ be a scalar field on the substrate. Consider the discrete Dirichlet-type action

$$S[\phi] \;:=\; \frac{1}{2} \sum_{(i,j)\in E} w_{ij}\,(\phi_i - \phi_j)^2 \;+\; \sum_{i\in V} U(\phi_i),$$

where U is a local potential.

A standard variation with respect to ϕ_k yields the discrete Euler–Lagrange condition

$$\sum_{j:(k,j)\in E} w_{kj}(\phi_k - \phi_j) \;+\; U'(\phi_k) \;=\; 0.$$

Define the weighted graph Laplacian

$$(\Delta_G \phi)_k \;:=\; \sum_{j:(k,j)\in E} w_{kj}(\phi_k - \phi_j).$$

Then the field equation becomes

$$\Delta_G \phi \;+\; U'(\phi) \;=\; 0,$$

which is the discrete analogue of an elliptic Euler–Lagrange equation $\Delta\Phi + U'(\Phi) = 0$.

3. Time-dependent extension and wave-like modes

To exhibit propagating modes, introduce a time parameter t and take $\phi = \phi(t)$ with kinetic term

$$S_{\rm dyn}[\phi] \;:=\; \int dt \left[\frac{1}{2}\sum_{i\in V} m_i\, \dot\phi_i^2 - \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(\phi_i - \phi_j)^2 - \sum_{i\in V} U(\phi_i)\right],$$

with inertial weights $m_i > 0$. The Euler–Lagrange equations are

$$m_i\, \ddot\phi_i \;+\; (\Delta_G\phi)_i \;+\; U'(\phi_i) \;=\; 0.$$

In the linear regime $U(\phi) = \frac{1}{2}\mu^2\phi^2$, this becomes a discrete Klein–Gordon equation

$$m_i\, \ddot\phi_i \;+\; (\Delta_G\phi)_i \;+\; \mu^2\phi_i \;=\; 0,$$

whose normal modes are eigenvectors of Δ_G. Thus, "particles/waves" correspond to stable eigenmodes of the relational Laplacian.

4. Coarse-graining and emergence of a continuum Laplacian

Partition V into blocks (cells) $\{C_\alpha\}$ representing coarse degrees of freedom and define the block field

$$\Phi_\alpha \;:=\; \frac{1}{|C_\alpha|} \sum_{i\in C_\alpha} \phi_i.$$

Under repeated coarse-graining, microscopic irregularities of the adjacency structure are suppressed. If, at large scales, the induced connectivity is statistically homogeneous and isotropic (in the sense that effective couplings depend only on a coarse notion of separation), then the renormalized quadratic form converges to the universal continuum Dirichlet form

$$\frac{1}{2}\sum_{(i,j)\in E} w_{ij}(\phi_i - \phi_j)^2 \;\longrightarrow\; \frac{Z}{2}\int d^d x\, |\nabla\Phi|^2,$$

so that

$$\Delta_G \;\longrightarrow\; Z\,\Delta$$

as an effective operator on an emergent space of (possibly scale-dependent) spectral dimension d. In the static regime, the Green's function of Δ yields inverse-distance behavior (e.g. $1/r$ in $d = 3$), showing how familiar continuum laws can arise from a discrete, non-metric substrate.

5. abc-type constraints as anti-fine-tuning selection rules

The additive edge set E_+ encodes cancellations through primitive relations $a + b = c$. If such relations were allowed to exhibit arbitrarily strong cancellation across scales while keeping multiplicative complexity bounded, the substrate could support pathologically fine-tuned, nearly self-similar configurations.

To model a global restriction against such persistent fine-tuning, impose an *abc*-type admissibility criterion on $\mathcal{T}$. For fixed $\varepsilon > 0$, retain only triples $(a, b, c) \in \mathcal{T}$ satisfying

$$c \;\le\; \mathrm{rad}(abc)^{\,1+\varepsilon},$$

where $\mathrm{rad}(n)$ is the product of distinct prime divisors of n. Operationally, this prunes additive edges that would implement unusually strong cancellations without corresponding multiplicative diversification. In this toy model, such pruning acts as an *anti-fine-tuning* selection principle: it limits the persistence of exact self-similarity generated by additive interactions, while leaving multiplicative scaling structure intact.

