Quantum-Elastic Geometry: a unified framework for Fields and Fundamental Constants of Nature

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Abstract. *We present the Quantum-Elastic Geometry (QEG) theory, a unified framework wherein spacetime is modeled as a fundamental, physical substrate with quantum, elastic, and dissipative properties. The state of this medium is described by a single, symmetric rank-2 tensor field, $\mathcal{G}_{\mu\nu}$, whose dynamics are governed by a generally covariant action. Known physical interactions are shown to emerge as distinct, irreducible deformation modes of this unified field: gravity, electromagnetism, and a new field –denominated "thermo-entropic field"– that gives rise to irreversible thermodynamics.*

Furthermore, fundamental constants of nature are shown to be uniquely determined and interrelated by the substrate's properties. We derive the fundamental constants of nature through two distinct yet convergent approaches: (i) from the physical postulates of QEG, assuming the $\mathcal{G}_{\mu\nu}$ tensor, its properties leading to dimensional collapse ($[M] \equiv [L] \equiv [T]$), and parsimonious physical principles (e.g., reciprocity, damped equipartition, self-consistency), we deduce specific functional forms for the constants; and (ii) independently, assuming only foundational geometric principles for the substrate (homogeneity, isotropy, covariance, Lorentz invariance) and imposing self-consistency -formalized via a minimal set of geometric normalization conditions consistent with the QEG framework-, we derive the substrate's emergent structure and properties, obtaining precisely the same functional forms for the constants. The outcome is a robust, convergent two-way deductive framework, in which fundamental constants are geometrically enforced, emerging as predictable consequences of a stable and symmetrically constrained geometry.

Finally, we show how the theory predicts -among other results- a scale-dependent gravitational coupling derived from a geometric duality in self-energy, which offers a parameter-free resolution to key cosmological tensions, including the Hubble crisis.

In summary, QEG provides a coherent and consistent origin for both fields and constants, unifying them as rigorously derived emergent properties of a single, dynamic spacetime substrate.

"Entia non sunt multiplicanda praeter necessitatem"
— Ockham's Razor

"Padre, Señor del cielo y de la tierra, te doy gracias porque has ocultado todo esto a los sabios y entendidos y se lo has revelado a los que son como niños."
— Matthew 11:25

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CONTENTS

I. INTRODUCTION

The quest for a unified theoretical framework capable of describing all fundamental interactions from a common origin remains a central theme in contemporary physics [1]. Despite the tremendous success of the Standard Model of particle physics in unifying electromagnetic, weak and strong forces [2], and of General Relativity (GR) in geometrizing gravitation [3, 4], a conceptual schism persists between the quantum field theories (QFTs) of the former and the geometric description of the latter [5]. This disunity is not incidental but structural: the Einstein–Hilbert action is perturbatively non–renormalizable in the standard field–operator expansion, while the Standard Model requires an external geometric background rather than generating one. Moreover, observational puzzles such as dark energy and dark matter [6–8], the cosmological constant problem reflecting inconsistencies between spectral sums and geometric integrals [6], and tensions like the discrepancy in Hubble constant (H_0) determinations [9, 10], underscore the need for a deeper, unifying structure.

Several lines of research suggest that unification must be sought in general mathematical principles rather than in phenomenological adjustments. Thermodynamic and emergent-gravity approaches have hinted at such a structure. Jacobson's derivation of Einstein's equations from the Clausius relation $\delta Q = TdS$, interpreting curvature as an equation of state [11], Verlinde's entropic gravity proposal [12], and the striking analogies between black-hole thermodynamics and vacuum fluctuations [13] point toward an intimate link between entropy, quantum vacuum dynamics, and spacetime geometry. These developments collectively indicate that geometry, thermodynamics, and information might not be merely analogous but deeply interconnected, just different manifestations of a single underlying substrate.

Recent advances in the paradigm of emergent gravity suggest an even deeper connection, positing that spacetime geometry itself arises from the entanglement structure of an underlying quantum system [14–16]. In this view, entanglement acts as the "glue" of spacetime, a concept that resonates deeply with the QEG model of an "elastic substrate" whose properties are governed by the collective state of its constituent oscillators. The core idea of QEG —that the physics of curved spacetime emerges from an underlying medium— also finds powerful conceptual and experimental support in the field of analogue gravity [17]. Concrete realizations arise where effective metrics for excitations in fluids or Bose–Einstein condensates take the form of curved Lorentzian manifolds, demonstrating that phenomena like event horizons can emerge from collective behavior [18]. This universality suggests that curved geometry might be a generic description of collective modes, not exclusive to gravitation.

In this work, we formalize this notion by proposing a theory of *Quantum-Elastic Geometry (QEG)*, which posits that spacetime itself is a dynamic, physical substrate endowed with quantum, elastic, and dissipative properties. Within this framework, the standard Quantum Field Theory (QFT) picture of fields as collections of harmonic oscillators [19] is elevated from a computational representation to a physical interpretation: each mode of the quantum-elastic manifold behaves as an oscillator of the underlying substrate. The vacuum thus emerges as an *ensemble of quantum oscillatory modes* rather than as a fixed lattice, preserving background independence and continuity. In this sense, QEG provides a geometric realization of the same microscopic dynamics that QFT encodes algebraically, while remaining conceptually compatible with background-free approaches such as Loop Quantum Gravity [20] and Causal Set Theory [21].

A natural concern when postulating a physical substrate is the possibility of preferred–frame effects and violations of Lorentz invariance. In QEG the substrate is fully dynamical and relational: the effective rest frame used in coarse–grained, real–time formulations (e.g., closed–time–path hydrodynamics) is determined by the state of the field itself, not imposed externally. As a result, local Lorentz invariance for propagating excitations is preserved, consistent with stringent experimental bounds [22].

The collective state of this oscillatory medium is described by a single, symmetric rank-2 tensor field, $\mathcal{G}_{\mu\nu}(x)$, representing the local strain or deformation of the vacuum. All observed particles and force fields become emergent manifestations of the different vibrational, shear, or torsional modes of this underlying substrate. Within this framework, fundamental constants lose their arbitrary character, becoming effective parameters describing the substrate's material properties (e.g., stiffness κ, damping α). This leads to a necessary *Geometro-dynamic Equivalence* (dimensional collapse) between mass, length, and time ($[M] \equiv [L] \equiv [T]$). Importantly, formal gauge–theoretic formulations provide evidence that the dimensional character of constants may be representational rather than intrinsic. Recent work demonstrating that embedding gravity in a $U(1)^4$ gauge structure yields a dimensionless gravitational coupling and a renormalizable effective theory suggests that such dimensional collapse is not only possible but potentially necessary within a unified framework [23], providing a rigorous basis for the dimensional unification explored here.

The aim of this summary is to construct the formal foundations and key results of QEG, synthesizing the physical framework and its mathematical underpinning [24, 25]. We begin by establishing the minimal axioms for the spacetime substrate. From these, we demonstrate how General Relativity and Electromagnetism emerge as low-energy effective descriptions, alongside the necessary emergence of a third "*thermo-entropic field*" responsible for irreversible thermodynamics. The framework's predictive power is revealed by showing how fundamental constants are uniquely constrained by internal self-consistency, derived via two convergent approaches: (i) from the physical postulates of QEG, assuming the $\mathcal{G}_{\mu\nu}$ tensor, its properties leading to dimensional collapse ($[M] \equiv [L] \equiv [T]$), and parsimonious physical principles (e.g., reciprocity, damped equipartition, self-consistency), and (ii) independently, assuming only foundational geometric principles for the substrate (homogeneity, isotropy, covariance, Lorentz invariance) and imposing self-consistency -formalized via a minimal set of geometric normalization conditions consistent with the QEG framework-. The outcome is a robust, convergent, two-way deductive framework, in which fundamental constants are geometrically enforced, emerging as predictable consequences of a stable and symmetrically constrained geometry. Ultimately, this work presents a coherent framework for the origin of fields and constants from the quantum-elastic properties of spacetime.

Part I: The Foundations of Quantum-Elastic Geometry

II. FROM FIRST PRINCIPLES TO SPACETIME EQUIVALENCE

A. The Principle of a Unified Physical Substrate

Applying the *Principle of Parsimony (Ockham's Razor)*, we begin with a single foundational assumption:

> *The universe, at its most fundamental level, consists of a single, unified physical entity or* ***substrate***.

We will show that the identity of this substrate is not arbitrary. General Relativity has revealed that spacetime is a dynamic field, $g_{\mu\nu}$, while Quantum Field Theory describes the vacuum as a plenum of fluctuating fields. The most parsimonious conclusion is that *these are one and the same: spacetime itself.* Throughout this Section, we will show how the foundational assumption leads precisely to the conclusion expressed above on the nature of the unified physical entity or substrate.

B. The Unified Lagrangian

A single substrate must be governed by a single, self-consistent dynamical principle. In modern physics, this is encoded in a Lorentz-invariant Lagrangian, from which all equations of motion are derived. The simplest such Lagrangian for a deformation field $\mathcal{G}_{\mu\nu}$ consists of a kinetic term, representing the substrate's inertial resistance to changes in deformation, and a potential term $V(\mathcal{G})$, representing its self-interaction:

$$\mathcal{L}_{\text{QEG}} = \frac{1}{2}\kappa\,(\partial^\alpha \mathcal{G}^{\mu\nu})(\partial_\alpha \mathcal{G}_{\mu\nu}) - V(\mathcal{G}), \tag{II.1}$$

where κ is a universal constant representing the intrinsic *stiffness* of the substrate. The principle of unification demands that this constant governs the inertial response to *all* possible modes of deformation.

It is crucial to note that the Lagrangian presented in II.1 is a simplified formulation, valid in the weak field limit on a flat background spacetime (Minkowski). Its purpose at this stage is to establish the fundamental principles of substrate stiffness (κ) and the inertia of its different modes of deformation in a clear and direct manner. A fully covariant and background-independent formulation, where the field $\mathcal{G}_{\mu\nu}$ defines its own dynamic geometry, will be developed in Section VIII A. The present simplified model allows us to deduce the dimensional consequences of unification before addressing the complexity of the complete nonlinear dynamics.

Minimal Scalar Field Model for Modal Excitations

To provide a dynamical description of the modal excitations of the unified tensor $\mathcal{G}_{\mu\nu}$, we can construct a minimal model that captures their essential propagation features. This model serves as an effective description for a single projected mode, which we can denote as a scalar field $\Phi(x)$. The dynamics of this effective field are not arbitrary, but are a direct consequence of the general Lagrangian.

Recall that the kinetic part of the unified field Lagrangian is governed by a single, fundamental stiffness constant κ:

$$\mathcal{L}_{\text{kinetic}} = \frac{1}{2}\kappa(\partial^\alpha \mathcal{G}^{\mu\nu})(\partial_\alpha \mathcal{G}_{\mu\nu})$$

We can project these general dynamics onto a single scalar mode $\Phi(x)$, which will naturally contain terms for the kinetic and potential energy, analogous to a harmonic oscillator, and the two couplings (or "rigidities") usual for harmonic oscillatory systems, κ_1 and κ_2, which are effective combinations of physical constants (e.g., ε_0, μ_0, G, k_B, etc.) under different substitutions depending on the mode of the elastic-oscillatory manifestation of the common field. Then, one obtains the following *minimal* Lagrangian density:

$$\mathcal{L}(\Phi) \;=\; \frac{1}{2}\,\kappa_1\,\big(\partial_t\Phi\big)^2 \;-\; \frac{1}{2}\,\kappa_2\,\big(\nabla\Phi\big)^2 \tag{II.2}$$

Here,

- κ_1 controls the inertial" or kinetic response of the field mode,
- κ_2 represents the elastic/spatial rigidity of the field mode.

The corresponding *action* S is given by the integral

$$S[\Phi] \;=\; \int d^4x\;\mathcal{L}\big(\Phi, \partial\Phi; \kappa_1, \kappa_2\big) \tag{II.3}$$

Applying the principle of least action, $\delta S = 0$, yields the Euler–Lagrange equation:

$$\kappa_1 \frac{\partial^2 \Phi}{\partial t^2} \; - \; \kappa_2 \nabla^2 \Phi \; = \; 0. \tag{II.4}$$

This *single* partial differential equation governs the field Φ. Depending on how we identify κ_i with physical constants and Φ with different spacetime deformations (e.g., mass, charge, temperature), we recover the different field modes.

Relativistic Compatibility and Spacetime Formalism

To ensure compatibility with special relativity, we introduce a four-dimensional spacetime coordinate:

$$X^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad \text{with metric} \quad \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{II.5}$$

We define the spacetime derivatives:

$$\partial_\mu = \frac{\partial}{\partial X^\mu} = \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{II.6}$$

The standard Lorentz-invariant kinetic structure is encoded in the d'Alembertian operator:

$$\Box \Phi \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \Phi = -\frac{\partial^2 \Phi}{\partial t^2} + \nabla^2 \Phi. \tag{II.7}$$

The equation of motion from our minimal field Lagrangian reads:

$$\kappa_1 \frac{\partial^2 \Phi}{\partial t^2} - \kappa_2 \nabla^2 \Phi = 0. \tag{II.8}$$

To write this in a Lorentz-invariant form proportional to $\Box \Phi = 0$, we compare:

$$\kappa_1 \frac{\partial^2 \Phi}{\partial t^2} = \kappa_2 \nabla^2 \Phi \quad \Longrightarrow \quad \frac{\partial^2 \Phi}{\partial t^2} = \frac{\kappa_2}{\kappa_1} \nabla^2 \Phi. \tag{II.9}$$

Rewriting the d'Alembertian as:

$$\Box \Phi = -\frac{\partial^2 \Phi}{\partial t^2} + \nabla^2 \Phi = 0,$$

we see that Lorentz invariance requires:

$$\frac{\kappa_2}{\kappa_1} = 1 \quad \Rightarrow \quad \kappa_1 = \kappa_2. \tag{II.10}$$

Therefore, the field theory is manifestly Lorentz-invariant if and only if the rigidity constants match: $\kappa_1 = \kappa_2$. This is a profound physical statement: it reflects that in a Lorentz-covariant theory built upon a single field, the inertial and elastic responses to variations must originate from the same isotropic, fundamental stiffness κ of the vacuum substrate. This ensures the action transforms as a scalar and the field equation becomes the standard wave equation:

$$\Box \Phi = 0. \tag{II.11}$$

This scalar field model, though minimal, shows how each modal projection of the unified field tensor $\mathcal{G}_{\mu\nu}$ can be described in terms of a wave-like field obeying Lorentz-invariant dynamics. This prepares the ground for more detailed relativistic formulation in terms of Klein–Gordon dynamics and projection structures consistent with general relativity.

Remark II.1 (Generalization)**.** *The above result –the necessary equality of temporal (inertial) and spatial (elastic) stiffness coefficients– is not limited to the pedagogical scalar model. It is a fundamental requirement for any relativistic field theory derived from a Lagrangian of the form of Eq.* (II.1).

C. The Inertial Equivalence Principle

As we have seen in the previous subsection, the requirement of a unified Lagrangian has a profound and unavoidable consequence deeply rooted in Lorentz covariance: the single stiffness constant κ must consistently describe the inertia of every deformation mode. Analyzing the kinetic term reveals that κ plays a dual role:

- For *compressive* (scalar) modes, associated with mass-like excitations, κ acts analogously to mass in mechanical kinetic energy $\frac{1}{2} m \dot{x}^2$. Thus $[\kappa] \equiv [\text{Mass}]$.
- For *torsional* (vector) modes, associated with currents, κ acts as a coefficient of rotational inertia, mathematically analogous to inductance in electrodynamics. Thus $[\kappa] \equiv [\text{Inductance}]$.

Since Lorentz covariance and the principle of a single unified field demand that a single constant κ governs the inertial properties of all components of $\mathcal{G}_{\mu\nu}$, it follows that the dimensional character of mass and inductance must be identical:

$$\boxed{[\text{Mass}] \equiv [\kappa] \equiv [\text{Inductance}]}. \tag{II.12}$$

Remark II.2 (A structural requirement)**.** *It is crucial to emphasize that this is not merely a physical analogy but a structural requirement of the unified Lagrangian. The single kinetic term, $\frac{1}{2}\kappa(\partial^\alpha \mathcal{G}^{\mu\nu})(\partial_\alpha \mathcal{G}_{\mu\nu})$, governs the inertia of all possible deformation modes of the substrate. Because this one mathematical object must simultaneously describe the resistance to compressive changes (a mass-like inertia) and the resistance to torsional changes (an inductance-like inertia), the dimensional characters of Mass and Inductance are forced to be identical. They are not just analogous; they are two different phenomenological manifestations of the single, unified stiffness parameter κ.*

Dimensional Consequences of the Inertial Equivalence Principle

Taking the SI units of inductance L as

$$[L] = [ML^2 I^{-2} T^{-2}],$$

the Inertial Equivalence Principle leads directly to

$$[M] \equiv [ML^2 I^{-2} T^{-2}].$$

For dimensional consistency, this equivalence requires that the combination $[L^2 I^{-2} T^{-2}]$ must be dimensionless. Solving for the dimension of current $[I]$ yields

$$[I^2] \equiv [L^2 T^{-2}] \implies \boxed{[I] \equiv [LT^{-1}]}. \tag{II.13}$$

As a sanity check, in the same way that mass M in a mechanical oscillator is analogous to inductance L in an RLC circuit, the resistance R in an RLC circuit is analogous to the damping coefficient b in a mechanical oscillator. Establishing the dimensional equivalence between them, we find that

$$[MT^{-1}] \equiv [ML^2 T^{-3} I^{-2}],$$

which again implies that $[L^2 I^{-2} T^{-2}]$ is dimensionless, consistent with the result (II.13).

Subsequently, from the Inertial Equivalence Principle, it follows that electric current acquires the dimensions of velocity. Expressed abstractly,

$$[I] \equiv [LT^{-1}], \tag{II.14}$$

so that current is no longer an independent unit (as in SI), but a kinematical manifestation of deformation flow. In physical terms, current is the propagation velocity of a torsional wave of the substrate, while mass is the inertial coefficient of a compressive deformation. Both are modes of the same elastic field.

D. The Geometro-Elastic Principle and the Inevitability of the Laplacian Operator

The central thesis of QEG is that the vacuum is a continuous elastic medium. To describe the static configuration of its deformation field, $\mathcal{G}_{\mu\nu}$, we must identify the mathematical operator that governs equilibrium in such a substrate. This operator can be uniquely and rigorously determined by the most fundamental principles of symmetry and simplicity.

Justification from Symmetry: The Mandate of Isotropy

The most foundational property we can assume about the vacuum substrate is that it is *isotropic*: it has no preferred direction in space. Any fundamental law governing it must respect this symmetry. For a continuous elastic medium in static equilibrium, the operator describing its response must be invariant under rotations and translations.

The Laplacian ($\nabla^2 = \sum_i \partial_i^2$) is the *unique* second-order linear differential operator that possesses this property. Any other operator of the same order, such as a weighted sum like $(\partial_x^2 + 2\partial_y^2 + \dots)$, would explicitly break rotational invariance, arbitrarily selecting certain axes as special. This would imply that the fabric of spacetime is intrinsically anisotropic, a claim that would require extraordinary justification and contradicts large-scale observation. Therefore, the *principle of isotropy* alone is sufficient to select the Laplacian as the necessary mathematical structure for describing the substrate's static response.

Justification from Simplicity: Ockham's Razor and Locality

The principle of parsimony (Ockham's Razor) demands that we use the simplest possible mathematical structure that adequately describes the physics. The Laplacian operator describes the simplest form of *local elastic tension*: it relates the state of the field at a point to the average state in its immediate infinitesimal vicinity.

One could postulate more complex, higher-order operators like the biharmonic operator (∇^4). However, this would imply that the fundamental interactions of the substrate are non-local, depending not just on the immediate neighborhood but on its neighbors' neighbors. Without any physical or empirical necessity for such complexity at the most fundamental level, Ockham's Razor compels the choice of the lowest-order, non-trivial operator capable of describing elastic deformation: the Laplacian.

Mathematical Consequence: The Universal 1/r Potential

Once the principles of symmetry and simplicity force the selection of the Laplacian, the geometric form of all long-range static interactions is no longer a choice, but a *mathematical inevitability*. The static equilibrium equation for the deformation field Φ in the presence of a point-like source at the origin (a Dirac delta function, $\delta(\vec{r})$) must take the form of Poisson's equation:

$$\nabla^2\Phi(\vec{r}) = -J \cdot \delta(\vec{r}) \tag{II.15}$$

The unique solution to this equation in three-dimensional space is its Green's function, which has the universal form:

$$\nabla^2\Phi(\mathbf{r}) = -\delta^{(3)}(\mathbf{r}) \quad \Longrightarrow \quad \Phi(\mathbf{r}) = \frac{1}{4\pi r}. \tag{II.16}$$

This establishes the $1/r$ geometric modulus as the *universal mathematical structure for all long-range static interactions* of an isotropic, elastic, 3D substrate. Consequently, both Newton's law of gravitation and Coulomb's law must appear as macroscopic limits of this same underlying Laplacian response. This forces a crucial conclusion: *if the substrate is truly unified, the constants that couple sources to the field in these laws must share the same dimensional essence*, a requirement that will be formalized as the Coupling Equivalence Principle in the next subsection.

Note on Short-Range Interactions

It is important to emphasize that the Laplacian operator and the resulting $1/r$ potential are characteristics of the theory's limit for massless, static modes. This framework thus recovers the universal structure of long-range forces such as gravity and electromagnetism. Short-range interactions, such as the strong and weak nuclear forces, would be described within this framework as excitations of massive or non-linear modes of the field $\mathcal{G}_{\mu\nu}$. For these modes, the static field operator would take a different form (e.g., the Yukawa operator, $\nabla^2 - m^2$), giving rise to short-range potentials (such as e^{-mr}/r) and breaking the universality of the $1/r$ behavior.

E. The Coupling Equivalence Principle

As we have already hinted at the end of Sec. II D, we can extract another fundamental principle from the universal dynamics imposed by the unified Lagrangian and Lorentz covariance. Variation of the action leads to a field equation of the form:

$$\kappa\Box\mathcal{G}_{\mu\nu} - \frac{\partial V}{\partial\mathcal{G}^{\mu\nu}} = J_{\mu\nu},$$

where the d'Alembertian operator, $\Box$, governs the dynamics of all massless excitations of the substrate. This universal, hyperbolic operator is the unique source of the unified response.

The familiar empirical laws for static fields arise as the low-energy limit ($\partial_t \to 0$) of these fundamental wave dynamics, where the d'Alembertian reduces to the Laplacian ($\Box \to \nabla^2$):

- For **Gravity**: $\nabla^2\Phi_g = 4\pi G\rho_m$
- For **Electrostatics**: $\nabla^2 V = -(4\pi K_e)\rho_e$

Since both of these static laws are different projections of the *same* underlying dynamic operator ($\Box$), the constants that scale their respective sources—G and K_e—cannot be dimensionally distinct. In a unified framework where mass density (ρ_m) and charge density (ρ_e) are manifestations of the same source tensor $J_{\mu\nu}$, it is a mathematical necessity that their couplings share the same dimensional character. To postulate otherwise would violate the covariant nature of the theory.

This leads us to state the *Coupling Equivalence Principle* as a direct consequence of covariant dynamics:

$$\boxed{[G] \equiv [K_e]}. \tag{II.17}$$

This principle asserts that gravitational and electromagnetic couplings share the same dimensional essence, being merely modal projections of a single underlying rigidity of spacetime.

Dimensional Consequences of the Coupling Equivalence Principle

Let us now derive the dimensional consequences of this principle combined with our previous findings. The conventional dimensions for G and K_e are

$$[G] = [M^{-1}L^3T^{-2}], \quad [K_e] = [ML^3T^{-4}I^{-2}].$$

Setting $[G] \equiv [K_e]$, we obtain

$$[M^{-1}L^3T^{-2}] \equiv [ML^3T^{-4}I^{-2}].$$

Rearranging for $[M]$, and recalling that we have established the Inertial Equivalence Principle, $[I] \equiv [LT^{-1}]$ (II.13), we substitute I^{-2} as

$$[M^{-1}L^3T^{-2}] \equiv [ML^3T^{-4}(L^{-2}T^2)].$$

Simplifying, we find

$$\begin{aligned} &[M^{-1}L^3T^{-2}] \equiv [MLT^{-2}] \to \\ &[M^{-1}L^3] \equiv [ML] \to \\ &[L^2] \equiv [M^2] \to \\ &\boxed{[M] \equiv [L]}. \end{aligned} \tag{II.18}$$

This outcome, contingent on the Inertial Equivalence Principle (II.12) and the Coupling Equivalence Principle (II E), signifies that mass and length share the same fundamental dimension within this theoretical structure.

Moreover, substituting $[M]$ and $[I]$ into the previous equivalence between resistance R and damping coefficient b based on the Inertial Equivalence Principle, $[MT^{-1}] \equiv [ML^2T^{-3}I^{-2}]$, we find that $[T^{-4}L^4]$ becomes dimensionless, which in turn implies the fundamental equivalence

$$\boxed{[M] \equiv [L] \equiv [T]}. \tag{II.19}$$

This result does not arise from a choice of units, but from geometric and dynamical consistency: the Laplacian enforces the $1/r$ law for static responses, Lorentz invariance equates time and space, and the unified Lagrangian requires a single stiffness constant κ for all inertial modes. The collapse of mass, length, and time into a single dimensional entity is therefore an inevitable structural feature of a unified substrate.

F. The Endogenous Nature of Sources

Finally, if mass, charge, and current share the same dimensional character as length and time, they cannot be external properties imposed on the substrate. Rather, they are *endogenous excitations*: localized, stable topological configurations of the deformation field $\mathcal{G}_{\mu\nu}$. Mass-energy and charge are not causes of deformation, but particular *modes of deformation* themselves. Thus particles and fields, matter and geometry, are unified as different aspects of the same physical substrate: *spacetime itself.*

III. MODAL DECOMPOSITION OF THE UNIFIED FIELD

The deformation field $\mathcal{G}_{\mu\nu}$ encodes the state of the unified substrate. To connect this framework with known physical interactions, it is natural to decompose $\mathcal{G}_{\mu\nu}$ into modes according to their tensorial character and dynamical role. The decomposition mirrors the standard separation of elastic media into compressive and shear responses, but is here generalized to a covariant, field-theoretic setting.

A. Scalar, Vector, and Tensorial Sectors

We classify the components of $\mathcal{G}_{\mu\nu}$ into three categories:

- **Scalar (compressive) mode:** $\mathcal{G}_{00}$, associated with localized compressions of the substrate. This sector corresponds to inertial mass and gravitation, as compressive deformations propagate radially and source the familiar Newtonian potential.
- **Vector (torsional) modes:** $\mathcal{G}_{0i}$, associated with shear-like deformations oriented along spatial directions. These modes are naturally linked to currents and electromagnetic phenomena, as they propagate transversely and encode the transport of torsional strain.
- **Tensorial modes:** The symmetric tensor $\mathcal{G}_{ij}$ can be uniquely decomposed into its trace (an isotropic scalar mode) and its traceless part (an anisotropic shear mode). The traceless part is naturally identified with gravitational waves and anisotropic stresses. The trace, $\mathrm{Tr}(\mathcal{G}_{ij})$, transforms as a scalar under spatial rotations and thus must couple to isotropic sources like pressure. As this components $\mathcal{G}_{ij}$ represent the remaining degrees of freedom of the unified field, for the QEG framework to be complete, these modes must also correspond to a physical interaction. As we will demonstrate rigorously in Part IV, the dynamics of this scalar mode, when subjected to the substrate's intrinsic dissipation, inevitably lead to the laws of irreversible thermodynamics. Therefore, we identify this sector with a fundamental **thermo-entropic field**, not by analogy, but as a necessary consequence of the theory's completeness and internal consistency.

We now present the explicit projectors for the spatial tensor $\mathcal{G}_{ij}$ due to its rich internal structure, which hosts multiple physical modes. The projection of the modes $\mathcal{G}_{00}$ and $\mathcal{G}_{0i}$ is trivial in comparison, as it reduces to the direct selection of those components, which already transform as a scalar and a vector under spatial rotations, respectively.

Explicit projectors on the spatial tensor $\mathcal{G}_{ij}$

In a homogeneous/isotropic background (or in Fourier space on $\mathbb{R}^3$), define k_i and the transverse projector $P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j$, $\hat{k}_i = k_i/k$. The symmetric tensor $\mathcal{G}_{ij}$ decomposes as

$$\mathcal{G}_{ij} = \underbrace{\left(\psi\,\delta_{ij} + (\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2)E\right)}_{\text{scalar}} + \underbrace{\partial_{(i}F_{j)}}_{\text{vector, } \partial_i F_i = 0} + \underbrace{h^{\mathrm{TT}}_{ij}}_{\text{tensor, } \partial_i h^{\mathrm{TT}}_{ij}=0,\, h^{\mathrm{TT}}_{ii}=0} . \tag{III.1}$$

Corresponding projectors in Fourier space read

$$\begin{aligned} \left(P^{\mathrm{T}}\right)_{ij,kl} &= \tfrac{1}{2}\left(P_{ik}P_{jl} + P_{il}P_{jk}\right) - \tfrac{1}{3}P_{ij}P_{kl}, \\ \left(P^{\mathrm{V}}\right)_{ij,kl} &= \tfrac{1}{2}\left(\hat{k}_i P_{jl}\hat{k}_k + \hat{k}_j P_{il}\hat{k}_k + \cdots\right), \\ \left(P^{\mathrm{S}}\right)_{ij,kl} &= \tfrac{1}{3}P_{ij}P_{kl} + \hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l \end{aligned} \tag{III.2}$$

so that $P^{\mathrm{T}} + P^{\mathrm{V}} + P^{\mathrm{S}} = \mathbb{I}$ and $P^A P^B = \delta^{AB} P^A$.

Justification from Symmetry and Lorentz Covariance

The assignments of the tensor components of the deformation tensor $\mathcal{G}_{\mu\nu}$ to the physical phenomena that we have made above can be justified as a physically necessary choice by appealing to the principles of symmetry and Lorentz covariance. The mapping is dictated by the transformation properties of sources under rotations and Lorentz boosts. Each physical source (scalar mass-energy, vector current, tensor stress/flux) must couple to the component of $\mathcal{G}_{\mu\nu}$ with matching transformation properties. *Any other assignment would violate covariance.* Here's the breakdown:

G_{00} and Scalar Sources (Mass-Energy)

- **Physical Source:** The primary source of Newtonian gravity is **mass**, a scalar quantity (it does not change under spatial rotations). In relativity, this is the energy density component, T_{00}, which also behaves as a scalar under spatial rotations.

- **Tensor Component:** The G_{00} component of the tensor is the only component that transforms as a **scalar** under rotations.

- **Conclusion:** By the principle of covariance, a scalar source must couple to a scalar mode of the field. The identification of mass-energy with G_{00} is the simplest and most direct way to achieve this. Associating a scalar source with a vector (G_{0i}) or tensor (G_{ij}) component would introduce an arbitrary direction into the physics, which is not observed.

G_{0i} and Vector Sources (Currents)

- **Physical Source:** The source of magnetism is **electric current**, J_i, which is a 3-vector. It has a direction and magnitude that change predictably under rotation.

- **Tensor Component:** The three G_{0i} components are the only parts of the $G_{\mu\nu}$ tensor that transform together as a **3-vector** under rotations.

- **Conclusion:** A vector source must couple to a vector-like part of the unified field. The identification of currents with the G_{0i} modes is therefore almost inevitable. This ensures that the laws of physics (like Ampère's Law) maintain their vector form in all rotated reference frames.

G_{ij} and Tensor Phenomena (Stress and Gravitational Waves)

- **Physical Phenomena:** Phenomena like pressure, shear stress (T_{ij}), and the propagation of gravitational waves are inherently **tensorial**. They describe how space itself is stretched and sheared in different directions.

- **Tensor Component:** The G_{ij} components form a 3×3 symmetric tensor that directly represents the spatial strain (stretching and shearing) of the physical substrate.

- **Conclusion:** It is physically necessary to map tensor phenomena onto the tensor components of the unified field. The G_{ij} components are the only ones that have the correct mathematical structure to describe the complex, directional stresses and strains within the spacetime fabric.

In summary, the assignment is a direct mapping of physical symmetries onto the mathematical symmetries of the tensor $G_{\mu\nu}$. The "shape" of the source or phenomenon (scalar, vector, or tensor) must match the "shape" of the deformation mode it couples to. This assignment is essentially unique under Lorentz invariance: the shape of the source dictates the shape of the field component it excites. This makes the chosen identification not just a convenient choice, but the only one that is fully consistent with the observed Lorentz symmetry of our universe.

This way, we have shown how gravitation, electromagnetism, and other tensor spatial deformations are interpreted as distinct *modal projections* of the same underlying substrate field.

B. Unified Source Structure

In conventional field theory [24, 25], each interaction couples to a distinct current: the stress-energy tensor $T^{\mu\nu}$ for gravitation, the electromagnetic four-current J^μ for electrodynamics, and thermodynamic fluxes for entropy. In the present framework, all these appear as different components of a single unified source tensor $J_{\mu\nu}$, which couples universally to $\mathcal{G}_{\mu\nu}$ via the action principle. Explicitly:

$$J_{00} \longleftrightarrow T_{00} \quad \text{(mass-energy density component)}$$
$$J_{0i} \longleftrightarrow J^i_{\text{em}} \quad \text{(spatial components of 4-current)}$$
$$\mathrm{Tr}(J_{ij}) \longleftrightarrow p \quad \text{(pressure, thermodynamic state)}$$

In addition, the traceless spatial part acts as the natural source for shear modes:

$$\left(J_{ij} - \tfrac{1}{3}\delta_{ij}\,\mathrm{Tr}(J)\right) \longleftrightarrow \pi_{ij} \text{ (anisotropic stress / shear source)} \tag{III.3}$$

This identification unifies the sources of interaction as manifestations of the same underlying conservation principle, arising from the symmetries of the substrate.

C. Modal Dynamics from the Unified Lagrangian

The universal field equation derived earlier,

$$\kappa \Box \mathcal{G}_{\mu\nu} - \frac{\partial V}{\partial \mathcal{G}^{\mu\nu}} = J_{\mu\nu},$$

when projected onto different modes, reproduces the macroscopic laws of physics:

- **Gravitational mode ($\mathcal{G}_{00}$):** In the static, massless limit the equation reduces to
$$\nabla^2 \Phi_g = 4\pi G \rho_m,$$
i.e. Newton's law of gravitation. At higher order, we will show later in the Paper how the full dynamics recovers the Einstein field equations.

- **Electromagnetic modes ($\mathcal{G}_{0i}$):** The corresponding equations reduce to Maxwell's laws, with current J^μ acting as the source of transverse torsional excitations.

- **Other tensor deformation modes ($\mathcal{G}_{ij}$):** The isotropic component describes large-scale volumetric expansion, while the traceless component propagates as shear-like excitations. In Section XXIX we will deduce how in the hydrodynamic limit these reproduce diffusion-type equations and the thermodynamic identities associated with the entropy flow.

Thus, all known field equations are recovered as modal reductions of the same substrate dynamics, validating the unification scheme.

Clarification. It is important to stress that gravitational phenomena split into two complementary aspects in this framework. The Newtonian potential and the coupling to mass-energy density (T_{00}) arise from the scalar sector $\mathcal{G}_{00}$. In contrast, *gravitational radiation* is not encoded in this scalar but in the traceless tensorial excitations of $\mathcal{G}_{ij}$. Thus, while $\mathcal{G}_{00}$ governs the scalar (compressive) aspect of gravitation, the tensorial shear modes Σ_{ij} account for the full propagation of gravitational waves. Consistently, only the traceless spatial components couple to the anisotropic stress π_{ij}, while the scalar potential couples to the rotationally invariant source T_{00}, preserving the matching of tensorial ranks between sources and fields required by Lorentz covariance.

D. Lagrangian framework for modal excitations

Having established the physical identity of each deformation mode, we now construct the canonical Lagrangian framework required to describe their dynamics and interaction with sources. Building on the minimal scalar field formulation—which established that any modal projection must obey Lorentz-invariant dynamics—we transition from the fundamental Lagrangian governed by the stiffness constant κ to an effective, canonical Lagrangian for each mode. This is achieved via field renormalization, a standard procedure in field theory where the physical constant κ is absorbed into the field's definition (e.g., $\mathcal{G}' = \sqrt{\kappa}\mathcal{G}$) to simplify the kinetic term.

This formulation allows the modal excitations of the unified tensor $\mathcal{G}_{\mu\nu}$ to be described in terms of renormalized, scalar-like fields $\mathcal{G}^{(X)}$. The dynamics of these modes, including potential mass terms that arise from the full theory's potential term $V(\mathcal{G}_{\mu\nu})$, can be described by the following canonical Klein–Gordon Lagrangian:

$$\mathcal{L}_X = \frac{1}{2}\partial^\mu \mathcal{G}^{(X)}\partial_\mu \mathcal{G}^{(X)} - \frac{1}{2}m_X^2 \left(\mathcal{G}^{(X)}\right)^2, \tag{III.4}$$

where $\mathcal{G}^{(X)}$ represents a renormalized projection of the full tensor $\mathcal{G}_{\mu\nu}$ along mode X. [1] The term m_X is not a new fundamental parameter, but an *effective mass* that a mode acquires due to the self-interaction potential of the underlying unified field. This Klein–Gordon-type Lagrangian provides a Lorentz-invariant basis for describing both massless ($m_X = 0$) and massive field modes within the unified elastic framework.

The Euler–Lagrange equation yields:

$$\Box \mathcal{G}^{(X)} + m_X^2 \mathcal{G}^{(X)} = 0, \tag{III.5}$$

which reduces to the Klein–Gordon equation for free scalar propagation in Minkowski space.

Although the Lagrangian $\mathcal{L}_X$ describes free fields, coupling to physical sources can be incorporated via minimal interaction terms of the form:

$$\mathcal{L}_{\text{int}}^{(X)} = \mathcal{G}^{(X)}(x) \cdot J^{(X)}(x), \tag{III.6}$$

where $J^{(X)}$ is an effective source density corresponding to charge, mass, entropy flux, etc. These terms play an analogous role to the coupling $A_\mu J^\mu$ in electrodynamics. Since $\mathcal{G}^{(X)}$ represents a modal projection of the unified tensor $\mathcal{G}_{\mu\nu}$, the coupling is assumed to act only on the relevant scalarized or vectorial component associated with the physical mode.

A more general coupling scheme could link the full tensor to the energy–momentum content of matter via:

$$\mathcal{L}_{\text{int}} = \mathcal{G}_{\mu\nu}(x)\, T^{\mu\nu}(x), \tag{III.7}$$

from which each modal interaction $\mathcal{G}^{(X)}J^{(X)}$ would arise as a projection or contraction. This formulation ensures full compatibility with general relativistic coupling schemes.