6. Role of the toy model

This construction provides an explicit hybrid demonstration of the central thesis. A background-free substrate can be encoded as a discrete relational graph whose microscopic generation mixes multiplicative (structure-preserving) and additive (structure-mixing) operations. A standard variational principle on this substrate produces a Laplacian field operator (and its dynamical wave-like extension). Coarse-graining yields a universal continuum Laplacian, while *abc*-type anti-cancellation constraints naturally enter as global selection rules restricting which fine-tuned additive patterns are admissible across scales.

C. Computational consistency check (finite-N validation)

To validate the internal consistency of the toy construction at finite size, we implemented the weighted arithmetic graph $G = (V, E)$ with $V = \{1, \ldots, N\}$ and

$$E = E_\times \cup E_+,$$

where $E_\times$ contains multiplicative edges $i \leftrightarrow pi$ for $p \in \mathcal{P}$, and E_+ contains additive triangles induced by primitive relations $a + b = c$ with $\gcd(a, b, c) = 1$.

We constructed the weighted Laplacian Δ_G from the quadratic Dirichlet form

$$Q[\phi] \;=\; \frac{1}{2} \sum_{(i,j)\in E} w_{ij}\,(\phi_i - \phi_j)^2,$$

and verified numerically that Δ_G is symmetric and positive semidefinite, with a near-zero eigenvalue corresponding to the constant mode (or one per connected component). In the dynamical extension, the normal modes are Laplacian eigenvectors, as expected from the discrete Euler–Lagrange equations.

To probe the role of arithmetic anti-fine-tuning, we compared spectra under different additive edge ensembles. For dense additive connectivity (all primitive triples with $c \leq N$), *abc*-type pruning removes only a negligible fraction of edges at small N, reflecting the rarity of high-quality additive cancellations. In a sparse regime (random subsampling of additive triangles), we further injected engineered "super-canceling" relations of the form $(1, 2^k - 1, 2^k)$ with enhanced weights; these modify low-lying Laplacian eigenvalues and thus the long-wavelength mode structure. An *abc*-type admissibility criterion naturally eliminates such edges, illustrating its interpretation as an anti-fine-tuning selection rule.

Overall, the numerical experiment supports the central claim: the Laplacian operator and its mode decomposition arise robustly from the variational principle, while deep arithmetic constraints can be implemented as global selection rules that suppress pathological cancellation patterns without affecting the universality of the coarse-grained continuum behavior.

D. Spectral and Green–Function Diagnostics of Emergence

1. Effective Green's Function and Inverse-Distance Behaviour

A direct diagnostic of the emergent geometry is provided by the Green's function associated with the coarse-grained Laplacian. On a connected component of the discrete substrate, consider the static equation

$$\Delta_G \phi = \delta_{i_0},$$

where δ_{i_0} denotes a localized source at a vertex i_0. After coarse-graining over blocks $\{C_\alpha\}$, the corresponding block field Φ_α defines an effective potential on the emergent space.

Numerical evaluation on finite realizations of the arithmetic graph shows that, at sufficiently large coarse-grained distances, the resulting potential decays monotonically and is well approximated by an inverse-distance profile, consistent with the Green's function of a Laplacian in an effective continuum dimension. Microscopic irregularities and fractal connectivity modify the short-distance behaviour, but do not affect the long-distance scaling, illustrating the universality of the Laplacian response under coarse-graining.

2. Spectral Dimension from the Heat Kernel

Complementary information on the emergent geometry is obtained from the spectral dimension, defined through the heat kernel trace

$$K(t) = \mathrm{Tr}\, e^{-t\Delta_G} = \sum_n e^{-t\lambda_n},$$

where $\{\lambda_n\}$ are the eigenvalues of the Laplacian on the largest connected component. The scale-dependent spectral dimension is given by

$$d_s(t) = -2\,\frac{d\log K(t)}{d\log t}.$$

In the arithmetic toy model, numerical evaluation of $d_s(t)$ exhibits a clear crossover behaviour. At short diffusion times t, the effective dimension reflects the irregular, partially fractal microstructure of the substrate. At larger t, $d_s(t)$ stabilizes to a nearly constant value compatible with an integer dimension, signalling the emergence of an effective smooth geometry at macroscopic scales.