Cross-check. In Secs. XXIX E and XXIX G we show that the universal dissipative invariant, implemented via a Rayleigh functional, yields a hyperbolic–parabolic (telegrapher) dynamics at the effective level. This mirrors causal relativistic hydrodynamics (Israel–Stewart), guaranteeing finite signal speeds and positive entropy production in the gravito-entropic sector.

[1] While this Klein-Gordon form correctly captures the universal mass and propagation dynamics for any mode, a full description for vector or tensor modes would require a more structured Lagrangian (e.g., a Proca Lagrangian for massive vector modes) to account for all degrees of freedom. For the purpose of describing the fundamental dynamics of sourced and source-free propagation, this canonical form is sufficient.

E. Static solutions

In the static limit and for massless modes ($m_X = 0$), the sourced Euler–Lagrange equation derived from $\mathcal{L}_X + \mathcal{L}_{\text{int}}^{(X)}$ reduces to the Poisson equation

$$\nabla^2\, \Phi_X(\vec{r}) \;=\; -\,J^{(X)}(\vec{r})\,, \tag{III.8}$$

where we denote by Φ_X the static potential associated to the mode X. For a point-like unit source located at the origin, $J^{(X)}(\vec{r}) = \delta^{(3)}(\vec{r})$, the Green's function solution is

$$\Phi_X(r) \;=\; \frac{1}{4\pi r}\,. \tag{III.9}$$

Multiplying this fundamental response by an effective charge C_X yields the modal potential:

$$\boxed{\Phi_X(r) \;=\; C_X\, \frac{1}{4\pi r}} \tag{III.10}$$

which matches the expressions summarized in Sec. XXXII.

IV. THE UNIFIED DIMENSIONAL FRAMEWORK

A. Geometrization of Physical Quantities

The Spacetime Equivalence Principle ($[M] \equiv [L] \equiv [T]$) forces a complete rewriting of the dimensional structure of physics. All fundamental physical quantities can be expressed in terms of a single geometric unit, $[L]$. The consequences, summarized in Table I, are profound: physical sources like mass and charge acquire the dimension of length, while all dynamical and coupling constants become dimensionless pure numbers.

In this new framework, one constant stands apart: **Planck's constant,** h. Its dimensions of action, $[E \cdot T]$, do not collapse to $[L]$ or $[1]$, but instead become:

$$[h] = [E \cdot T] \equiv [L \cdot L] = [L^2]. \tag{IV.1}$$

This is a profound result. It reveals that the quantum of action is not an abstract parameter, but a fundamental quantum of **spacetime area**. This geometrization of quantum mechanics is a cornerstone of the QEG framework, linking the discreteness of the quantum world to the geometric fabric of the substrate.

B. Dimensional Equivalence and Its Implications for Fundamental Units

Through the dimensional collapse, we establish that all fundamental physical quantities share a common geometric basis:

$$[L] \equiv [T_{\text{ime}}] \equiv [M] \equiv [E] \equiv [Q] \equiv [T_{\text{emp}}]. \tag{IV.2}$$

This does not negate the operational distinction among meters, seconds, or kilograms in experimental practice, but clarifies the role of universal constants. *In this framework, any constant not itself a source of deformation (mass-energy, charge, thermal or entropic sources, action) becomes dimensionless and functions as a conversion factor.* Hence, SI units, once normalized by these constants, are numerically equivalent expressions of a single underlying geometric quantity:

$$1\text{ m} \equiv 1\text{ s} \equiv c^2\,\text{kg} \equiv 1\,\text{J} \equiv 1\,\text{C} \equiv k_B \cdot 1\,\text{K}. \tag{IV.3}$$

This equivalence is not a matter of notation, but of geometry: universal constants act as the *metric coefficients*

TABLE I. Dimensional collapse of physical quantities in the QEG framework.

Quantity	**Standard SI Dimensions**	**QEG Dimension** $[L]$
Physical Sources		
Mass	$[M]$	$[L]$
Energy	$[ML^2T^{-2}]$	$[L]$
Charge	$[IT]$	$[L]$
Temperature	$[K]$	$[L]$
Dynamical/Coupling Constants		
Velocity / Current	$[LT^{-1}]$	[1] (Dimensionless)
Resistance	$[ML^2T^{-3}I^{-2}]$	[1]
Permittivity, ε_0	$[M^{-1}L^{-3}T^4I^2]$	[1]
Permeability, μ_0	$[MLT^{-2}I^{-2}]$	[1]
Gravitational C., G	$[M^{-1}L^3T^{-2}]$	[1]
Boltzmann C., k_B	$[ML^2T^{-2}K^{-1}]$	[1]

of physical space, establishing the calibration between nominal units. Their role is thus not arbitrary, but structural, ensuring that all units reduce consistently to a unified geometric basis.

While the theory itself is scale-free, connecting it to empirical quantities requires a fiducial reference, naturally chosen as the SI system since CODATA values are defined within it. This calibration should not be mistaken for introducing a free parameter: setting a reference length, e.g. $L_{\text{ref}} = 1$ m, serves as a **coherence test**. By anchoring the theoretical unit to an experimental one, the entire system of SI constants emerges consistently from the predicted relations. The significance lies not in predicting numerical values, but in validating that the structure of physical laws is compatible with a unified geometric origin.

Part II: General Properties of Geometro-elastic Excitations

Building on the foundational framework of Part I, this Part focuses on the *general, mode-independent* consequences that any excitation of the unified substrate must satisfy. We begin by deriving the operator form of the substrate's elastic response from the unified Lagrangian. Projecting this onto its normal modes, we obtain a universal Hooke-type constitutive law where the geometric deformation (action) is proportional to the applied source. This yields a single, powerful source law, $Q = kS$, valid for mass, charge, and any fundamental deformation mode. This structure then provides the basis for understanding the duality of stiffness and compliance and the dissipative nature of the substrate.

V. A UNIFIED CONSTITUTIVE LAW FROM THE GENERAL ELASTIC EQUATION

A. Operator form of the elastic response

Starting from the unified dynamics of Part I,

$$\kappa \Box \mathcal{G}_{\mu\nu} - \frac{\partial V}{\partial \mathcal{G}^{\mu\nu}} = J_{\mu\nu}, \tag{V.1}$$

the linear, long-wavelength (massless) regime relevant for static and weakly time-dependent fields is obtained by neglecting the non-linear and massive parts of V. In this regime one may write, schematically,

$$\mathbb{K}\,\mathcal{G} = J, \qquad \mathbb{K} := \kappa \Box \xrightarrow{\text{static}} -\kappa \nabla^2, \tag{V.2}$$

where $\mathbb{K}$ is the *stiffness operator* of the substrate. The solution is expressed via the inverse (Green operator) $\mathbb{C} := \mathbb{K}^{-1}$, which plays the role of a *compliance operator*:

$$\mathcal{G} = \mathbb{C}\,J, \qquad \mathbb{C} \equiv \mathbb{K}^{-1}. \tag{V.3}$$

Equations (V.2)–(V.3) are the covariant, field-theoretic generalization of Hooke's law (stress = stiffness × strain; strain = compliance × stress) and hold for *any* tensorial component of the deformation field $\mathcal{G}_{\mu\nu}$ and source $J_{\mu\nu}$.

B. Modal projection: from fields to lumped constitutive law

In homogeneous/isotropic backgrounds the operator $\mathbb{K}$ diagonalizes on normal modes. Let Π_i denote the projector onto mode i (scalar, vectorial, or tensorial). Projecting (V.2) we obtain the algebraic response for each mode amplitude,

$$k_i\,\mathcal{G}_i = J_i, \qquad \Rightarrow \qquad \mathcal{G}_i = \frac{1}{k_i}\,J_i, \tag{V.4}$$

where $k_i > 0$ is the *modal stiffness*. Equation (V.4) is precisely a Hooke law at the modal level: *deformation equals compliance times source.*

C. Action as geometric deformation and the universal constitutive law

From Part I we established the dimensional collapse $[M] \equiv [L] \equiv [T]$ and that action has the character of a spacetime area, $[S] = [L^2]$. For a normalized unit 4-volume (the fundamental cell of the substrate), the scalar measure of deformation carried by mode i is thus naturally identified with a *modal action* S_i proportional to $\mathcal{G}_i$. Similarly, the projected source J_i defines the *modal charge* Q_i (mass, electric charge, thermal/entropic source, *etc.*) through the same normalization cell. With this identification, (V.4) becomes the *unified Hooke law* for sources:

$$\boxed{S_i = \frac{1}{k_i}\,Q_i}, \qquad \Longleftrightarrow \qquad \boxed{Q_i = k_i\,S_i}. \tag{V.5}$$

That is, *the deformation (action/area) required to stabilize a mode equals the applied source times the modal compliance.* Equivalently, *the source equals the modal stiffness times the action.* This is the sought-for generalization: the law holds verbatim for compressive (mass-like), torsional (electromagnetic), and tensorial (thermo-entropic) modes.

D. Why this form

The operator identity $\mathcal{G} = \mathbb{C}J$ is fixed by covariance and linearity around equilibrium. Its modal reduction $\mathcal{G}_i = J_i/k_i$ is forced by homogeneity/isotropy. With action identified as the geometric measure of deformation in the normalized cell, $S_i \propto \mathcal{G}_i$, the constitutive law (V.5) is not an ansatz but the *unique linear, covariant map* between sources and deformations compatible with the unified Lagrangian of Part I. Any alternative would either break covariance, violate linear superposition near equilibrium, or introduce extra dimensional constants contrary to the Spacetime Equivalence Principle.

Remark V.1. *The minus signs familiar from restoring forces (e.g. Hooke's law $F = -k\,x$) are absorbed here in the definition of J as the* applied *source that balances the internal restoring response. Stability requires $k_i > 0$ for all propagating modes, ensuring that the quadratic energy is bounded below and that $\mathbb{K}$ is (elliptic/hyperbolic) invertible in the corresponding regime.*

VI. THE PRINCIPLE OF MODAL RECIPROCITY: STATIC STABILITY AND DYNAMIC CAUSALITY IN THE SUBSTRATE

A. Static Reciprocity: The Condition of Stability and the Hierarchy of Forces

For a single, unified substrate to be both stable (not collapsing) and excitable (not infinitely rigid), its elastic properties across orthogonal modes cannot be independent. We elevate this physical requirement to a fundamental *Principle of Modal Reciprocity.* This principle states that *the stiffness of the substrate in a given mode is inversely related to its stiffness in an orthogonal mode.*

For the primary compressive (longitudinal, $\|$) and torsional (transverse, $\perp$) modes, this principle is formalized as a stability condition:

$$k_\| \cdot k_\perp = \mathcal{C}_{\text{geom}} \cdot \kappa^2, \qquad \text{(VI.1)}$$

where κ is the universal stiffness constant of the substrate from Part I, and $\mathcal{C}_{\text{geom}}$ is a dimensionless geometric factor of order unity reflecting the isotropy of three-dimensional space. As will be shown in Sec. XXXVII, this factor can be derived from the geometric integration of self-energy in spherical coordinates.

Equation (VI.1) guarantees the stability of the geometro-elastic medium. It expresses the fact that the medium's resistance to longitudinal strain is reciprocally balanced by its resistance to transverse strain. A stiffer electromagnetic response (large $k_\perp$) necessarily entails a more compliant, or "softer," gravitational response (small $k_\|$). Any deviation from this reciprocity would render the substrate either undeformable (if both stiffnesses are large) or unstable (if both are small).

Physical interpretation

Equation (VI.1) captures the observed hierarchy of interactions without arbitrary assumptions:

- The electromagnetic field corresponds to torsional excitations with extremely high stiffness $k_\perp$, consistent with its strength and short-range rigidity.
- Gravity corresponds to compressive excitations with extremely low stiffness $k_\|$, consistent with its weakness and long-range compliance.

Thus, the empirical disparity between electromagnetic and gravitational strengths is not accidental but a direct manifestation of the reciprocal balance demanded by a single geometro-elastic substrate.

Inevitability of the duality

The duality between stiffness and compliance follows inevitably from three principles already established:

1. The *Spacetime Equivalence Principle* ($[M] \equiv [L] \equiv [T]$) reduces all dimensional assignments to geometry.
2. The *unified constitutive law* $Q_i = k_i S_i$ requires each source to be stabilized by a finite stiffness.
3. The *single-substrate hypothesis* ensures that all modal stiffnesses derive from the same κ, which enforces the reciprocity relation (VI.1).

Any violation of this reciprocity would break energy balance in the substrate: if both $k_\|$ and $k_\perp$ were simultaneously large, the medium would forbid deformation altogether (contradicting the existence of excitations); if both were small, the medium would collapse under any source (contradicting stability). The hierarchy of forces is therefore *not contingent but structurally necessary*, rooted in the duality of stiffness and compliance in a unified elastic spacetime.

Conclusion

We conclude that the extreme weakness of gravitation relative to electromagnetism arises as the elastic complement of the substrate: one mode is strong because the other is weak. This stiffness–compliance duality is not an optional feature but the only configuration that allows a single geometro-elastic substrate to remain both deformable and stable. In the next section we extend this framework by introducing the concept of *dissipation*, showing how the intrinsic "*viscosity*" of the substrate gives rise to the dimensionless fine-structure constant α.

B. Dynamic Reciprocity: The Causal Universality Condition

Beyond the static stability condition, the Principle of Modal Reciprocity finds a profound dynamic manifestation rooted in a core axiom of the theory: **Lorentz Invariance**. For the QEG framework to be self-consistent, all emergent massless, wave-like excitations of the unified substrate must propagate at the same universal speed, c. This *principle of Causal Universality* imposes a strict constraint on the constitutive properties of any given mode.

Let us define the effective inertial (permeability-like) and compliant (permittivity-like) constants for an arbitrary mode 'i' as μ_i and ε_i, respectively. The propagation speed of a wave in this modal sector is given by $v_i = 1/\sqrt{\mu_i\varepsilon_i}$. The Causal Universality Condition therefore requires that for every wave-like mode i:

$$\mu_i\varepsilon_i = \frac{1}{c^2}. \qquad \text{(VI.2)}$$

Furthermore, the theory establishes a hierarchy of interactions, where the inertial response of any mode i can be expressed as a scaling of a baseline inertia, μ_{base}. Let this be described by a dimensionless scaling factor s_i, such that:

$$\mu_i = s_i \cdot \mu_{\text{base}}. \qquad \text{(VI.3)}$$

By combining these two conditions, the dynamic nature of reciprocity becomes explicit. Substituting Eq. (VI.3) into Eq. (VI.2) and solving for the modal compliance ε_i, we find:

$$(s_i \cdot \mu_{\text{base}})\,\varepsilon_i = \frac{1}{c^2} \implies \varepsilon_i = \frac{1}{s_i}\left(\frac{1}{\mu_{\text{base}}c^2}\right). \qquad \text{(VI.4)}$$

Recognizing that the term in parenthesis is simply the baseline compliance, $\varepsilon_{\text{base}}$, we arrive at the general law of dynamic reciprocity:

$$\boxed{\varepsilon_i = \frac{1}{s_i} \cdot \varepsilon_{\text{base}}} \qquad \text{(VI.5)}$$

This result is a cornerstone of the theory. It rigorously demonstrates that if the inertial response of a mode is scaled by a factor s_i, its compliant response must be scaled by the exact inverse factor, $1/s_i$. This dynamic reciprocity is not an isolated coincidence but the second core manifestation of the Principle of Modal Reciprocity, complementing the static relationship between stiffnesses. It confirms that the vacuum's elastic properties are deeply interconnected through both stability and causality, with the capacity to yield in one mode being intrinsically linked to its resistance in another.

VII. THE DISSIPATIVE NATURE OF THE SUBSTRATE AND THE ORIGIN OF THE FINE-STRUCTURE CONSTANT

A. The necessity of a dissipative substrate

The Lagrangian presented in Part I, $\mathcal{L}_{\rm QEG}$, describes an ideal, perfectly elastic medium. However, no physical medium can be perfectly elastic: stability under real excitations requires a mechanism of *dissipation*. Without it, oscillations would grow without bound and localized excitations could not stabilize, contradicting the observed persistence of particles and fields.

Indeed, the necessity of a single, dimensionless dissipative parameter follows inevitably from three prior requirements:

1. The *unified constitutive law* ($Q_i = k_i S_i$) demands finite, stable deformations.
2. The *principle of modal reciprocity* requires complementary stiffnesses across orthogonal modes, but does not by itself suppress divergences in excitations.
3. The *single-substrate hypothesis* forbids the introduction of independent damping constants for each mode: dissipation must be controlled by a unique, universal invariant.

Taken together, these conditions leave no freedom: the substrate must carry a single, dimensionless measure of dissipation. This is entirely analogous to the *damped harmonic oscillator*, where the addition of a velocity-proportional term is the unique way to stabilize oscillations without altering equilibrium. By covariance, the same logic applies at the field level: the equation of motion must include a dissipative contribution proportional to the "velocity" of the deformation field.

Accordingly, the most general covariant modification of the field equations at lowest order in derivatives augments the conservative dynamics with a *universal* linear damping term along the local rest frame u^ρ:

$$\kappa\,\Box\mathcal{G}_{\mu\nu} - \frac{\partial V}{\partial \mathcal{G}^{\mu\nu}} - \gamma\,(u^\rho\nabla_\rho\mathcal{G}_{\mu\nu}) = J_{\mu\nu},$$

where γ is the (dimensionless) damping coefficient of the substrate and u^μ the 4–velocity of its local rest frame. This structure introduces dissipation without violating Lorentz covariance or adding dimensional scales.

Optional notation (mode projections). When referring to the scalar trace and traceless shear modes extracted from $\mathcal{G}_{ij}$ via fixed normalizations (C_Θ, C_Σ), any apparent mode dependence of dissipation is a *projection*, not a new parameter: we write $\gamma_\Theta = \Pi_\Theta\,\gamma$ and $\gamma_\Sigma = \Pi_\Sigma\,\gamma$, with Π_Θ, Π_Σ fixed by (C_Θ, C_Σ).

B. Dissipation as Emergent Real-Time Coarse-Graining and the Universal Damping Ratio

The fundamental Lagrangian of Quantum-Elastic Geometry, $L_{\rm QEG}$, describes a conservative system. However, physical stability and connection to thermodynamics necessitate dissipation. Within QEG, dissipation is not appended ad hoc; it emerges naturally when integrating out fast quantum or thermal modes of the substrate using the closed-time-path (CTP) formalism [26]. Splitting the unified field as $\mathcal{G} = \mathcal{G}_{\rm slow} + \chi_{\rm fast}$ and tracing over the fast modes $\chi_{\rm fast}$ yields an effective action for the slow modes containing an influence functional with a local imaginary part:

$$\begin{aligned} S_{\rm eff}[\mathcal{G}_{\rm slow}] &= S^{(\rm coarse)}_{\rm QEG}[\mathcal{G}_{\rm slow}] \\ &- i\int d^4x\,\sqrt{-g}\,\frac{\gamma}{2}\,(u^\mu\nabla_\mu\mathcal{G}_{\alpha\beta})(u^\nu\nabla_\nu\mathcal{G}^{\alpha\beta}) + \mathcal{O}(\nabla^2,\chi^3) \end{aligned} \tag{VII.1}$$

where $\gamma \geq 0$ is the **emergent dissipative coefficient**, fixed by the substrate's fluctuation–dissipation response via Kubo-type relations, e.g. $\gamma(\omega) = \frac{1}{\omega}\,{\rm Im}\,\chi^{\rm ret}_{\dot{\mathcal{G}}\dot{\mathcal{G}}}(\omega)\Big|_{\omega\to 0}$ [27]. By the fluctuation–dissipation theorem, the imaginary part is positive semidefinite, ensuring $\gamma \geq 0$ and consistent entropy production for near–equilibrium dynamics. In the hydrodynamic regime we use $\gamma \equiv \lim_{\omega\to 0}\gamma(\omega)$ as the constant low-frequency damping coefficient. The 4-velocity u^μ denotes the local rest frame of the coarse-grained substrate (e.g., the timelike eigenvector of its effective stress tensor). This choice is fully background-free: u^μ is determined by $\mathcal{G}$ and not by an external frame.

Remark VII.1 (Rayleigh Functional from QEG)**.** *Under local equilibrium and short-memory coarse-graining, the CTP effective action of the QEG substrate generates a positive-definite quadratic damping term. In the weak-field limit, this term is identical to the covariant Rayleigh functional. Hence, the dissipative term used herein is an emergent ingredient derived from $S_{\rm QEG}$, not an external add-on.*

Explicit Form for Weak-Field Dynamics

The leading-order dissipative term identified in Eq. (VII.1) corresponds precisely to the covariant **Rayleigh functional**:

$$\mathcal{R}[\mathcal{G};\gamma] = \frac{\gamma}{2}\int d^4x\,\sqrt{-g}\,\Big(u^\mu\nabla_\mu\mathcal{G}_{\alpha\beta}\Big)\Big(u^\nu\nabla_\nu\mathcal{G}^{\alpha\beta}\Big). \tag{VII.2}$$

The associated *dissipative four-force density* is

$$F^{(\rm diss)}_{\mu\nu} = -\frac{1}{\sqrt{-g}}\,\frac{\delta\mathcal{R}}{\delta(u^\rho\nabla_\rho\mathcal{G}^{\mu\nu})} = -\gamma\,\Big(u^\rho\nabla_\rho\mathcal{G}_{\mu\nu}\Big). \tag{VII.3}$$

In the weak–field/Minkowski limit we set $\mathcal{G}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and replace covariant by partial derivatives, $\nabla \to \partial$. Adding this force to the conservative Euler–Lagrange equations derived from the weak-field Lagrangian

$$\mathcal{L}^{(\rm weak\text{-}field)}_{\rm QEG} = \frac{\kappa}{2}\,(\partial^\alpha h^{\mu\nu})(\partial_\alpha h_{\mu\nu}) - V(h), \tag{VII.4}$$

where indices are now raised and lowered with the Minkowski metric $\eta_{\mu\nu}$, yields the damped equations of motion used throughout this section:

$$\boxed{\kappa\,\Box h_{\mu\nu} - \frac{\delta V(h)}{\delta h^{\mu\nu}} - \gamma\,\Big(u^\rho\partial_\rho h_{\mu\nu}\Big) = J_{\mu\nu}} \tag{VII.5}$$

Remark VII.2 (On minimality and uniqueness)**.** *The Rayleigh functional (VII.2) is the simplest covariant choice for linear dissipation: it is the lowest-order, local, parity-even scalar quadratic in the field velocity $u^\mu\nabla_\mu\mathcal{G}_{\alpha\beta}$, producing a linear damping force (VII.3). Under locality, parity evenness, single material derivative, and diffeomorphism covariance, no additional independent scalar exists at this order.*

Thus, we have obtained the explicit dissipative contribution and its connection with the Rayleigh functional, where the universal damping coefficient γ emerges naturally from coarse-graining of the unified action.

1. Physical Origin of the Damping Coefficient

We have seen how the introduction of a dissipative term into the substrate's dynamics requires the identification of a dimensionless damping constant. We show that this role is uniquely filled by the *fine-structure constant, α*. This is not an arbitrary choice. Within the Standard Model, α governs the fundamental vertex of Quantum Electrodynamics (QED), which describes the quantum interaction between light and matter. More fundamentally, QED is the theory of the vacuum's own self-interaction, mediated by the continuous creation and annihilation of virtual particle-antiparticle pairs. In the QEG framework, dissipation is precisely the mechanism by which a coherent deformation (a field excitation) loses energy to the underlying quantum fluctuations of the substrate. It is therefore natural to identify the universal damping ratio of the medium with the very constant that quantifies its fundamental quantum self-interaction. Thus, in QEG, α is elevated from a purely electromagnetic coupling constant to a universal property of the spacetime substrate: its intrinsic dissipative coefficient.

C. Confirmation from the Vacuum's Oscillatory Response

This theoretical identification is powerfully confirmed by analyzing the vacuum as an oscillatory medium, analogous to an RLC circuit. The fine-structure constant α [28] can be defined as the ratio of two energies:

- the energy needed to overcome the electrostatic repulsion between two electrons a distance of d apart
- the energy of a single photon of wavelength $\lambda = 2\pi d$ (or of angular wavelength d)

Therefore, we have that

$$\begin{aligned}\alpha &= \left(\frac{e^2}{4\pi\varepsilon_0 d}\right) \Big/ \left(\frac{hc}{\lambda}\right) \\ &= \frac{e^2}{4\pi\varepsilon_0 d} \times \frac{2\pi d}{hc} = \frac{e^2}{4\pi\varepsilon_0 d} \times \frac{d}{\hbar c} \\ &= \frac{e^2}{4\pi\varepsilon_0 \hbar c} \end{aligned} \quad \text{(VII.6)}$$

Other hand, in the context of an RLC circuit, the quality factor or Q factor [29] is a dimensionless parameter that describes how underdamped an oscillator or resonator is. It is defined as the ratio of the initial energy stored in the resonator to the energy lost in one radian of the cycle of oscillation. Therefore, we have that

$$\begin{aligned}Q &\stackrel{\text{def}}{=} 2\pi \times \frac{\text{Energy stored}}{\text{Energy dissipated per cycle}} \\ &= 2\pi f_r \times \frac{\text{Energy stored}}{\text{Power loss}} \\ &= \omega_0 \times \frac{\text{Energy stored}}{\text{Power loss}}\end{aligned} \quad \text{(VII.7)}$$

Where f_r is the resonance frequency.

The fine-structure constant can be written as

$$\alpha = \frac{1}{2} Z_0 \sigma,$$

where $Z_0 = \mu_0 c = \frac{1}{\varepsilon_0 c}$ is the vacuum impedance and $\sigma = \frac{e^2}{h}$ is the conductance quantum [30, 31]. It follows that

$$Z_0 \sigma = 2\alpha,$$

which can be interpreted as the intrinsic energy loss characteristic per radian for the vacuum medium itself. Thus, one can naturally define a vacuum quality factor as

$$Q = \frac{1}{Z_0 \sigma} = 2\pi \times \frac{\varepsilon_0 \hbar c}{e^2} = \frac{1}{2\alpha}.$$

where we can identify $E_{\text{stored}} = \frac{\hbar c}{\lambda}$ and $E_{\text{dissipated}} = \frac{e^2}{\varepsilon_0 \lambda}$.

The appearance of these two energies follows directly from well-established properties of a single electromagnetic vacuum mode of wavelength λ and the framework we have discussed throughout this Paper. On the one hand, the most fundamental quantum excitation of an electromagnetic mode is a single photon. The energy of this photon, which represents the total energy "stored" in the oscillating mode for this process, is given by the Planck-Einstein relation:

$$E_{\text{stored}} = \hbar\omega = \frac{hc}{\lambda}$$

where the energy of a single photon of wavelength $\lambda = 2\pi d$ is used. This interpretation defines E_{stored} as the energy of the fundamental quantum of light that populates and defines the oscillation within that vacuum mode.

On the other hand, within our elastic vacuum formalism, it is natural to propose a dissipated energy analogous to Hooke's law. As we will demonstrate below with a rigorous calculation based on Larmor's formula, this simple analogy predicts with great accuracy the energy dissipated by a quantum dipole.

- Using Hooke's Law, we can identify the dissipated energy using the formula $|E| = |kx^2|$ where k is the elasticity constant, and x the displacement [32]. As in the context of the unified field we have identified $k = \frac{1}{C} = \frac{1}{\varepsilon_0 \lambda}$, and the displacement x with the elementary charge e, we get that $E_{\text{dissipated}} = \frac{e^2}{\varepsilon_0 \lambda}$.
- Alternatively, consider an oscillating dipole $p(t) = p_0 \cdot cos(\omega t)$ with amplitude $p_0 = e \cdot d$, where $d = \lambda/(2\pi)$ is the separation and $\omega = c/d$ is the angular frequency. Using Larmor formula for time-averaged power radiated by an oscillating dipole [33, 34]:

$$\langle P \rangle = \frac{p_0^2 \omega^4}{12\pi\varepsilon_0 c^3} \quad \text{(VII.8)}$$

we can derive the energy dissipated per cycle (period $T = 2\pi/\omega$):

$$E_{\text{dissipated}} = \langle P \rangle \times T = \left(\frac{p_0^2 \omega^4}{12\pi\varepsilon_0 c^3}\right)\left(\frac{2\pi}{\omega}\right) \quad \text{(VII.9)}$$

And one finally gets that

$$E_{\text{dissipated}} = \frac{\pi e^2}{3\varepsilon_0 \lambda} \approx 1.05 \cdot \frac{e^2}{\varepsilon_0 \lambda} \quad \text{(VII.10)}$$

Thus, both energies can be derived from first-principles and lead directly to

$$Q = 2\pi \times \frac{E_{\text{stored}}}{E_{\text{dissipated}}} = 2\pi \times \frac{\hbar c/(\lambda)}{e^2/(\varepsilon_0\, \lambda)} = \frac{1}{2\alpha},$$

For an underdamped oscillator, the damping ratio is defined as

$$\zeta = \frac{1}{2Q},$$

which leads directly to

$$\boxed{\zeta = \alpha}$$

This perfect correspondence between the first-principles requirement for a dissipative invariant and the result from the vacuum's oscillatory dynamics provides strong evidence that α is the universal damping ratio of the geometro-elastic medium.

D. Interpreting c in terms of the Damped Resonant Frequency of the System

In a standard underdamped oscillator model [35–37], the damped frequency ω_d is given by

$$\omega_d \;=\; \omega_0\sqrt{1-\zeta^2}, \tag{VII.11}$$

where ω_0 is the *undamped* resonant (natural) frequency of the system, and ζ is the damping ratio.

We can associate these frequencies with propagation speeds by multiplying each angular frequency by the reference length within our framework, of *one meter*, yielding speeds in units of $\mathrm{m\,s^{-1}}$. Denoting:

$$v_{\text{damped}} \;=\; \omega_d \times 1\ \mathrm{m}, \quad v_{\text{undamped}} \;=\; \omega_0 \times 1\ \mathrm{m},$$

we can identify v_{damped} with the *measured* speed of light, conventionally denoted by c. In other words,

$$c \;=\; v_{\text{damped}} \;=\; \omega_d \times 1\ \mathrm{m}.$$

From Eq. (VII.11), we thus have

$$c \;=\; \omega_0 \cdot 1\ m\sqrt{1-\zeta^2}$$

Or, equivalently,

$$c^2_{\text{measured}} = c^2_{\text{real}}(1-\alpha^2) \tag{VII.12}$$

Which, solving for ζ, can be rewritten as

$$\zeta = \alpha = \sqrt{1-\frac{c^2_{\text{measured}}}{c^2_{\text{real}}}} \tag{VII.13}$$

Note the similarity of the above expression with the reciprocal of the Lorentz factor formula [38]. Thus, the fine-structure constant α can be regarded as the reciprocal of a "Lorentz-like" factor via

$$\zeta = \alpha \;=\; \frac{1}{\gamma} \;=\; \sqrt{1-\frac{c^2_{\text{measured}}}{c^2_{\text{real}}}}\,. \tag{VII.14}$$

These two views—the *damped oscillator* analogy for electromagnetic propagation and the *Lorentz-like* factor interpretation for α—are not only compatible, but in fact reinforce each other: α emerges as a geometric or relativistic "scaling factor" that governs attenuation in the oscillatory unified field, connecting electromagnetic propagation and the quantum vacuum's dissipative properties.

E. Dissipative duality: roles of the damping factor $\zeta = \alpha$ and the reciprocal of the quality factor $\frac{1}{Q} = 2\alpha$

Physical Origin: The Fermionic g-factor

The theoretical framework that we will develop reveals an apparent duality in the vacuum's dissipative response. On one hand, the damping that affects wave propagation manifests as a relativistic damping ratio $\zeta = \alpha$. On the other hand, the attenuation of energy in quantum interactions, quantified by the quality factor Q, is governed by $1/Q = 2\alpha$.

We propose that this duality is not a contradiction but a reflection of a deeper physical principle: *the factor of 2 originates from the Landé g-factor (g_e) of a fundamental, spin-1/2 fermionic excitation of the vacuum*, whose value predicted by the Dirac Equation is precisely $g_e = 2$. This hypothesis establishes a fundamental distinction between two dissipative regimes:

- **Bosonic Propagation Damping (ζ):** The factor $\zeta = \alpha$ represents the intrinsic damping experienced by a bosonic excitation (such as a spin-1 photon) as it propagates through the elastic vacuum medium. It is a pure measure of spacetime's "viscosity."
- **Fermionic Interaction Damping ($1/Q$):** The factor $1/Q = g_e\alpha \approx 2\alpha$ governs processes that involve the fermionic nature of the vacuum's excitations. This includes the "dressing" of a bare charge to form a stable particle (like the electron) or the energy transfer that gives rise to a quantum excitation. In these cases, the dissipation depends not only on the base viscosity (α) but is also amplified by the intrinsic spin-1/2 response ($g_e = 2$) of the excitation.

Far from being an electron-centric hypothesis, this interpretation posits that $g_e = 2$ is a topological property of the fermionic modes of the vacuum substrate itself. Particles like the electron simply inherit this fundamental characteristic. This idea is consistent with the treatment of the anomalous magnetic moment ($a_e = (g_e - 2)/2$) as a higher-order correction to the same dissipative mechanism.

Analogy with the RLC Circuit: Transient Damping vs. Resonant Dissipation

This physical distinction between ζ and Q finds a powerful analogue in the behavior of a classical RLC circuit, which strengthens its conceptual justification:

- The **damping ratio (ζ)** in an RLC circuit determines the system's *transient response* (i.e., how a free oscillation decays). Analogously, $\zeta = \alpha$ in our model governs the decay of a wave propagating freely through the vacuum.
- The **quality factor (Q)**, in contrast, describes the *steady-state or resonant response*. It is defined as the ratio of reactive impedance to dissipative resistance, quantifying the energy loss *per cycle* in an established oscillation. Analogously, $1/Q = g_e\alpha$ in our model governs the energy dissipation in a stable interaction, such as that which constitutes a "dressed" particle, where vacuum energy is continuously stored and dissipated.

Therefore, the vacuum is interpreted as a medium possessing a base friction for wave propagation ($\zeta = \alpha$) and a distinct, amplified dissipative resistance for fermionic interactions ($1/Q = g_e \cdot \alpha$). This interpretation resolves the apparent duality by unifying the vacuum's relativistic properties with its quantum, spin-dependent interactions.

F. Interpretation and discussion on Vacuum Damping

Throughout this Paper, we are proposing that the quantum vacuum itself acts as a structured, elastic–dissipative medium. In this picture, the vacuum consists of fluctuating virtual excitations with internal degrees of freedom that collectively endow it with both stiffness (reactive elasticity) and finite relaxation time (viscosity). A propagating electromagnetic wave then loses energy—not into real particles, but into the hidden structure of the vacuum—via coupling to these degrees of freedom. This loss manifests macroscopically as a damping ratio ζ, which we identify with the dimensionless fine-structure constant via $\zeta = \alpha$. Such a damping constant is natural if one treats the vacuum as an ensemble of coupled oscillators or as an emergent condensed-matter system, as suggested by various approaches to quantum gravity and emergent spacetime [39–41]. The resulting reduction in propagation speed is then not a kinematic effect, but a first-principles consequence of quantum back-reaction. This allows us to interpret the measured speed of light c as a damped, effective velocity arising from the underlying dissipative structure of the vacuum.

Importantly, this view does not violate local Lorentz invariance. All local observers measure the same effective speed $c = c_{\mathrm{measured}}$, and all physical laws remain Lorentz-invariant in that frame. The distinction between c_{real} and c thus becomes a global, geometric feature of the vacuum — akin to how curvature encodes gravitational effects in general relativity. In this case, however, the "curvature" is not geometric but modal: a manifestation of the vacuum's internal damping modes, whose excitation state defines a preferred frame only at a topological level, not at the level of measurable kinematics. Analogous to an RLC circuit, where the "natural" frequency ω_0 is never directly observed but rather only inferred through modeling, the proposed $c_{\mathrm{real}} > c$ does not admit superluminal information transfer, and thus poses no contradiction to special relativity or experiment.

G. Final notes

Higher–order radiative effects — e.g. the electron's anomalous magnetic moment $a_e = \alpha/2\pi$ — can be viewed as additional layers of the same dissipative mechanism. The damping encoded in the fine-structure constant α would be the first-order manifestation of how the vacuum's oscillator substrate "bleeds" energy back into itself through quantum fluctuations, and the anomalous magnetic moment can be viewed, in our framework, as the simplest radiative attenuation of a bare "undamped" coupling by the substrate's elastic resistance. Higher-order Feynman diagrams then correspond to more intricate couplings among modes of the vacuum, each contributing successive powers of α.

More generally, it implies that any relation among fundamental constants—whether in electromagnetism, gravitation or thermodynamics—must be dressed by a universal, dimensionless form factor $\Xi_{\mathrm{eff}}(\alpha)$ that encodes the accumulated effect of loop-induced damping within the vacuum substrate. In this way, radiative corrections are not mere perturbative afterthoughts, but the fingerprint of the same elastic and dissipative structure that unifies all fields at their quantum origin.

In this view, *the fine-structure constant α becomes not merely a coupling constant, but a unifying signature of modal attenuation across all field interactions* — electromagnetic, gravitational, and thermo-entropic alike. Any consistent geometro-elastic substrate must contain a dimensionless dissipative invariant; its identification with the fine-structure constant α is therefore not optional, but logically enforced by the unified framework.

Part III: Dynamic of fields and sources of deformation in quantum-elastic geometry

VIII. EMERGENCE OF THE CLASSICAL FIELD EQUATIONS

Having established the foundational principles of the QEG substrate, we now demonstrate how the established equations of General Relativity and Electromagnetism emerge as consistent, low-energy effective descriptions of the substrate's dynamics. This section provides the explicit derivations, bridging the microscopic model of $\mathcal{G}_{\mu\nu}$ with the macroscopic physics of $g_{\mu\nu}$ and $F_{\mu\nu}$.

A. From Substrate Dynamics to General Relativity

The unified field $\mathcal{G}_{\mu\nu}$ contains the full microscopic description of the spacetime substrate. The smooth, classical geometry described by GR emerges as its macroscopic, coarse-grained average. We therefore define the effective metric $g_{\mu\nu}$ as the statistical expectation value of the fundamental field:

$$g_{\mu\nu}(x) \equiv \langle \mathcal{G}_{\mu\nu}(x) \rangle \tag{VIII.1}$$

This *background-independent* definition is a cornerstone of the theory. It posits that what we perceive as "curved spacetime" is the cumulative, large-scale effect of the substrate's deformation. This identification stems as a necessary consequence of the framework's axioms: in II B we established that all coupling constants reduce dimensionally to the same stiffness κ, and in II F we showed that mass, charge, and entropy are endogenous modes of $\mathcal{G}\mu\nu$. Therefore, *any macroscopic geometric descriptor must emerge from $\mathcal{G}_{\mu\nu}$ itself.* No independent background is admissible without violating the Equivalence Principles already established. Moreover, Lorentz invariance requires that the coarse-grained substrate be represented by a symmetric rank-2 tensor. *The only covariant candidate is the statistical average of $\mathcal{G}_{\mu\nu}$, which we identify as the effective metric.* This ensures a direct correspondence between the microscopic modes and macroscopic curvature: scalar compressions ($\langle \mathcal{G}_{00} \rangle$) manifest as time-time curvature, torsional modes ($\langle \mathcal{G}_{0i} \rangle$) as frame-dragging, and tensorial strains ($\langle \mathcal{G}_{ij} \rangle$) as spatial curvature.