3. Effect of abc-Type Pruning

When additive edges associated with unusually strong cancellations are artificially enhanced, the low-lying spectrum of Δ_G becomes distorted, leading to anomalously localized or fine-tuned modes. Imposing an *abc*-type admissibility criterion removes precisely these pathological additive connections.

Numerically, this pruning restores smooth spectral behaviour while leaving the large-scale Green's function profile and the asymptotic spectral dimension unchanged. This confirms that *abc*-type bounds act as global anti-fine-tuning selection rules: they suppress unstable microscopic configurations without affecting the universal macroscopic physics.

4. Interpretation

Together, the Green's function profile and the flow of the spectral dimension provide concrete, computable diagnostics of the emergence mechanism proposed in this work. They demonstrate that a background-free, arithmetic–topological substrate can support a fractal microstructure while still giving rise, through coarse-graining, to universal Laplacian dynamics and effective continuum behaviour. Deep arithmetic constraints enter naturally as consistency conditions ensuring the stability of this emergence against pathological fine-tuning.

E. Topological Defects as Protected Eigenmodes (Explicit Demonstration)

1. Bipartite relational substrate and chiral symmetry

To exhibit a genuinely *topologically protected* mode, it is convenient to consider a bipartite (two-sublattice) relational substrate. Let $V = V_A \sqcup V_B$ and let the microscopic couplings connect only A-sites to B-sites. This enforces a chiral (sublattice) symmetry: there are no on-site terms.

Define a first-order (Dirac-type) operator D mapping fields on V_A to fields on V_B, and consider the quadratic action

$$S[\psi] \;=\; \langle D\psi, D\psi\rangle,$$

so that the associated Euler–Lagrange operator is the positive semidefinite Laplacian-like operator

$$\Delta \;=\; D^\dagger D.$$

A zero mode of D yields an eigenmode of Δ at eigenvalue 0. Such zero modes can be *topologically protected* when the substrate supports distinct gapped phases characterized by a winding number.

2. Domain wall and protected midgap mode (SSH prototype)

Consider the 1D bipartite chain (SSH model) with unit cells $n = 1, \ldots, L$ and sites (A_n, B_n). Couplings are of two types: (i) intra-cell $A_n \leftrightarrow B_n$ with strength t_1, and (ii) inter-cell $B_n \leftrightarrow A_{n+1}$ with strength t_2. In the bulk, the system is gapped for $t_1 \neq t_2$ and admits a $\mathbb{Z}$ winding number ν (equivalently a Zak phase).

Create a *domain wall* by choosing $(t_1, t_2) = (t_1^{(L)}, t_2^{(L)})$ on the left half and $(t_1, t_2) = (t_1^{(R)}, t_2^{(R)})$ on the right half with $\nu_L \neq \nu_R$. Then bulk–boundary correspondence implies the existence of a localized midgap eigenmode at the interface. This mode is protected: it cannot be removed by any local perturbation that preserves chiral symmetry and keeps the bulk gap open.

3. Robustness and physical interpretation

Numerically, the domain wall produces an eigenvalue $|E| \ll 1$ (midgap/zero mode) with an eigenvector exponentially localized at the defect. Under symmetry-preserving disorder (random fluctuations of couplings without on-site terms), the mode persists and remains localized. It can disappear only if the perturbation breaks chiral symmetry (e.g. on-site potentials) or closes the gap.

In the present framework, the domain wall is a microscopic topological defect of the relational substrate, while the protected eigenmode is a robust, particle-like excitation emerging endogenously from topology rather than from fine-tuned dynamics.

Thus, topologically protected eigenmodes provide a concrete mechanism by which quantization emerges as sector individuation in a variational framework, rather than as an independent postulate.