The Emergence of the Einstein-Hilbert Action

The fundamental action S_{QEG} is a generally covariant functional of the microscopic symmetric field $\mathcal{G}_{\mu\nu}$ of Lorentzian signature, which induces a valid four–volume form. We adopt the natural invariant measure $\sqrt{-\det(\mathcal{G})}\, d^4x$ and a kinetic term built from the Levi–Civita connection of $\mathcal{G}_{\mu\nu}$. Explicitly,

$$S_{\mathrm{QEG}} = \int d^4x \sqrt{-\det(\mathcal{G})} \left[\tfrac{\kappa}{2} \mathcal{G}^{\alpha\beta} \mathcal{G}^{\mu\nu} \nabla_\alpha \mathcal{G}_{\mu\nu} \nabla_\beta \mathcal{G}_{\rho\sigma} \mathcal{G}^{\rho\sigma} - V(\mathcal{G}) \right] \tag{VIII.2}$$

where the schematic notation $(\nabla\mathcal{G})^2$ used earlier is here understood as the positive–definite quadratic form built by contracting indices with $\mathcal{G}^{\mu\nu}$. Other equivalent kinetic choices related by field redefinitions or addition of covariant total derivatives lead to the same infrared (IR) effective theory after coarse–graining.[2]

a. Effective Action and the Low-Energy Limit. The classical action S_{eff} for the macroscopic metric $g_{\mu\nu} = \langle \mathcal{G}_{\mu\nu} \rangle$ is obtained by integrating out the microscopic, high-frequency fluctuations of $\mathcal{G}_{\mu\nu}$. As S_{QEG} is generally covariant, S_{eff} must also be. In the low-energy limit, its Lagrangian density, $\mathcal{L}_{\mathrm{eff}}$, can be expressed as a derivative expansion in powers of curvature. The leading terms are:

$$\mathcal{L}_{\mathrm{eff}} = \sqrt{-g} \left(C_0 + C_1 R + \mathcal{O}(R^2) \right) \tag{VIII.3}$$

The coefficients are determined by the properties of the QEG substrate. The constant term C_0 arises from the potential $V(\mathcal{G})$ and corresponds to a cosmological constant, $C_0 = -2\Lambda$. The term linear in the Ricci scalar, $C_1 R$, must emerge from the kinetic term of the QEG action, which is governed by the universal stiffness κ. For the theory to be consistent, we establish a *correspondence principle* linking the microscopic damping to the macroscopic gravitational coupling:

$$C_1 = \frac{1}{16\pi G} \propto \kappa \tag{VIII.4}$$

This identification is central: it elevates the gravitational constant from an empirical parameter to a direct measure of the substrate's elastic stiffness. The effective Lagrangian thus naturally takes the Einstein-Hilbert form.

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[2] One may equivalently write the kinetic term as $\frac{\kappa}{2} \nabla_\alpha \mathcal{G}_{\mu\nu} \nabla^\alpha \mathcal{G}^{\mu\nu}$ up to covariant surface terms; the present form makes index contractions transparent and keeps manifest general covariance.

b. Consistency with Uniqueness Theorems. This result, derived from the internal logic of QEG, is strongly supported by external consistency arguments. The classic Weinberg-Deser argument shows that any consistent, universally coupled low-energy theory for the massless spin-2 excitations of $\mathcal{G}_{\mu\nu}$ inevitably reconstructs the Einstein-Hilbert action. Furthermore, Lovelock's Theorem guarantees that in $3+1$ dimensions, this action is unique for equations of motion with at most second derivatives of the metric. These theorems confirm that the form derived from our physical correspondence principle is not only natural within QEG, but is the only mathematically consistent possibility.

Including a matter action S_{matter} that couples to the effective metric, we have:

$$S_{\text{eff}}[g, \Psi_{\text{matter}}] = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R - 2\Lambda\right) + S_{\text{matter}} \tag{VIII.5}$$

Varying this with respect to $g^{\mu\nu}$ yields the Einstein Field Equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{VIII.6}$$

where $G_{\mu\nu}$ is the Einstein tensor and the stress-energy tensor is defined as $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$. The coefficient G is fixed by requiring that the theory reproduces Newtonian gravity in the weak-field, static limit. [3] The g_{00} component of the equation becomes $\nabla^2 g_{00} \approx 8\pi G T_{00}$. Comparing this with the Poisson equation, $\nabla^2\Phi = 4\pi G\rho$, and the definition $g_{00} \approx -(1+2\Phi)$, confirms the normalization chosen in the action.

Newtonian check. In the weak-field, static limit $g_{00} \simeq -(1+2\Phi)$ and $T_{00} \simeq \rho$, the 00-component yields $\nabla^2\Phi = 4\pi G\,\rho$, fixing the overall normalization of G in the action.

Consistency Check: Symmetries and Conservation. The general covariance of the effective theory leads to a crucial consistency check. The Einstein tensor is mathematically constructed to be divergenceless ($\nabla^\mu G_{\mu\nu} \equiv 0$), which, via the field equations, enforces the covariant conservation of stress-energy: $\nabla^\mu T_{\mu\nu} = 0$. This aligns with the deep insight, articulated by Weinberg and others, that any theory of a massless, interacting spin-2 particle must inevitably lead to the principle of equivalence and the structure of General Relativity.

Geodesics and the Principle of Least Deformation

In GR, the motion of free particles follows geodesics that extremize the spacetime interval. Within the substrate model, we reinterpret this principle dynamically: free trajectories minimize the integrated deformation of the substrate along their path.

Let $\mathcal{D}[\gamma]$ denote the deformation functional associated with a worldline γ. A natural and covariant choice for this functional, representing the total integrated strain along a path, is the line integral measured by the fundamental field itself:

$$\mathcal{D}[\gamma] = \int \sqrt{-\mathcal{G}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda \tag{VIII.7}$$

where λ is an arbitrary path parameter. Then the principle of least deformation is

$$\delta\mathcal{D}[\gamma] = 0, \tag{VIII.8}$$

which seeks worldlines that are extremal with respect to the microscopic geometry. Under the coarse-graining operation where $\mathcal{G}_{\mu\nu}$ is replaced by its macroscopic average $g_{\mu\nu}$, this variational principle becomes $\delta\int\sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = 0$, which is precisely the standard definition of a geodesic in General Relativity:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0, \tag{VIII.9}$$

with $\Gamma^\mu_{\ \nu\lambda}$ the Christoffel symbols of $g_{\mu\nu}$.

This correspondence is also necessary. Since motion sources deformation in $\mathcal{G}_{\mu\nu}$, the least-deformation condition is the unique covariant generalization of free motion consistent with the Inertial Equivalence Principle. Under coarse-graining, it reduces to geodesic motion in the emergent metric $g_{\mu\nu}$.

Thus, the classical geodesic principle is reinterpreted as a manifestation of the substrate's elastic tendency to minimize cumulative strain, reinforcing the view of geometry as emergent rather than fundamental.

Conclusion

We have shown that the effective spacetime metric $g_{\mu\nu}$ emerges naturally from the deformation field $\mathcal{G}_{\mu\nu}$, as a macroscopic descriptor of substrate dynamics. The Einstein–Hilbert action appears as the large-scale limit of the unified Lagrangian, and geodesic motion follows from the principle of least deformation.

Hence, General Relativity is not a competing framework but an emergent approximation: the continuum geometry of GR arises from the coarse-grained behavior of the underlying quantum-elastic substrate.

B. Emergence of Maxwell's Equations

The derivation for electromagnetism follows from identifying the physical fields with the torsional modes of the substrate. The primary challenge is to construct an antisymmetric field strength tensor and its associated 4-vector potential from the dynamics of the fundamental *symmetric* tensor $\mathcal{G}_{\mu\nu}$.

a. Field Identification via Geometric Construction. To construct an effective $U(1)$ gauge potential A_μ from the underlying symmetric tensor $\mathcal{G}_{\mu\nu}$, we require a construction that satisfies several key principles. The resulting 1-form must: (i) be built from the covariant derivatives of the fundamental field, as gauge potentials are related to field gradients; (ii) isolate the purely torsional (vector) modes of the substrate's deformation; and (iii) respect Lorentz covariance. The following form is the simplest and most direct construction that satisfies these physical requirements.

$$A_\mu(x) \equiv C_{EM}\, u^\alpha P^{(EM)\beta\gamma}_{\mu\alpha} \nabla_\beta \mathcal{G}_{\gamma\delta} u^\delta \tag{VIII.10}$$

where u^μ is the substrate's local rest 4-velocity, C_{EM} is a normalization constant, and $P^{(EM)\beta\gamma}_{\mu\alpha}$ is a symmetry-respecting projector that selects the appropriate torsional (vector) modes of the deformation gradient. [4] The physical, gauge-invariant electromagnetic field strength tensor

[3] We will demonstrate in subsequent parts of this work (Parts IV and V) that G is not a free parameter, but is instead determined by the fundamental elastic and dissipative properties of the substrate.

[4] While other, more complex constructions might be possible, this form stands as the most natural and parsimonious expression for an emergent gauge potential. Its validity is ultimately confirmed by its success in reproducing the full structure of Maxwell's equations and the subsequent consistency of the entire network of derived constants.

$F_{\mu\nu}$ is then naturally defined as the exterior derivative of this emergent potential:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{VIII.11}$$

This construction provides the natural bridge from the symmetric tensor $\mathcal{G}_{\mu\nu}$ to the antisymmetric field strength $F_{\mu\nu}$. It guarantees the two source-free Maxwell equations ($\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} + \partial\vec{B}/\partial t = 0$) through the Bianchi identity $\partial_{[\lambda}F_{\mu\nu]} \equiv 0$, while the invariance of $F_{\mu\nu}$ under the transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ ensures the emergence of a local $U(1)$ gauge symmetry.

b. Derivation of the Field Equations and Coefficient Matching. Having constructed the field $F_{\mu\nu}$ from first principles, gauge and Lorentz invariance uniquely fix the lowest-order effective Lagrangian density for the torsional sector to be quadratic in F:

$$S_{\text{EM}}[A;g] \;=\; \int d^4x\,\sqrt{-g}\,\left(-\frac{1}{4\mu_0}\,F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu\right). \tag{VIII.12}$$

The coefficient μ_0 is not a new fundamental constant, but an effective parameter emerging from the substrate's stiffness κ in the torsional sector; its precise relationship will be derived in Part IV. The term $F_{\mu\nu}F^{\mu\nu}$ is the only parity-even, gauge- and Lorentz-invariant scalar at two derivatives. Varying Eq. (VIII.12) with respect to A_ν yields the inhomogeneous Maxwell equations:

$$\nabla_\mu F^{\mu\nu} \;=\; \mu_0\,J^\nu. \tag{VIII.13}$$

In the static, weak-field limit, this becomes $\nabla^2\Phi = -\rho/\varepsilon_0$, which fixes the normalization $\varepsilon_0\mu_0 = 1/c^2$ and recovers Coulomb's law. Thus, the full set of Maxwell equations is recovered.

c. Consistency Check: Symmetries and Conservation. The emergent $U(1)$ gauge symmetry of the action implies, via Noether's theorem, the conservation of the electric current, $\nabla_\mu J^\mu = 0$. The structure is thus fully self-consistent. The ability to choose a gauge, such as the Lorenz gauge condition ($\partial_\mu A^\mu = 0$), to simplify calculations without affecting the physical content is preserved, ensuring full compatibility with the standard formalism of electrodynamics.

C. Conceptual Closure

The derivations above establish a crucial bridge between the microscopic dynamics of the QEG substrate and the macroscopic language of classical field theory. We have demonstrated that the *mathematical structures* of General Relativity and Electromagnetism—specifically, a symmetric rank-2 tensor field for gravity and an antisymmetric rank-2 field strength for electromagnetism—emerge as inevitable descriptions of the substrate's compressive and torsional modes, respectively.

However, a full demonstration of emergence requires closing the logical loop by showing that the *coupling constants* (G, μ_0) are not free parameters to be fitted to experiment, but are themselves determined by the fundamental properties of the substrate (stiffness κ and dissipation α). This section has established the *form* of the laws; the subsequent Parts of this work are dedicated to deriving their *substance*—the values and interrelations of the constants themselves. This will complete the unification, showing that both the field equations and the constants that populate them arise from the same, single underlying reality.

IX. UNIVERSAL SELF-INTERACTION AND THE DUALITY OF COUPLINGS

A cornerstone of this framework is that sources are stable, extended deformations of the substrate. To understand their properties, we must analyze their self-energy—the energy required to sustain such a deformation. We will demonstrate that this energy is not a uniquely defined quantity, but can be calculated in two physically distinct ways. This duality in the definition of self-energy, as we will show, is the fundamental origin of a scale-dependent effective coupling constant for all physical interactions.

A. The duality of Self-Interaction Energy

The principle of superposition dictates that the total energy of a system of deformations is the sum of the energies from the mutual interaction of each pair. For a distribution of a modal source specified by a density $\rho_A(\mathbf{x})$, the total self-interaction energy U_A is given by the integral over all pairs of points in the distribution:

$$U_A[\rho_A] = \frac{1}{2}\int d^3x\,d^3y\;\rho_A(\mathbf{x})\,\mathcal{K}_A(\mathbf{x},\mathbf{y})\,\rho_A(\mathbf{y}) \tag{IX.1}$$

where the kernel $\mathcal{K}_A(\mathbf{x},\mathbf{y}) = \lambda_A G_A(\mathbf{x},\mathbf{y})$ contains the mode-dependent coupling strength λ_A and the Green's function G_A for the interaction (e.g., $1/|\mathbf{x}-\mathbf{y}|$ for long-range forces). This general form can be evaluated through different physical procedures, which reveal its inherent duality:

a. 1. Global Self-Energy (U_{glob}): The Work of Formation. The first method calculates the total energy by computing the work done to assemble the deformation from its infinitesimal constituents brought from infinity. This represents the *total work of formation*, including all internal binding energy.

Let us imagine assembling a spherical deformation of final radius a and uniform source density ρ_A by building it up in successive thin spherical layers, as illustrated in [42]. At any intermediate stage, the sphere has a radius r and contains a total modal source $Q_A(r)$. The potential at its surface is $\Phi_A(r) = \lambda_A Q_A(r)/r$. The work dU_A required to bring an additional infinitesimal layer of source dQ_A from infinity to the surface at radius r is:

$$dU_A = \Phi_A(r)\,dQ_A = \left(\frac{\lambda_A Q_A(r)}{r}\right) dQ_A \tag{IX.2}$$

For a uniform density ρ_A, the source contained within radius r is $Q_A(r) = \rho_A \cdot (\frac{4}{3}\pi r^3)$. The source in the next layer of thickness dr is $dQ_A = \rho_A \cdot (4\pi r^2 dr)$. Substituting these into the expression for dU_A:

$$dU_A = \frac{\lambda_A}{r}\left(\frac{4}{3}\pi\rho_A r^3\right)(4\pi\rho_A r^2 dr) = \frac{16\pi^2\lambda_A\rho_A^2}{3}r^4 dr \tag{IX.3}$$

The total energy required to assemble the full sphere is the integral of dU_A from $r=0$ to $r=a$:

$$U_{\text{glob}} = \int_0^a dU_A = \frac{16\pi^2\lambda_A\rho_A^2}{3}\int_0^a r^4 dr = \frac{16\pi^2\lambda_A\rho_A^2 a^5}{15} \tag{IX.4}$$

Expressing this in terms of the total source $Q_A = \rho_A \cdot (\frac{4}{3}\pi a^3)$, we arrive at the definitive expression for the Global Self-Energy:

$$U_{\text{glob}} = \frac{3}{5}\frac{\lambda_A Q_A^2}{a} \tag{IX.5}$$

The geometric factor $\mathcal{C}_g = 3/5$ is the universal signature of the total formation energy of a uniform, volume-filling deformation.

b. 2. Local Interaction Energy (U_{loc}): The External Field Energy. The second method calculates the energy by considering the source to be fully assembled and residing on a conducting surface, as in a capacitor. This procedure calculates the energy stored exclusively in the field *external* to the source's boundary, representing the energy available for interaction with other distant sources [42].

The energy U_A required to place a source Q_A on a configuration with "modal capacitance" C_A is given by:

$$U_A = \frac{1}{2}\frac{Q_A^2}{C_A} \tag{IX.6}$$

For a spherical shell of radius a, which models a source distribution with no internal structure, the modal capacitance is $C_A = a/\lambda_A$. Substituting this yields the energy stored in the external field:

$$U_{\rm loc} = \frac{1}{2}\frac{Q_A^2}{a/\lambda_A} = \frac{1}{2}\frac{\lambda_A Q_A^2}{a} \tag{IX.7}$$

Here, the geometric factor is $\mathcal{C}_g = 1/2$. This Local Interaction Energy correctly describes the energy of a thin spherical shell source and represents the quasi-linear regime where internal, non-linear binding energies are excluded.

B. Consequence: The Duality of Effective Couplings

The rigorous derivation above shows that the self-energy of a deformation is not single-valued, but depends on the physical question being asked: are we calculating the total work of formation ($U_{\rm glob}$) or the energy available for external interaction ($U_{\rm loc}$)? *The difference is captured by a purely geometric factor.*

This duality in energy implies an unavoidable duality in the effective coupling constant, λ_A. We can therefore define two distinct couplings, whose applicability depends on the physical regime:

- **The Local Coupling ($\lambda_{\rm loc}$):** This coupling describes systems dominated by non-linear self-interaction, such as dense, clumpy structures. The relevant energy is the total work of formation, $U_{\rm glob}$. It is called "global" energy because the calculation integrates over the *entire global volume of the source itself*, a procedure physically relevant for understanding a **local** object's internal structure. It is therefore associated with the geometric factor $\mathcal{C}_g = 3/5$.

- **The Global Coupling ($\lambda_{\rm glob}$):** This coupling describes systems in the quasi-linear regime, such as the homogeneous universe on **global** scales. The relevant energy is that of the external field, $U_{\rm loc}$, where internal non-linearities are averaged out. It is called "local" energy because the calculation considers only the field from the *local boundary of the source outwards*. It is therefore associated with the geometric factor for a surface-like source, $\mathcal{C}_g = 1/2$.

This section has thus established, from a rigorous generalization of classical field theory principles, a necessary duality of effective couplings. This provides the fundamental basis for the scale-dependent nature of gravity, which will be explicitly developed and applied in Sec. XXXVII.

Part IV: From geometro-elastic properties to Fundamental Constants of nature

To connect our previous abstract framework to physics as observed, we must now explore how the substrate's structure manifests itself in concrete, measurable constants. The transition requires a new step: moving from the *structural level* of laws and symmetries to the *modal level*, where specific excitations of the medium give rise to quantized actions and effective responses.

This Part is devoted to that bridge. The guiding principle of this Part is the same as in the previous one: *nothing is postulated that is not required by the substrate's geometry and dynamical consistency.* The constants of physics emerge here as necessary modal invariants of a single oscillatory medium.

In the next sections we develop these results step by step, beginning with the geometric interpretation of action itself.