X. DISCUSSION AND RELATION TO CONCRETE REALIZATIONS

The aim of this work has been to derive the *structural form* of physical law from minimal operational principles, rather than to propose a specific microphysical model. Starting from multiplication, integration, and variation as generators of extension, accumulation, and locality, we obtained a constrained blueprint: any consistent "field reality" must be formulated in terms of local densities accumulated into a scalar action and constrained by stationarity. This section summarizes the main conceptual results, clarifies what is structural versus contingent, and explains how concrete theories may be read as realizations of the same invariant architecture.

A. Summary of structural results

The analysis supports the following conclusions:

- **Extension is multiplicative.** Multiplication is the primitive operation that generates geometric extension; physically meaningful quantities therefore require products of local degrees of freedom.
- **Accumulation is integral.** Integration is the unique consistent accumulation of local extension across a domain, elevating the action to the primary global quantity.
- **Local laws are variational.** Differentiation/functional variation is the only operation that can extract local constraints from globally accumulated quantities, making Euler–Lagrange-type dynamics inevitable.
- **Completeness requires an expansive sector.** Restricting the action to purely contractive (restoring) contributions defines a special subclass; allowing the full algebraic logic compels an expansive sector with an intensive–extensive product structure contributing intrinsically to the global action.[7]
- **Force-like modes are derived.** Gravitational-like and electromagnetic-like responses arise as geometric projections of derivatives (radial vs. azimuthal/holonomy) of the expansive sector, not as independent primitives.
- **Couplings are translators.** Apparent coupling constants can be interpreted as scaling/normalization factors associated with projection, dimensional reduction, and regime sampling of a common invariant content.

These results are structural: they depend on extension, accumulation, and locality alone, not on a prior choice of spacetime ontology, particle taxonomy, or gauge group.[8]

Remark X.1. *Just as zero is the neutral element of addition—the primitive operation of accumulation—the physical vacuum may be understood as the neutral configuration of the substrate: a state of vanishing deformation that nonetheless encodes the full potential for geometric extension under multiplicative composition.*

Remark X.2. *Since multiplication defines geometric extension as the sole fundamental magnitude, the distinction between different dimensions / categories that humans use to measure and describe reality (mass, energy, length, time, etc) is revealed as an artifact of scale and units conversion, not ontology. Consequently, physical constants of nature are ratios of the substrate's geometric properties, encoding conversion between equivalent geometric descriptions of the same underlying extension and accumulation structure.*

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[7] Furthermore, the generative nature of addition (accumulation) acting upon a multiplicative geometric substrate acts as a driver of spectral expansion. Just as additive operations on integers generate new prime content ('additive innovation'), the integration of geometric densities intrinsically drives the generation of entropy and cosmic expansion, forcing the existence of the thermo-entropic sector.

[8] The same minimality philosophy can be expressed in terms of *structural orthogonality*: rather than positing specific entities, one defines independence by the vanishing of overlap under a sufficiently rich family of evaluations. In this view, "physical distinctness" is captured by invariant orthogonality conditions, while dynamics emerges from how invariant overlaps change under accumulation and variation. This provides a mathematically economical notion of independence compatible with the invariant-based architecture emphasized throughout.

B. Minimality of axioms and what is *not* assumed

A key methodological point is that we do not assume: (i) a specific microscopic substrate, (ii) specific particle species, (iii) a preferred interaction list, or (iv) a particular quantization postulate. Instead, the framework isolates a small set of unavoidable requirements: (a) local extension must be generated, (b) global content must be accumulated, and (c) local constraints must be extracted from the accumulated content. [9] Everything else is contingent: it belongs to the choice of realization (field content, potentials, defect structure, boundary conditions, and regime structure).

In this sense, the blueprint provides a diagnostic: it separates what is forced by the operational cycle from what is a modeling choice.

C. Quantization as individuation of accumulation

Section VIII introduced the idea that accumulation has physical content only if it admits individuation at a minimal resolution scale. The key logical point is not that the continuum integral is "wrong", but that it must be understood as an effective description of a discrete accumulation process when $S \gg S_0$. This viewpoint naturally aligns with phase-based constructions (winding/holonomy) in which topological labels discretize admissible sectors.

In a concrete realization, discreteness can arise because finite-action (or finite-energy) conditions split configuration space into disconnected sectors labeled by topological invariants. In that case, "quantum numbers" correspond to sector labels, while "masses" correspond to variational minima inside each finite-action sector. This mechanism supplies a structural route to quantization grounded in the variational architecture itself.

Remark X.3. *This discrete resolution is not an arbitrary add-on. As shown in our complementary arithmetic analysis, the multiplicative (prime-power) spectral basis—the natural basis for geometric extension—is intrinsically incompatible with the additive (digit / binary) basis—the natural basis for accumulation by summation. This incompatibility yields an* arithmetic uncertainty principle*: a system cannot be simultaneously localized in its multiplicative configuration and in its additive accumulation history. When translated into the variational setting, this structural limit manifests as a non-zero minimal resolution of accumulated action, hence necessitating a quantum of action $S_0 \neq 0$ as the individuating unit of accumulation.*

D. Regimes, multiscale geometry, and fractal effective measures

The regime dependence discussed previously (local/global, IR/UV) is an unavoidable consequence of integral-based accumulation: different domains and boundary constraints sample the same invariant density differently. A natural strengthening of this idea is that the *effective measure* controlling accumulation may itself become scale-dependent. In that case, departures from smooth integer-dimensional scaling can be interpreted as an emergent regime of the underlying structure, rather than as a new ontology.

Operationally, the blueprint remains unchanged: local extension is still generated multiplicatively, accumulated into an action, and constrained by variation. What changes across scales is the effective sampling of the invariant content, which can manifest as running effective couplings and modified power-law behavior (including fractal or multifractal patterns) in appropriate coarse-grained regimes.

Remark X.4. *Within the present framework, second-order wave operators arise naturally from the quadratic variational structure of the action. However, for modes carrying intrinsic orientation or torsional topological charge, the preservation of phase and orientation information suggests a linearization of the underlying second-order operator. In this sense, Dirac-type equations appear not as independent postulates, but as the linear compatibility conditions for oriented modes propagating on a quadratic substrate.*

E. Relation to concrete realizations: what a realization must provide

The framework is intentionally abstract: it does not fix the detailed form of the expansive sector, nor the numerical values of projection factors. A concrete realization must therefore specify:

- an explicit *underlying field structure* whose scalar/invariant content defines the accumulated action,
- a *projection dictionary* identifying radial (gradient) and azimuthal (transport/holonomy) modes as derived contractive responses,
- a *regime structure* (linear/quadratic, massive, nonlinear) explaining localization, finite range, and stability,
- boundary/regularity conditions that define the admissible class of variations and (when relevant) split configuration space into discrete sectors.

Within these constraints, many distinct models can qualify as realizations.

F. Quantum-elastic and gravito-entropic realizations (example class)

Quantum-elastic / gravito-entropic field theories provide an instructive example class. In these realizations, a unified deformation structure (e.g. a symmetric tensorial substrate) admits projected modes that reproduce long-range gravity-like and electromagnetism-like sectors as elastic responses in the linear regime, while an expansive scalar sector encodes thermo-entropic dynamics and vacuum-like contributions. In addition, nonlinear self-interaction potentials can produce massive (finite-range) propagation and confinement-like filamentary minima without introducing new fundamental fields beyond the substrate degrees of freedom.

From the perspective of the present paper, the interest of such realizations is conceptual: they make explicit the dictionary between (i) a primitive expansive sector, (ii) derived contractive projected modes, (iii) regime-dependent effective couplings, and (iv) quantization via finite-action topological sectors and circulation/holonomy constraints. In particular, the appearance of quantized action in defect sectors and the emergence of discrete excitation families can be read as concrete implementations of the individuation principle of accumulation.

G. Outlook: programmatic implications

By shifting focus from "forces" to operations and invariants, the framework suggests a program:

[9] In this view, the foundational notion is not "force" or "interaction", but *structural independence*, formalizable as orthogonality under a complete family of evaluations. Physical independence is thus recovered as a geometric condition of the substrate.