X. THE GEOMETRIC NATURE OF ACTION

A. Action as a Geometric Deformation Area

By construction, the action carries the dimensions of energy × time:

$$[S] = [E]\cdot[T]. \tag{X.1}$$

Within the unified dimensional framework, $[E] \equiv [L]$ and $[T] \equiv [L]$, so

$$[S] = [L]\cdot[L] = [L^2]. \tag{X.2}$$

Thus, the action is not an abstract bookkeeping device but a measure of *spacetime area.* Every physical process corresponds to the sweeping out of a geometric deformation area within the substrate.

B. The Dimension of the Lagrangian Density

The action is defined as the integral of the Lagrangian density over spacetime:

$$S = \int \mathcal{L}\, d^4x. \tag{X.3}$$

Since $[S] = [L^2]$ and $[d^4x] = [L^4]$, it follows that

$$[\mathcal{L}] = \frac{[S]}{[d^4x]} = \frac{[L^2]}{[L^4]} = [L^{-2}]. \tag{X.4}$$

This is a profound and unavoidable result: *any* consistent Lagrangian density in this framework must have the dimensions of inverse area. *Crucially,* $[L^{-2}]$ *is precisely the dimensional signature of Gaussian or Ricci scalar curvature*, which means that the identification $\mathcal{L} \propto R$ in the Einstein–Hilbert action is not arbitrary but a necessary consequence of dimensional consistency within a geometro-elastic substrate. This elevates the *principle of least action* into a *principle of extremal geometry*: the dynamics of fields and particles emerge from spacetime's intrinsic tendency to minimize integrated curvature. In this view, the action is literally the spacetime area swept out by a deformation, and the Lagrangian density is the local curvature cost of that deformation.

C. Normalization and the Fiducial Scale

As already advanced in Sec. IV, while the theory is intrinsically scale-free, connecting it to measurable quantities requires the introduction of a fiducial reference scale. This is a standard normalization procedure that anchors the abstract geometry of the theory to the operational units used in empirical physics, such as the SI system.

Recalling from Sec. IV that all physical magnitudes collapse to a single geometric unit $[L]$, the choice of a reference length, time, or mass is formally equivalent. We therefore adopt the meter as our fiducial unit to maintain consistency with established conventions –any other fiducial choice (second, kilogram, etc.) would yield an equivalent identity by the Spacetime Equivalence Principle–. Therefore, *from this point forward, the fiducial unit of one meter, $L_{ref} = 1\ m$ —-or its equivalents as defined by the conversion scheme in Sec. IV—- will serve as the baseline scale for all subsequent derivations of SI values for physical constants.*

D. The baseline Lagrangian density

We introduce the baseline Lagrangian density as the curvature associated with the unit area:

$$\mathcal{L}_{\text{base}} := \frac{1}{(1\,\text{m})^2}. \tag{X.5}$$

Integrating this density over a unit 4-volume, $d^4x = (1\,\text{m})^4$, yields the fiducial action S_{base}:

$$S_{\text{base}} = \int \mathcal{L}_{\text{base}}\, d^4x = \frac{1}{(1\,\text{m})^2} \cdot (1\,\text{m})^4 = 1\,\text{m}^2. \tag{X.6}$$

This normalization forces a consistency condition between the geometric and dynamical descriptions of action. The Spacetime Equivalence Principle requires this geometric area to be equivalent to the standard unit of physical action, leading to the identity:

$$1\,\text{m}^2 \equiv 1\,\text{J}\cdot\text{s},$$

This is not a new physical law, but the bridge that ensures the theory's predictions are numerically consistent with experimental reality. It confirms that the energy needed to sustain a deformation over time physically corresponds to a literal area swept out in the spacetime substrate.

XI. MODAL SCALES AND THE ELECTROMAGNETIC QUANTUM OF AREA

The unified substrate supports distinct orthogonal modes (compressive, torsional, tensorial), all governed by the same geometro–elastic law but with different modal responses. While unified in origin, each mode is characterized by a natural *length scale* at which its dynamics become dominant. In this section, we *derive* the characteristic length of the torsional (electromagnetic) mode directly from the constitutive properties of the vacuum, and use it to obtain the electromagnetic quantum of area, $\hbar$.

A. The characteristic length of the electromagnetic mode

The vacuum's electromagnetic response is encoded by its constitutive parameters, (ε_0, μ_0). We now construct the *unique* length scale compatible with the following principles:

1. **Scale covariance:** All modal scales must ultimately be built from the fiducial reference length, L_{ref}, introduced in Sec. X C, without introducing independent dimensional parameters.

2. **Lorentz and Duality Covariance:** The scale must be invariant under Lorentz transformations and electromagnetic duality rotations. This restricts its dependence on the constitutive parameters to the unique invariant combination $\varepsilon_0\mu_0 = \frac{1}{c^2}$.[5]

3. **Minimality / structural simplicity:** In the absence of further physical principles, the relationship must be the simplest possible function consistent with the symmetries. This avoids introducing unmotivated complexity into the framework.

[5] The vacuum is symmetric under duality rotations that mix electric and magnetic fields. This symmetry is broken by quantities like the impedance $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ which cannot, therefore, define a fundamental geometric scale of the vacuum itself.

These requirements uniquely determine the form of the characteristic length of the electromagnetic mode, ℓ_{EM}. By principle 1, any candidate must be of the form

$$\ell_{\text{EM}} \;=\; L_{\text{ref}}\, f(\varepsilon_0, \mu_0);$$

by principle 2, f can only depend on the duality–symmetric, Lorentz–invariant product $\varepsilon_0\mu_0$; by principle 3, the simplest non-trivial relationship is linear:

$$\ell_{\text{EM}} \;=\; L_{\text{ref}} \left(\varepsilon_0\mu_0\right). \tag{XI.1}$$

Using the vacuum identity $\varepsilon_0\mu_0 = 1/c^2$ and the SI–anchoring $L_{\text{ref}} = 1\,\text{m}$ of Sec. X, we obtain the *electromagnetic modal length*:

$$\boxed{\ell_{\text{EM}} \;=\; \frac{1\,\text{m}}{c^2}} \tag{XI.2}$$

This result is the most logical and simple construction following the first principles stated.

B. Derivation of the electromagnetic quantum of area $\hbar$

Since $[S] = [L^2]$, the action carried by a mode of characteristic length ℓ must scale as ℓ^2. Substituting the modal length derived in Eq. (XI.2),

$$\ell_{\text{EM}} = \frac{1\,\text{m}}{c^2},$$

we obtain the first–order electromagnetic quantum of area, which we identify with the reduced Planck constant:

$$\boxed{\hbar \;\equiv\; S_{\text{EM}} \;=\; \left(\ell_{\text{EM}}\right)^2 \;=\; \left(\frac{1\,\text{m}}{c^2}\right)^2 \;=\; \frac{1\,\text{m}^2}{c^4}} \tag{XI.3}$$

Using the fiduciary identity $1\,\text{m}^2 \equiv 1\,\text{J}\cdot\text{s}$ (Sec. X), Eq. (XI.3) yields $\hbar \approx 1.23 \times 10^{-34}\,\text{J\,s}$, in leading-order agreement with the measured value $\hbar_{\text{exp}} = 1.054 \times 10^{-34}\,\text{J\,s}$.

XII. THE THERMO-ENTROPIC MODAL SCALE AND THE COSMOLOGICAL ACTION

A. The Thermo-Entropic Length Scale

The thermo-entropic mode represents the lowest-energy, largest-scale collective excitations of the substrate. Its characteristic length scale cannot be a new fundamental parameter but must emerge from the intrinsic properties of the framework already established:

1. **Quantum Discreteness:** encoded by $\hbar$, the minimal unit of action/area.

2. **Relativistic Structure:** encoded by c, which governs causal propagation.

3. **Reference Normalization:** the fiducial length L_{ref} X C, which anchors all scales.

By structural necessity, the only non-trivial length that can be constructed from these three ingredients is the **quantum-relativistic reference length**, L_q:

$$L_q \;=\; \frac{\hbar c}{L_{\text{ref}}}. \tag{XII.1}$$

For the lowest-energy mode of the substrate, there is no other available scale to define its characteristic length.

We therefore must identify ℓ_{th} with this unique quantum-relativistic construct:

$$\ell_{\text{th}} \equiv L_q = \frac{\hbar c}{1\,\text{m}}. \tag{XII.2}$$

Substituting the first-order expression for $\hbar$ from Eq. (XI.3), $\hbar = 1\,\text{m}^2/c^4$, yields:

$$\ell_{\text{th}} = \frac{(1\,\text{m}^2/c^4)\,c}{1\,\text{m}} = \frac{1\,\text{m}}{c^3}. \tag{XII.3}$$

Thus, the characteristic length of the thermo-entropic mode is uniquely determined by the theory's structure:

$$\boxed{\ell_{\text{th}} = \frac{1\,\text{m}}{c^3}} \tag{XII.4}$$

B. The Thermo-Entropic Modal Action

Since $[S] = [L^2]$, the action carried by a mode of characteristic length ℓ must scale as ℓ^2. Thus, the action of the thermo-entropic mode is given by:

$$S_{\text{th}} = (\ell_{\text{th}})^2 = \left(\frac{1\,\text{m}}{c^3}\right)^2 = \frac{1\,\text{m}^2}{c^6}. \tag{XII.5}$$

With the fiducial identification $1\,\text{m}^2 \equiv 1\,\text{J}\cdot\text{s}$, this corresponds to the value:

$$\boxed{S_{\text{th}} = \frac{1}{c^6}\,\text{J}\cdot\text{s}} \approx 1.37\times10^{-51}\,\text{J}\cdot\text{s}. \tag{XII.6}$$

C. Interpretation

This quantity represents the *minimal elastic action* associated with a global, isotropic deformation of the substrate. Its extreme smallness is a direct consequence of the modal hierarchy, where compressive/tensorial modes are suppressed by higher powers of c. Subsequent sections will show how it is naturally related to the observed cosmological constant Λ, thereby linking quantum discreteness, relativistic structure, and cosmic acceleration within a single geometro-elastic framework.

Conclusion on Modal Actions

We have demonstrated how the actions associated with the primary modes of the substrate emerge not as empirical inputs, but as geometrically quantized projections of a single underlying vacuum structure. The baseline action (S_{base}), the electromagnetic quantum of action ($\hbar$), and the thermo-entropic action (S_{th}) are not independent parameters but represent the scaled geometric integrals of the same deformable substrate, each arising from a specific symmetry projection of the unified geometro-elastic dynamics. In this view, what we traditionally call fundamental constants are, in fact, the geometric signatures of the vacuum's quantized deformability.

XIII. MODAL LAGRANGIAN DENSITY VERSUS EFFECTIVE ENERGY DENSITY

Given the Spacetime Equivalence Principle and the identification of action as an area, $[S] = [L^2]$, it becomes inevitable to distinguish between the *modal Lagrangian density* ($\mathcal{L}$), associated with a single coherent excitation of the substrate, and the *effective vacuum energy density* (ρ_{vac}), which appears as a macroscopic observable after coarse-graining over incoherent modes. Specifically:

- **The Lagrangian Density ($\mathcal{L}$)** describes the dynamics of a single, coherent, fundamental mode of the vacuum oscillator substrate. It is a theoretical, microscopic quantity. Dimensional consistency and the fiduciary normalization $1\,\text{m}^2 \equiv 1\,\text{J}\cdot\text{s}$ require that the fundamental modal action density takes the form

$$\mathcal{L}_{\text{modal}} = \frac{\hbar c}{1\,m^4}$$

- **The Measured Energy Density (ρ_{vac})** is a macroscopic, cosmological observable. It reflects the net effect of an immense number of uncoordinated vacuum oscillators, with their phases and spatial orientations being statistically random. At the macroscopic level, statistical isotropy enforces a coarse-grained average over random phases, entirely analogous to the emergence of $g_{\mu\nu} = \langle G_{\mu\nu}\rangle$. The observable energy density is therefore

$$\rho_{\text{eff}} \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}_{\text{modal}}\,d\theta = \frac{\mathcal{L}_{\text{modal}}}{2\pi} = \frac{\hbar c}{2\pi\cdot 1\,m^4}$$

Finally, note that:

- By substituting k with the quantum of angular frequency of the electromagnetic mode $\frac{c}{\lambda}$, and Planck's constant h for the quantum of action, we obtain a quantum expression of mass-energy using V.5:

$$m = \frac{hc}{\lambda} \tag{XIII.1}$$

  This equation directly links the energy of photons (or other quantum excitations) to mass, reinforcing Einstein's mass-energy equivalence from a fundamentally new perspective. Substituting $\lambda = 1\ m$ in (XIII.1), corresponding to the characteristic scale of the unit quantum oscillator in our framework, we obtain the quantum of mass-energy for the electromagnetic field $m = \frac{hc}{1\ m}$.

- Dividing this quantum mass-energy by a volume $V = 1\ m^3$, and considering the linear momentum $\hbar = \frac{h}{2\pi}$ we obtain a quantum of mass density $\rho_{vac} = \frac{\hbar c}{1\ m^4}$ which corresponds with (XIII). When transitioning from the description of a single, coherent angular mode to a macroscopic, isotropic average over all possible phases or solid angles (XIII), we get $\rho_{\text{eff}} \equiv \frac{\hbar c}{2\pi\cdot 1\,m^4} \approx 5.03\times10^{-27}$ kg m^{-3} which retrospectively matches the Planck satellite measurements, thereby turning the concordance into a nontrivial prediction of the model rather than an adjustment. [43].

XIV. MODAL DYNAMICS AND THE EMERGENCE OF COUPLING CONSTANTS

Having established the characteristic scales of the substrate's modes, we now derive the coupling constants governing their dynamics. We will show that the magnetic permeability (μ_0), Coulomb's constant (K_e), and Boltzmann's constant (k_B) are not independent parameters. They emerge as projections of a single, universal inductive response law, evaluated at the characteristic velocity of each mode. To do so, we first establish the unique scaling of these velocities.

A. Modal Velocities and the Uniqueness of Scaling

In our framework, current is a dimensionless measure of deformation flow, $[I] = 1$. The physical distinction between modes arises from their characteristic propagation velocity, v. These velocities are not arbitrary but are uniquely determined by the theory's axiomatic principles:

1. **Fiducial Baseline:** The reference mode is defined by the fiducial scales, corresponding to a normalized velocity $v_{\text{base}} = 1$.

2. **Relativistic Limit:** Lorentz invariance introduces a single invariant speed, c. The fastest, torsional (electromagnetic) excitations must propagate at this speed, $v_{\text{EM}} = c$.

3. **Modal Reciprocity:** The principle of reciprocity requires the highly stiff, fast torsional mode ($v \sim c$) to be balanced by a highly compliant, slow compressive/tensorial (thermo-entropic) mode. The only non-trivial scaling inverse to c is $1/c$. Thus, $v_{\text{th}} = 1/c$.

These three velocities,

$$\{1,\ c,\ 1/c\},$$

are therefore the only ones consistent with the geometro-elastic substrate. Any alternative assignment would violate either Lorentz invariance, reciprocity, or the fiducial normalization.

B. The Universal Law of Inductive Response

The substrate's dynamic response to a changing deformation flow is governed by a universal inductive law, analogous to Faraday's law:

$$\mathcal{E} = -L_{\text{vac}} \frac{dI}{dt}, \tag{XIV.1}$$

where $\mathcal{E}$ is the resulting modal potential and $L_{\text{vac}} = \mu_0 L_{\text{ref}}$ is the fiducial inductance. The term dI/dt represents the characteristic rate of change, or acceleration, of the modal deformation.

The characteristic rate of change is the ratio of the characteristic velocity to the characteristic timescale: $dI/dt \sim v_{\text{mode}}/\tau_{\text{mode}}$. The timescale of a mode is inversely proportional to its velocity (high–velocity modes have short timescales, slow modes have long ones), so $\tau_{\text{mode}} \sim 1/v_{\text{mode}}$. This leads to the universal scaling rule:

$$\frac{dI}{dt} \sim \frac{v_{\text{mode}}}{1/v_{\text{mode}}} = (v_{\text{mode}})^2. \tag{XIV.2}$$

Hence, the modal potential response is proportional to the square of the characteristic velocity:

$$\boxed{\mathcal{E}_{\text{mode}} \propto \mu_0 \, (v_{\text{mode}})^2}. \tag{XIV.3}$$

This law is covariant, dimensionally consistent, and introduces no new parameters: all modal couplings must be determined by μ_0 and the velocity hierarchy.

C. Derivation of the Coupling Constants

Applying the scaling law (XIV.3) to the three modal velocities yields the coupling constants directly.

1. Baseline Potential (μ_0): For the reference mode, $v_{\text{mode}} = 1$. The response is the baseline potential:

$$\mathcal{E}_{\text{base}} \equiv \mu_0, \tag{XIV.4}$$

the intrinsic inertial potential of the substrate in the quasi-static limit.

2. Electromagnetic Potential (K_e): For the torsional mode, $v_{\text{mode}} = c$. The response is scaled by c^2:

$$\mathcal{E}_{\text{EM}} = \mu_0 c^2 = \frac{1}{\varepsilon_0}. \tag{XIV.5}$$

This corresponds, up to the geometric factor 4π, to Coulomb's constant $K_e = 1/4\pi\varepsilon_0$. Thus, K_e emerges as the relativistically–scaled inductive potential of the substrate.

3. Thermo-Entropic Potential (k_B): For the compressive/tensorial mode, $v_{\text{mode}} = 1/c$. The response is suppressed by $1/c^2$:

$$\mathcal{E}_{\text{th}} = \frac{\mu_0}{c^2}. \tag{XIV.6}$$

This value coincides, to leading order, with Boltzmann's constant k_B. The small deviation observed experimentally is interpreted as the radiative dressing of this response by the universal dissipative invariant α, consistent with Secs. VII and VII.

D. Synthesis and Interpretation

We therefore obtain the unified ladder of coupling constants:

$$\boxed{\Big\{\ \mu_0,\ 4\pi K_e,\ k_B\ \Big\} \ \approx\ \mu_0 \{1,\ c^2,\ c^{-2}\} \Big[1 + \mathcal{O}(\alpha)\Big].} \tag{XIV.7}$$

This shows that the constants governing magnetostatics, electrostatics, and thermodynamics are not independent. They are modal projections of the same inductive response of the vacuum, evaluated at the three characteristic velocities allowed by the geometro-elastic substrate.

Conceptual significance. This result provides a unified operational principle: *electromotive response emerges from the substrate's geometric resistance to deformation.* What experimental physics has treated as three unrelated constants are revealed here as facets of a single property. μ_0 sets the baseline inductive stiffness; K_e is its relativistic amplification at high frequency ($v = c$); and k_B is its reciprocal attenuation in the slow, dissipative regime ($v = 1/c$). In this sense, magnetostatics, electrostatics, and thermodynamics are unified as different *modal responses of the same geometro-elastic medium.*

XV. SYNTHESIS OF CONSTANTS AND THE VACUUM'S CONSTITUTIVE EQUATION

Building on the principles derived in the preceding sections, we now demonstrate their ultimate convergence. We will show that the modal invariants of the substrate are not independent but are locked into a single, self-consistent relationship by the theory's structure. This synthesis is achieved by applying the principles of fermionic dressing and damped equipartition.

A. The Electro-Thermal Identity

We first establish a direct link between the elementary charge and the thermal potential of the vacuum.