- **Classification problem:** classify admissible realizations by their invariant content, projection dictionaries, and sector topology (rather than by assumed interaction lists).
- **Quantitative closure:** derive numerical scaling factors (effective couplings) from geometric normalization, boundary/regularity constraints, and sector stability.
- **Multiscale predictions:** identify observational signatures of regime transitions (including possible scale-dependent effective dimensionality) as changes in the sampling of the same invariant content.

Regardless of realization, the operational cycle—extension by multiplication, accumulation by integration, locality by variation—remains the irreducible skeleton on which any consistent formulation of physical law must be built.

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[1] R. Abraham and J. E. Marsden. *Foundations of Mechanics.* Benjamin/Cummings, 1978.

[2] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. *Physical Review*, 115(3):485–491, 1959.

[3] J. Ambjørn and R. Loll. Non-perturbative lorentzian quantum gravity, causality and topology change. *Nuclear Physics B*, 536:407–434, 1998.

[4] M. Atiyah. Geometry and physics. *Bulletin of the American Mathematical Society*, 81:1–31, 1975.

[5] L. E. Ballentine. *Quantum Mechanics: A Modern Development.* World Scientific, 1998.

[6] J. D. Bekenstein. Black holes and entropy. *Physical Review D*, 7:2333–2346, 1973.

[7] J. S. Bell. On the einstein podolsky rosen paradox. *Physics*, 1:195–200, 1964.

[8] M. Born. Zur quantenmechanik der stoßvorgänge. *Zeitschrift für Physik*, 37:863–867, 1926.

[9] R. Bott and L. W. Tu. *Differential Forms in Algebraic Topology*, volume 82 of *Graduate Texts in Mathematics.* Springer, 1982.

[10] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics I.* Springer, 1987.

[11] J. Butterfield. On the persistence of particles. *Foundations of Physics*, 37:415–458, 2007.

[12] G. Calcagni. Fractal universe and quantum gravity. *Physical Review Letters*, 104:251301, 2010.

[13] H. B. Callen. *Thermodynamics and an Introduction to Thermostatistics.* John Wiley & Sons, 2 edition, 1985.

[14] P. M. Chaikin and T. C. Lubensky. *Principles of Condensed Matter Physics.* Cambridge University Press, 1995.

[15] R. Clifton and H. Halvorson. Entanglement and open systems in algebraic quantum field theory. *Studies in History and Philosophy of Modern Physics*, 32:1–31, 2001.

[16] A. Connes. *Noncommutative Geometry.* Academic Press, 1994.

[17] R. Courant and D. Hilbert. *Methods of Mathematical Physics*, volume 1. Wiley, 1989.

[18] P. A. M. Dirac. Quantised singularities in the electromagnetic field. *Proceedings of the Royal Society of London. Series A*, 133(821):60–72, 1931.

[19] R. P. Feynman and A. R. Hibbs. *Quantum Mechanics and Path Integrals.* McGraw–Hill, 1965.

[20] A. Fine. Hidden variables, joint probability, and the bell inequalities. *Physical Review Letters*, 48:291–295, 1982.

[21] A. M. Gleason. Measures on the closed subspaces of a hilbert space. *Journal of Mathematics and Mechanics*, 6:885–893, 1957.

[22] H. Goldstein, C. Poole, and J. Safko. *Classical Mechanics.* Addison-Wesley, 2002.

[23] J. Goldstone. Field theories with superconductor solutions. *Nuovo Cimento*, 19:154–164, 1961.

[24] R. Haag. *Local Quantum Physics.* Springer, 1996.

[25] R. Healey. Quantum theory: A pragmatist approach. *British Journal for the Philosophy of Science*, 63:729–771, 2012.

[26] F. W. Hehl and Y. N. Obukhov. *Foundations of Classical Electrodynamics: Charge, Flux, and Metric.* Birkhäuser, Boston, 2003.

[27] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. *Reviews of Modern Physics*, 81:865–942, 2009.

[28] C. J. Isham. *Lectures on Quantum Theory.* Imperial College Press, 1995.

[29] J. D. Jackson. *Classical Electrodynamics.* Wiley, 1999.

[30] T. Jacobson. Thermodynamics of spacetime: The einstein equation of state. *Physical Review Letters*, 75:1260–1263, 1995.

[31] E. T. Jaynes. Information theory and statistical mechanics. *Physical Review*, 106:620–630, 1957.

[32] T. W. B. Kibble. Topology of cosmic domains and strings. *Journal of Physics A: Mathematical and General*, 9(8):1387–1398, 1976.

[33] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*, volume 1. Wiley, 1963.

[34] C. Lanczos. *The Variational Principles of Mechanics.* Dover Publications, 1970.

[35] L. D. Landau and E. M. Lifshitz. *Mechanics*, volume 1 of *Course of Theoretical Physics.* Pergamon Press, 1976.

[36] L. D. Landau and E. M. Lifshitz. *Statistical Physics.* Pergamon Press, 1980.

[37] R. Landauer. Irreversibility and heat generation in the computing process. *IBM Journal of Research and Development*, 5:183–191, 1961.

[38] J. Maldacena and L. Susskind. Cool horizons for entangled black holes. *Fortschritte der Physik*, 61:781–811, 2013.

[39] N. Manton and P. Sutcliffe. *Topological Solitons.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004.

[40] N. D. Mermin. The topological theory of defects in ordered media. *Reviews of Modern Physics*, 51(3):591–648, 1979.

[41] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation.* W. H. Freeman, 1973.

[42] J. Moreno Borrallo. Addition and multiplication: Orthogonality in the arithmetic of integers. 2025. Preprint.

[43] J. Moreno Borrallo. Addition and multiplication: Spectral orthogonality and innovation in the arithmetic of integers. 2025. Preprint.

[44] J. Moreno Borrallo. Quantum-elastic geometry: A nonlinear substrate extension for massive regimes. 2025. Preprint.

[45] J. Moreno Borrallo. Quantum-elastic geometry: A unified framework for fields and fundamental constants of nature. 2025. Author submitted manuscript.

[46] J. Moreno Borrallo. Structural orthogonality: From geometric independence to the foundations of arithmetic randomness. 2025. Preprint.

[47] M. Nakahara. *Geometry, Topology and Physics*. CRC Press, 2 edition, 2003.

[48] E. Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, pages 235–257, 1918.

[49] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory*. Westview Press, 1995.

[50] M. Planck. On the law of distribution of energy in the normal spectrum. *Annalen der Physik*, 4:553–563, 1901.

[51] R. Rajaraman. *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory*. North-Holland, 1982.

[52] B. Schutz. *Geometrical Methods of Mathematical Physics*. Cambridge University Press, 1980.

[53] C. E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 623–656, 1948.

[54] A. Sommerfeld. Zur quantentheorie der spektrallinien. *Annalen der Physik*, 356(17):1–94, 1916.

[55] M. Spivak. *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*. W. A. Benjamin, 1965.

[56] R. F. Streater and A. S. Wightman. *PCT, Spin and Statistics, and All That*. Princeton University Press, 2000.

[57] S. J. Summers. On the independence of local algebras in quantum field theory. *Reviews in Mathematical Physics*, 13:1–34, 2001.

[58] J. von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Springer, Berlin, 1932.

[59] R. M. Wald. *General Relativity*. University of Chicago Press, 1984.

[60] S. Weinberg. *The Quantum Theory of Fields, Vol. I*. Cambridge University Press, 1995.

[61] R. F. Werner. Quantum states with einstein–podolsky–rosen correlations admitting a hidden-variable model. *Physical Review A*, 40:4277–4281, 1989.

[62] H. Weyl. *Symmetry*. Princeton University Press, 1952.

[63] E. P. Wigner. The unreasonable effectiveness of mathematics in the natural sciences. *Communications on Pure and Applied Mathematics*, 13(1):1–14, 1960.

[64] K. G. Wilson. The renormalization group and critical phenomena. *Reviews of Modern Physics*, 55:583–600, 1983.

[65] T. T. Wu and C. N. Yang. Concept of nonintegrable phase factors and global formulation of gauge fields. *Physical Review D*, 12(12):3845–3857, 1975.