1. **Principle of Fermionic Dressing:** The observed elementary charge, e, is the bare structural charge of the substrate, q_s, dressed by the dissipative response of the vacuum's spin-1/2 modes. As established in Sec. VII E, this is governed by the factor $g_e\alpha \approx 2\alpha$:

$$e = q_s \cdot (g_e \alpha). \tag{XV.1}$$

2. **The Bare Charge from Constitutive Properties:** The bare charge q_s can be constructed independently from the vacuum's constitutive properties. In Sec. XIV, we identified μ_0 with the substrate's baseline electromotive potential. The corresponding fiducial capacitance is $C_{\text{fid}} = \varepsilon_0 \cdot (1\,\text{m})$. The bare charge is their product:

$$q_s = C_{\text{fid}} \cdot V_{\text{fid}} = \varepsilon_0 \cdot (1\,\text{m}) \cdot \mu_0 = \frac{1\,\text{m}}{c^2}. \tag{XV.2}$$

Note that this is precisely the characteristic length of the electromagnetic mode (XI.2), confirming how the torsional deformation $\ell_{\rm EM}$ manifests as the minimal quantum of charge.

3. **Synthesis:** In Sec. XIV, we also derived the thermo-entropic potential as $k_B \equiv \mu_0/c^2$, which implies $1/c^2 \equiv k_B/\mu_0$. Substituting this into the expression for q_s gives $q_s \approx (k_B \cdot 1\,{\rm m})/\mu_0$. Inserting this into the dressing principle yields:

$$e \equiv \left(\frac{k_B \cdot 1\,{\rm m}}{\mu_0}\right)(g_e\alpha) \quad \Longrightarrow \quad \boxed{\mu_0 \cdot e \equiv k_B \cdot 1K \cdot 2\alpha}, \tag{XV.3}$$

where the fiducial meter X C (equivalent to 1 K IV) normalizes the relation. This equation forms the **electro-thermal identity** of the vacuum.

B. The Thermo-Quantum Identity

We now establish a key thermodynamic principle of the QEG framework: *the effective thermal energy available to a normal mode of the geometro-elastic substrate is suppressed by its quality factor, Q*. We demonstrate the robustness of this "Damped Equipartition Principle" by outlining two independent derivations from fundamental statistical mechanics.

a. 1. Energy Balance Approach. Consider a single, underdamped normal mode of the substrate modeled as a harmonic oscillator with quality factor $Q_A = 1/(2\zeta_A)$, in thermal equilibrium with a bath at temperature T. In thermal equilibrium (classical limit) a single quadratic oscillator has $\langle E_{\rm kin}\rangle = \frac{1}{2}k_BT$, $\langle E_{\rm pot}\rangle = \frac{1}{2}k_BT$, hence

$$\langle E_{\rm stored}\rangle \;=\; k_BT\,. \tag{XV.4}$$

From (XV.4) one gets the energy irreversibly lost to the bath per *cycle* as:

$$\Delta E_{\rm diss}^{\rm (cycle)} \;=\; \frac{2\pi}{Q_A}\,k_BT \tag{XV.5}$$

At equilibrium the bath must inject the *same* amount of energy back into the mode over a cycle, by detailed balance. It is convenient to quotient out the trivial 2π of phase advance and work *per natural radian* (the natural coarse-graining time in a quasi-monochromatic steady state). We therefore define the *available thermal energy per radian*

$$\boxed{E_{\rm eff}^{(A)} \;\equiv\; \frac{\Delta E_{\rm diss}^{\rm (cycle)}}{2\pi} \;=\; (k_BT)\,\frac{1}{Q_A}} \tag{XV.6}$$

b. 2. Fluctuation-Dissipation Approach. The same result is obtained independently from the Fluctuation-Dissipation Theorem (FDT). Consider a single normal mode A described (in generalized coordinates) by a damped harmonic oscillator with natural frequency ω_0, damping ratio $\zeta_A \ll 1$, and stiffness scale κ_A (as in Sec. X). The causal susceptibility (coupling Hamiltonian $-f(t)\,x$) is

$$\begin{aligned}\chi_A(\omega) \;&=\; \frac{1}{\kappa_A\left(\omega_0^2 - \omega^2 - i\,2\zeta_A\omega_0\omega\right)},\\ \chi_A''(\omega) \;&=\; \frac{2\zeta_A\omega_0\omega}{\kappa_A\left[(\omega_0^2-\omega^2)^2 + (2\zeta_A\omega_0\omega)^2\right]}.\end{aligned} \tag{XV.7}$$

The fluctuation–dissipation theorem (classical limit) for our conventions reads

$$\begin{aligned}S_{xx}(\omega) \;&=\; \frac{2k_BT}{\omega}\,\chi_A''(\omega)\\ \langle x^2\rangle \;&=\; \frac{1}{2\pi}\int_0^\infty S_{xx}(\omega)\,d\omega\,,\end{aligned} \tag{XV.8}$$

which reproduces $\langle E_{\rm pot}\rangle = \frac{1}{2}\kappa_A\omega_0^2\langle x^2\rangle = \frac{1}{2}k_BT$. The average power dissipated by the mode due to the thermal fluctuations is

$$P_{\rm diss} \;=\; \int_0^\infty d\omega\;\omega\;\chi_A''(\omega)\,\frac{k_BT}{\pi} \;=\; \frac{\omega_0}{Q_A}\,k_BT\,, \tag{XV.9}$$

where the last equality follows by evaluating the narrow-band Lorentzian integral using (XV.7) and $Q_A^{-1} = 2\zeta_A$. Multiplying by the period $T_0 = 2\pi/\omega_0$ yields exactly (XV.5), hence dividing by 2π gives (XV.6). This completes an independent FDT proof of

$$\boxed{E_{\rm eff}^{(A)} = (k_BT)/Q_A}$$

Quantum correction. Beyond the classical limit the factor k_BT is replaced by $\frac{\hbar\omega}{2}\coth(\hbar\omega/2k_BT)$. All results below persist with this replacement; at the fiducial scale we remain in the classical regime by construction.

Application to the Vacuum and the Thermo-Quantum Identity

We now apply this principle to the vacuum itself. As established in Sec. VII, processes involving the "dressing" of localized, spin-1/2 excitations (i.e., fermions) are governed by a quality factor determined by the fine-structure constant α and amplified by the Dirac g-factor, $g_e = 2$:

$$Q_{\rm ferm} = \frac{1}{g_e\alpha} \approx \frac{1}{2\alpha} \tag{XV.10}$$

The effective thermal energy available from the vacuum bath to fuel such a fermionic process is therefore:

$$E_{\rm eff}^{\rm (ferm)} = (k_BT)\frac{1}{Q_{\rm ferm}} = (k_BT)(g_e\alpha) \tag{XV.11}$$

A stable, resonant quantum of the substrate (i.e., a particle) at a characteristic wavelength λ requires that this available energy matches the quantum of excitation, $E_q = hc/\lambda$. This *energy matching condition*, $E_{\rm eff}^{\rm (ferm)} = E_q$, evaluated at the fiducial scale of the framework ($T \equiv 1\,{\rm K}, \lambda = 1\,{\rm m}$), yields one of the central results of our theory:

$$\boxed{(k_B \cdot 1\,{\rm K})\cdot(2\alpha) = \frac{hc}{1\,{\rm m}}} \tag{XV.12}$$

This equation, which we term the **Thermo-Quantum Identity**, forms a cornerstone of the synthesis of constants developed in Sec. XV. It connects the thermodynamic properties of the vacuum (k_B), its dissipative quantum nature (α), and its fundamental quantum of action (h).

C. The Vacuum's Constitutive Equation

The electro-thermal and thermo-quantum identities are two sides of the same coin. Combining them yields a single, powerful synthesis that interlocks the three domains of physics:

$$\boxed{\mu_0 \cdot e \equiv k_B \cdot 1\;K \cdot 2\alpha \equiv \frac{hc}{1\,{\rm m}}} \tag{XV.13}$$

where the second-order terms have been omitted, and a more accurate approximation would be given by

$$\begin{aligned}&\mu_0 \cdot e_q(2\alpha + \frac{\alpha}{2\pi} + \dots)\\ &\equiv k_B \cdot 1\,{\rm K}(2\alpha + \dots)\\ &\equiv \frac{hc}{1{\rm m}}(1 + 2\alpha + \dots)\end{aligned} \tag{XV.14}$$

This is the **Vacuum's Constitutive Equation**. It is not a numerical coincidence, but the necessary structural closure condition of the Quantum-Elastic Geometry framework. It reveals the fundamental constants of nature not as arbitrary measured values, but as deeply interconnected parameters whose ratios are fixed by the unified, dissipative, and quantized properties of the spacetime substrate itself.

XVI. THE NATURE OF THE GRAVITATIONAL CONSTANT G

A. Static Origin: Deriving G from Modal Reciprocity and Self-Energy

The *Principle of Modal Reciprocity*, established in Sec. VI, posits that the gravitational compliance (G) is inversely proportional to the electromagnetic stiffness (K_e), scaled by a geometric factor:

$$G = \mathcal{C}_{\text{geom}} \cdot \frac{1}{K_e} = \mathcal{C}_{\text{geom}} \cdot 4\pi\varepsilon_0 \qquad \text{(XVI.1)}$$

Our task is to determine $\mathcal{C}_{\text{geom}}$ from first principles. As anticipated in Sec. VI A, this factor is derived from the geometric integration of self-energy.

In Sec. IX, we rigorously derived the universal geometric factor associated with the self-energy of a dense, volume-filling spherical source. We showed that the total work of formation (U_{glob}) for such a configuration is characterized by a geometric factor of $\mathcal{C}_g = 3/5$. Since the standard Newtonian constant of gravitation, G_N, is precisely the coupling that governs these localized, self-interacting systems (e.g., stars and galaxies), it is a requirement of consistency to identify its geometric coefficient with this value:

$$\mathcal{C}_{\text{geom}} = \frac{3}{5} \qquad \text{(XVI.2)}$$

Substituting this directly into the reciprocity principle gives the definitive expression for the standard Newtonian constant of gravitation:

$$\boxed{G_N \equiv G_{\text{loc}} = \left(\frac{3}{5}\right) 4\pi\varepsilon_0} \qquad \text{(XVI.3)}$$

Evaluating with physical constants,

$$G = \left(\frac{3}{5}\right) 4\pi\varepsilon_0 \;\approx\; 6.676 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2},$$

in excellent agreement with the measured Newtonian constant. The numerical success of this derivation confirms the proposed reciprocity principle: the vacuum's longitudinal compliance (gravitational softness, G) is inversely proportional to its transverse stiffness (electromagnetic tension, K_e). This reveals a profound symmetry where gravity and electromagnetism emerge as orthogonal projections of the same underlying field elasticity. The empirical disparity between their strengths is not accidental but a structural necessity: the vacuum is extremely compliant to gravitational deformation precisely because it is so immensely rigid electromagnetically. In essence, the vacuum acts as a geometric impedance surface, whose tension and compliance must balance, thereby setting the apparent strengths of the fundamental forces.

B. Dynamic Origin of G: Gravity as a Quadratically-Damped Vacuum Response

Having established the static interpretation of G as a compliance modulus of the vacuum, we now turn to its *dynamic origin*. We demonstrate that G is not a fundamental constant, but a *direct, emergent consequence* of the dissipative structure of the quantum-elastic substrate. The derivation follows directly from three principles already established in the QEG framework:

1. The vacuum possesses a universal baseline stiffness, quantified by the magnetic permeability μ_0, which sets the primordial interaction scale XIV C.

2. Dissipation is encoded in the *structural damping tensor* $\zeta_{\mu\nu}$, with scalar norm $\|\zeta_{\mu\nu}\| = \alpha$, the fine-structure constant VII.

3. Gravity couples to the energy–momentum tensor $T_{\mu\nu}$, i.e. to quadratic field invariants, and therefore must arise as a *second-order dissipative response.*

From Primary Stiffness to Effective Coupling. At the bare level, the vacuum stiffness is μ_0. The presence of dissipation filters this bare coupling through a *dissipative transfer function* $\mathcal{T}(\alpha)$:

$$G = \mu_0 \cdot \mathcal{T}(\alpha).$$

For first-order processes (e.g. charge dressing), $\mathcal{T}(\alpha) \propto \alpha$. However, because gravity couples to $T_{\mu\nu}$, its response requires *two interactions* with the dissipative medium (mass deformation → dissipative modulation → induced force). Thus,

$$\mathcal{T}^{(2)}(\alpha) \propto \alpha^2.$$

In the absence of other fundamental geometric or normalization factors, the principle of parsimony dictates that this proportionality is direct, setting the constant to unity in a first-order approximation. Therefore, we have that

$$\mathcal{T}^{(2)}(\alpha) \equiv \alpha^2$$

Final Constitutive Relation. This uniquely fixes the effective gravitational coupling as

$$\boxed{G \;\equiv\; \mu_0\,\alpha^2} \qquad \text{(XVI.4)}$$

which is not a conjecture but a *necessary consequence* of: (i) μ_0 as primordial stiffness, (ii) α as damping norm, and (iii) the quadratic nature of gravitational coupling.

Tensorial Formulation. Promoting α^2 to its tensorial origin, we write

$$G\,g_{\mu\nu} = \mu_0\,\zeta_{\mu\alpha}\,\zeta^{\alpha}{}_{\nu}, \qquad \text{(XVI.5)}$$

which expresses gravity as the effective longitudinal stiffness generated by the quadratic action of the structural damping tensor. In analogy with elasticity, if $\mathcal{G}_{\mu\nu}$ is the strain, then $\zeta_{\mu\nu}$ represents the induced internal stress. Gravity is thus the macroscopic manifestation of the vacuum's dissipative geometry.

Numerical Validation. Evaluating with physical constants,

$$G = \mu_0\alpha^2 \approx (4\pi{\times}10^{-7}){\cdot}(0.007297)^2 \;\approx\; 6.69{\times}10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2},$$

in excellent agreement with the measured Newtonian constant.

Conclusion. We conclude that G is not a free parameter, but the *quadratically-damped residual of the electromagnetic substrate stiffness.* This interpretation anchors gravity within the same constitutive law as electromagnetism and thermodynamics, completing the triad of elastic responses of the vacuum.

XVII. UNIFIED INTERPRETATION OF THE FUNDAMENTAL FORCES OF GRAVITY AND ELECTROMAGNETISM

Having derived the nature of the fundamental coupling constants μ_0, K_e, k_B and G, we can now reinterpret the force laws they govern, revealing that they are not disparate laws, but complementary modal projections of a single underlying mechanism: *momentum exchange through the elastic and dissipative vacuum.*

A. Newton's and Coulomb's Laws as Damped Momentum Transfer

From the previously derived first-order expressions, such as $G \equiv \mu_0 \cdot \alpha^2$ and $k_B \equiv \mu_0/c^2$, it follows that:

$$G \equiv k_B \cdot \alpha^2 \cdot c^2$$

Using this expression, we can rewrite Newton's law to reveal its thermo-entropic nature. This force is mediated by the exchange of *damped, longitudinal momentum*, and expressed in terms of damped relativistic momenta, where $\zeta = \alpha$ is the norm of the structural damping tensor:

$$F_g \equiv k_B \cdot \frac{(Mc \cdot \zeta) \cdot (mc \cdot \zeta)}{r^2} \tag{XVII.1}$$

Thus, the terms in the numerator correspond to effective, damped relativistic momenta, whose propagation is modulated by the damping structure encoded in $\zeta_{\mu\nu}$. Gravity emerges as the effective resistance to the coherent alignment of projected momenta through the vacuum's dissipative tensorial geometry.

By symmetry, the Coulomb force can be reformulated as the exchange of *undamped, transverse momentum*:

$$F_e = K_e \cdot \frac{Q_1 Q_2}{r^2} = \frac{\mu_0 \cdot c^2}{4\pi} \cdot \frac{Q_1 Q_2}{r^2} = \mu_0 \cdot \frac{(Q_1 c)(Q_2 c)}{4\pi r^2} \tag{XVII.2}$$

This structure mirrors the gravitational expression, with Qc playing the role of transverse modal momentum and the vacuum magnetic permeability μ_0 acting as the transverse field stiffness. Thus, both Newton's and Coulomb's laws appear as complementary modal projections of the same unified vacuum tensorial response.

B. Physical Interpretation: the Structural Differences

The distinct mathematical forms of Eqs. XVII.1 and XVII.2 are not accidental; they reflect profound differences in the physical nature of the momentum exchange. Comparing the rewritten laws for the gravitational (Eq. XVII.1) and Coulomb (Eq. XVII.2) forces reveals two fundamental structural differences that provide deep insight into their distinct physical natures.

a. 1. The Selective Role of Damping. A crucial difference is the explicit presence of the damping factor, ζ, in the gravitational force, while it is absent in the static Coulomb force. This does not imply that electromagnetism is an undamped phenomenon—indeed, we have argued that the propagation of light is damped. Rather, it reveals the fundamental character of each interaction:

- The **Coulomb force** is a *conservative interaction* between static charges. Its formulation does not involve dissipation. The dissipative effects of electromagnetism arise in dynamic phenomena, such as radiation.
- The **gravitational force**, when expressed in this thermo-entropic form, is revealed to be an inherently *dissipative and entropic interaction.* It is not a static potential force in the classical sense, but the result of momentum exchange through a dissipative medium. Therefore, it *must* explicitly include the damping factor ζ as part of its fundamental definition.

This suggests that damping is a key feature that distinguishes the entropic (longitudinal) modes of the vacuum from the conservative (transverse) static modes.

b. 2. The Geometry of Propagation (r^2 vs. $4\pi r^2$). The second key difference lies in the geometry of the denominator.

- The **Coulomb force** includes the factor $4\pi r^2$, the surface area of a sphere. This reflects the isotropic, wave-like nature of the interaction, where the influence of a point charge propagates outwards uniformly in all directions, spreading over a spherical surface. This is characteristic of a **transverse wave**.
- The re-expressed **gravitational force**, by contrast, lacks the 4π solid angle factor. Its r^2 dependence represents a direct interaction between two points. This supports the interpretation of a **longitudinal interaction**, where the force acts as a direct "pressure" or "tension" along the line connecting the two masses, rather than as a field radiating spherically.

Remark XVII.1 (Geometric Interpretation)**.** *The distinction between the denominators r^2 and $4\pi r^2$ should not be read as an empirical asymmetry in the spatial geometry of the two forces, but rather as a difference in* modal propagation*. Both interactions obey Gauss-like flux conservation and are experimentally consistent with isotropic $1/r^2$ decay. The factor 4π is absorbed into the definition of the corresponding coupling constant (G or K_e) and thus carries no independent physical weight.*

In the present unified framework, the absence of the explicit 4π in the gravitational term expresses a longitudinal projection *of the vacuum's stress tensor—an exchange of compressive momentum directly along the line of centers—whereas the electromagnetic interaction represents a* transverse, rotational projection *whose influence is isotropically distributed over the spherical wavefront. Hence, the two geometries differ not in their spatial metric but in the internal symmetry of their modal excitation: tension versus shear. This interpretation preserves all standard field observables while clarifying the complementary tensorial roles of the gravitational and electromagnetic modes.*

In summary, both interactions reflect momentum exchange across a structured medium, but what differs is the symmetry of the exchange. The immense difference in their strengths arises not from arbitrarily different coupling constants, but from the physical properties of the vacuum itself:

- The gravitational force emerges as a highly suppressed, entropy-weighted longitudinal momentum flow. Gravity is weak not because its coupling is small, but because the vacuum is extremely rigid against this type of compressional deformation, and the Boltzmann constant k_B quantifies the high entropic cost required.
- The electromagnetic force, in contrast, reflects a much more efficient, transversely mediated momentum exchange, amplified by the vacuum's comparatively soft resistance to shear-like deformations, a process governed by μ_0.

XVIII. SYNTHESIS: UNIFYING THE VACUUM'S ELASTIC AND QUANTUM PROPERTIES

The ultimate test of this framework is its internal consistency. We will now demonstrate that the relationships derived are not only mutually compatible but also lead to profound connections between the vacuum's elastic, electromagnetic, and quantum characteristics.

A. A Consistency Condition for the Vacuum

We have derived the gravitational constant G from two independent perspectives, grounded in the vacuum's elastic properties:

- **Static Origin:** Based on a principle of reciprocity, where G is the geometrically-scaled compliance of the vacuum:

$$G = \left(\frac{3}{5}\right) 4\pi\varepsilon_0$$

- **Dynamic Origin:** Based on a model of quadratic damping, where G is the primary vacuum stiffness, quadratically suppressed by dissipation:

$$G = \mu_0 \alpha^2$$

For this framework to be self-consistent, these two expressions for G must be equal. Equating them reveals a profound **consistency condition** that the vacuum's properties must obey:

$$\boxed{\left(\frac{3}{5}\right) 4\pi\varepsilon_0 \equiv \mu_0 \alpha^2} \tag{XVIII.1}$$

This equation is a powerful prediction of the theory. Using the relation $Z_0^2 = \frac{\mu_0}{\varepsilon_0}$, we can rewrite this condition as:

$$Z_0 \equiv \frac{\sqrt{\frac{3}{5}4\pi}}{\alpha} \approx 376.28\ \Omega \tag{XVIII.2}$$

where one can identify the second-order term and write the more exact expression

$$Z_0 \equiv \frac{\sqrt{\frac{3}{5}4\pi}}{\alpha}\left(1 + \frac{\alpha}{2\pi} + \dots\right) \approx 376.73\ \Omega$$

Another direct and powerful corollary is a relationship between α, G, and μ_0. Rearranging Eq. (XVIII.1) yields:

$$\alpha \equiv \sqrt{\frac{G}{\mu_0}} \tag{XVIII.3}$$

This simple expression confirms that α is not an independent constant, but a *geometric attenuation coefficient* derived from the ratio of the vacuum's longitudinal and transverse stiffnesses, and elevates the fine-structure constant from a mere electromagnetic coupling parameter to a *universal measure of vacuum damping topology*, governing how all interactions manifest.

B. A Unified Emergence Mechanism for Gravity and Charge

The internal consistency of our framework reveals a profound structural symmetry between the origin of the gravitational constant G and the squared elementary charge e^2. Both emerge from the interplay of primordial vacuum properties (inertia and action) filtered by the universal damping invariant α.

a. Gravitational Coupling (Bosonic/Elastic Origin): As derived from the vacuum's dynamic and dissipative properties (Sec. XVI B), the gravitational coupling arises from the *primordial inertial stiffness* μ_0 being quadratically suppressed by the *bosonic propagation damping* proposed in Sec. VII E, $\zeta = \alpha$:

$$G_{\text{Loc}} \equiv \mu_0 \cdot \zeta^2 = \mu_0 \alpha^2 \tag{XVIII.4}$$

b. Electromagnetic Coupling (Fermionic/Quantum Origin): We can derive the analogous expression for e^2. Starting from the constitutive identities for the elementary charge e (Sec. XV A) and the quantum of action $\hbar$ (Sec. XVIII.24):

$$e \equiv \frac{2\alpha \cdot 1\,\text{m}}{c^2} \tag{XVIII.5}$$

$$\hbar \equiv \frac{1\,\text{m}^2}{c^4} \tag{XVIII.6}$$

Squaring Eq. (XVIII.5) and substituting Eq. (XVIII.6) yields a direct relationship between charge, action, and dissipation:

$$e^2 = 4\alpha^2 \cdot \frac{(1\,\text{m})^2}{c^4} = 4\alpha^2 \cdot \hbar \tag{XVIII.7}$$

This can be suggestively rewritten using the *fermionic damping factor* VII E $\frac{1}{Q} \equiv \zeta' \equiv 2\alpha$:

$$e^2 \equiv \hbar \cdot (2\alpha)^2 = \hbar \cdot (\zeta')^2 \tag{XVIII.8}$$

The parallel between Eq. (XVIII.4) and Eq. (XVIII.8) is striking and reveals a universal mechanism:

$$\boxed{G = \mu_0 \cdot \alpha^2 \quad \text{and} \quad e^2 = \hbar \cdot (2\alpha)^2} \tag{XVIII.9}$$

Both gravity and charge are emergent properties determined by the same squared damping factor, α^2, acting as a "*universal dissipative filter*." The fundamental interactions and charges become "*echoes*" of more primordial vacuum properties (μ_0 and $\hbar$), filtered through the same dissipative process. The difference in their nature and magnitude arises from their distinct "bare" origins:

- **Gravity** emerges from the vacuum's primordial *inertial stiffness* (μ_0), suppressed by the fundamental bosonic damping α^2.
- **Charge** emerges from the vacuum's primordial *quantum of action* ($\hbar$), suppressed by a damping factor $(2\alpha)^2$, whose factor of 2 is consistent with the $g_e = 2$ spin response associated with fermionic interactions.

This shared emergence mechanism is one of the most powerful pieces of evidence for the coherence of the proposed framework, suggesting a deep unity in the principles that govern the cosmos.

C. A Bridge to General Relativity: An Ohm's Law for the Vacuum

Additionally, our framework provides a remarkable consistency check with General Relativity. Substituting $G \equiv \mu_0 \cdot \alpha^2$ one can check that

$$G \cdot c \equiv \mu_0 \cdot c \cdot \alpha^2 = Z_0 \cdot \alpha^2 \approx \frac{1}{50.13}\ \Omega \tag{XVIII.10}$$

Note that we can set $\frac{1}{50.13} \equiv \frac{1}{16\pi}$, to have that $G \equiv \frac{1}{16\pi c}$. This implies that (i) the product $G \cdot c$ defines a fundamental 'resistive-like' constant of the vacuum, which we denote as the *natural resistance* X_N; and (ii) the relationship $G \equiv \frac{1}{16\pi c}$ can be interpreted as an Ohm's Law for the vacuum, $V = I \cdot R$, where:

- **Potential (V):** The gravitational constant G, within our dimensionally collapsed framework, acquires the role of a fundamental potential or electromotive force, as $G \equiv \mu_0 \cdot \zeta^2$ and $[\mu_0] = [V]$ (XIV C).
- **Current (I):** The term $1/c$ represents the natural, scaled current of the vacuum for the thermo-entropic mode (XIV C).
- **Resistance (R):** This implies that the vacuum possesses an intrinsic, dimensionless resistance $R_{\text{vac}} = 1/(16\pi)$.

This vacuum resistance is not an arbitrary number, but can be derived from first principles by decomposing it into two meaningful factors:

$$R_{\text{vac}} = \frac{1}{16\pi} = \frac{1}{4} \cdot \frac{1}{4\pi} \tag{XVIII.11}$$

The two terms have clear physical origins:

1. **The Geometric Resistance ($1/4\pi$):** This factor arises directly from the Green's function of the 3D Laplacian operator (II D). It represents the fundamental geometric impedance of three-dimensional space, quantifying how the influence of a point source is diluted as it spreads over a spherical solid angle.

2. **The Canonical coupling factor (1/4):** This factor is the canonical normalization constant required for any standard gauge field theory, as seen in the electromagnetic Lagrangian $\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Its presence here suggests that gravity, as an *emergent effective field*, must inherit the canonical normalization of the underlying field theory framework.

Therefore, the relationship we propose is not a numerical coincidence, but a profound statement about the structure of the vacuum.

D. First-Principles Prediction of the Fine-Structure Constant

A profound consequence of the vacuum's self-consistency is that the value of the fine-structure constant, α, becomes uniquely determined by the geometric and topological constraints of the framework. In the preceding sub-subsections, we derived two independent, constitutive laws for the vacuum impedance:

1. From the consistency between the static and dynamic origins of the gravitational constant G, we obtained (Eq. XVIII A):

$$Z_0 \equiv \frac{\sqrt{\frac{3}{5}4\pi}}{\alpha} \tag{XVIII.12}$$

2. From Eq. XVIII.10), we have that:

$$Z_0 \equiv \frac{1}{16\pi\alpha^2} \tag{XVIII.13}$$

For the theoretical framework to be internally consistent, these two expressions for Z_0 must be equal. Equating Eq. XVIII.12 and Eq. XVIII.13 forces a single, unique value for the fine-structure constant:

$$\frac{\sqrt{\frac{3}{5}4\pi}}{\alpha} \equiv \frac{1}{16\pi\alpha^2} \tag{XVIII.14}$$

Solving for α yields:

$$\alpha \equiv \frac{1}{16\pi\sqrt{\frac{3}{5}4\pi}} \tag{XVIII.15}$$

where one can identify the second-order term and write the more exact expression

$$\alpha \equiv \frac{1}{16\pi\sqrt{\frac{3}{5}4\pi}}(1 + \frac{\alpha}{2\pi} + \dots)$$

The first-order predicted value matches the experimentally measured CODATA value of $\alpha \approx 1/137.036$ with remarkable accuracy, and is highly significant. It shows that the value of the fine-structure constant is not an arbitrary input, but rather the precise value required to harmonize the vacuum's fundamental properties: its geometric impedance arising from the 3D Laplacian ($\frac{1}{4\pi}$), its canonical field structure inherited from gauge theories ($\frac{1}{4}$), and the geometry of its self-interaction energy ($\sqrt{\frac{3}{5}4\pi}$). This elevates the status of α from a mere electromagnetic coupling parameter to the primary geometric and topological constant of the unified vacuum.

As a final note, for the above interpretation to be consistent, then $\frac{1}{\sqrt{\frac{3}{5}4\pi}}$ must have dimensions of reciprocal of a resistance. We will uncover and check that this is indeed the case in (XVIII E).

E. Impedance vs. Dissipative Resistance and the quality factor Q

The robustness of this theoretical framework can be further tested by examining the consistency of the classical RLC oscillator analogy, which has served as a powerful conceptual guide. Consider the standard formula for the quality factor of a series RLC circuit at resonance:

$$Q = \frac{\omega_0 L}{R} \tag{XVIII.16}$$

Within our framework, we have already established the following vacuum parameters:

- The vacuum quality factor, derived from quantum conductance, is $Q = 1/(2\alpha)$ (VII.13).
- The characteristic inductive term is $\omega_0 L = (\frac{c}{1\,\mathrm{s}})(\mu_0 \cdot 1\,\mathrm{m})$. Given the spacetime equivalence where $1\,\mathrm{s} \equiv 1\,\mathrm{m}$, this simplifies to $\omega_0 L = \mu_0 c$, which is precisely the definition of the vacuum's wave impedance, Z_0.

Substituting these established quantities into the classical Q-factor formula allows us to solve for the effective resistance, R_Q, that must govern this specific dissipative process:

$$\frac{1}{2\alpha} = \frac{Z_0}{R_Q} \tag{XVIII.17}$$

This consistency requirement leads directly to a new constitutive law for the vacuum:

$$R_Q = 2\alpha Z_0 \equiv \frac{Z_0}{Q} \equiv 2 \cdot \sqrt{\frac{3}{5}4\pi} \tag{XVIII.18}$$

This result is profound. It reveals that the vacuum possesses two distinct, conceptually different resistive properties:

1. Wave Impedance (Z_0): Z_0 is the vacuum's characteristic impedance to wave propagation. It is a reactive property that governs the ratio of the electric and magnetic field strengths in an electromagnetic wave.

2. Dissipative Resistance (R_Q): This newly derived quantity, R_Q, is the vacuum's effective resistance governing the *rate of energy dissipation per cycle* in an oscillation, as quantified by the quality factor. It is a fundamentally dissipative, rather than reactive, property.

Equation XVIII.18 provides the explicit relationship between these two properties. It states that the dissipative resistance of the vacuum is its fundamental wave impedance modulated by its own universal quality factor, Q. This is physically intuitive: the resistance to energy loss should be proportional to both the resistance to wave propagation and the intrinsic friction of the medium.

Note that using Eq. XVIII.18, we can express $\frac{3}{5}4\pi = \frac{1}{4}R_Q^2$. Substituting this into the left-hand side of the master consistency condition (Eq. XVIII.1) yields:

$$\frac{1}{4}\varepsilon_0 R_Q^2 \equiv \mu_0 \alpha^2 \tag{XVIII.19}$$

This final expression is a remarkably powerful statement of unification. The dimensional consistency of Eq. XVIII.19 holds in both the SI system and the dimensionally-collapsed framework of this theory, further cementing its robustness.

F. Dynamic Origin of Permittivity from Vacuum Power Principles

Having established the distinct roles of wave impedance (Z_0) and dissipative resistance (R_Q), we can now propose a dynamic origin for the vacuum's electric permittivity, ε_0. We move beyond the static picture of permittivity as a passive capacity to store fields and instead derive it from the vacuum's fundamental power dissipation and resistance properties.

First, we define a *Unitary Vacuum Power*, P_{unit}, as the power dissipated when the vacuum's intrinsic structural voltage ($V \equiv \mu_0$) is applied across a unitary resistance ($R = 1\,\Omega$). According to Joule's Law:

$$P_{unit} = \frac{V^2}{R} \equiv \frac{\mu_0^2}{1\,\Omega} \tag{XVIII.20}$$

In this framework, P_{unit} represents the baseline rate of energy transfer or dissipation inherent to the vacuum's primary potential.

Next, we postulate that the vacuum's permittivity, ε_0, which represents its compliance or ability to "permit" an electric field, is an emergent property. It arises from this unitary power being modulated by the vacuum's own internal friction. The most natural choice for this friction is the **dissipative resistance**, R_Q, as it governs energy loss per oscillatory cycle. We therefore propose the following constitutive law:

$$\varepsilon_0 \equiv P_{unit} \cdot R_Q = \frac{\mu_0^2}{1\ \Omega} R_Q \equiv \mu_0^2 \cdot 2\sqrt{\frac{3}{5}4\pi} \tag{XVIII.21}$$

where one can identify the second-order term and write the more exact expression

$$\varepsilon_0 \equiv \left(\mu_0^2 \cdot 2\sqrt{\frac{3}{5}4\pi}\right)(1 + 2\alpha + \dots)$$

G. The Gravito-Entropic Power Equivalence

Building upon this last relationship, we can establish a direct link between the vacuum's primary inertial properties and its large-scale cosmological dissipation. Eq. XVIII.21 yields a direct equivalence:

$$\frac{1}{4}\mu_0^2 \equiv 2\pi\varepsilon_0 \cdot \alpha \equiv G_{glob} \cdot \zeta \tag{XVIII.22}$$

This equation describes how the vacuum's capacity to store inertial or reactive energy in its transverse (electromagnetic) modes is intrinsically related to the vacuum's compliance to longitudinal deformation (G_{glob}) and its damping factor (ζ). Ultimately, Eq. XVIII.22 signifies a profound equilibrium: *the vacuum's capacity to store inertial energy in its transverse mode is perfectly balanced by the power it dissipates in its longitudinal mode.* This relationship links the electromagnetic and gravitational sectors through a fundamental equilibrium.

H. The Elementary Charge as a high-order stiffness of the Vacuum

An additional insight into the vacuum's fundamental properties arises when substituting the geometric definition of α from Eq. XVIII.15 into the expression $e \equiv \frac{2\alpha \cdot 1\ m}{c^2}$ (XV A):

$$e \equiv \frac{2 \cdot 1\ m}{c^2}\left(\frac{1}{16\pi\sqrt{\frac{3}{5}4\pi}}\right) = \frac{1\ m}{c^2 \cdot 8\pi\sqrt{\frac{3}{5}4\pi}}$$

Using the fundamental relation $\mu_0\varepsilon_0 = 1/c^2$ to replace c^2, and substituting the permittivity ε_0 from Eq. XVIII.21, yields:

$$\begin{aligned} e &\equiv \mu_0\left(\mu_0^2 \cdot 2\sqrt{\frac{3}{5}4\pi}\right) \cdot \frac{1\ m}{8\pi\sqrt{\frac{3}{5}4\pi}} \\ &= \frac{1\ m \cdot 2\mu_0^3\sqrt{\frac{3}{5}4\pi}}{8\pi\sqrt{\frac{3}{5}4\pi}} = \frac{\mu_0^3 \cdot 1\ m}{4\pi} \to \boxed{e \equiv \frac{\mu_0^3 \cdot 1\ m}{4\pi}} \end{aligned} \tag{XVIII.23}$$

where one can identify the second-order term and write the more exact expression

$$e \equiv \frac{\mu_0^3 \cdot 1\ m}{4\pi} \cdot (1 + 2\alpha + \dots)$$

This result shows how charge is a direct, emergent property of the vacuum itself, determined solely by its most fundamental characteristic—the transverse inertial stiffness μ_0—projected through the geometry of three-dimensional space ($\frac{1}{4\pi}$).

The cubic dependence ($e \propto \mu_0^3$) signifies that charge is a non-linear excitation of the vacuum field. While the vacuum's primary elastic response is linear (as seen in wave propagation), the formation of a stable, quantized charge represents a higher-order, self-interaction phenomenon. It is a measure of the vacuum's capacity to sustain a localized, persistent deformation against its own inertial resistance.

Importantly, this formulation elevates the magnetic permeability μ_0 to the status of the single, primary constant of the electromagnetic sector. The elementary charge e and the electric permittivity ε_0 are both derived from it, establishing a clear hierarchy of fundamental constants.

Finally, equation XVIII.23 defines vacuum's capacity to manifest charge as a function of its intrinsic stiffness, contributing a big step into the geometrization of physics by demonstrating that not only forces, but also the sources of those forces, arise from the fabric of spacetime.

I. The Quantum of Action as an Emergent Property of Vacuum Compliance

Following the derivation of the elementary charge from the vacuum's inertial stiffness (μ_0), we show how the quantum of action, $\hbar$, can also be derivable from the vacuum's complementary elastic property: its compliance, or permittivity (ε_0). This demonstration serves to ground not only the sources of forces, but the very granularity of quantum mechanics, in the tangible, elastic properties of the vacuum.

Our starting point is the modal action for the electromagnetic field derived in Eq. XVIII.24, which established the dimensionality of action as a spacetime area:

$$\hbar \equiv \frac{1\,\mathrm{m}^2}{c^4} \tag{XVIII.24}$$

To express $\hbar$ in terms of permittivity, we use the fundamental relationship $c^4 = 1/(\mu_0^2\varepsilon_0^2)$ into Eq. (XVIII.24), which yields:

$$\hbar \equiv (1\,\mathrm{m}^2) \cdot (\mu_0^2\varepsilon_0^2)$$

Replacing the inertial term μ_0^2 with its equivalent expression derived from the vacuum's dissipative properties, as given in Eq. XVIII.21, reveals $\hbar$ as a pure function of vacuum compliance and geometry:

$$\hbar \equiv (1\,\mathrm{m}^2) \cdot \left(\frac{\varepsilon_0}{2\sqrt{\frac{3}{5}4\pi}}\right) \cdot \varepsilon_0^2$$

$$\hbar \equiv \frac{\varepsilon_0^3 \cdot (1\,\mathrm{m}^2)}{2\sqrt{\frac{3}{5}4\pi}} \tag{XVIII.25}$$

This leads to a new fundamental expression for the first-order term of the reduced Planck constant:

$$\boxed{\hbar \equiv \frac{\varepsilon_0^3 \cdot (1\,\mathrm{m}^2)}{2\sqrt{\frac{3}{5}4\pi}}} \tag{XVIII.26}$$

The cubic dependence on permittivity ($\hbar \propto \varepsilon_0^3$) is highly significant. It indicates that quantum action arises from a complex, volumetric self-interaction of the vacuum's compliance. It can be physically pictured as the total *"elastic energy potential"* that can be stored in a unit of spacetime area, a potential that is non-linearly dependent on the medium's softness.

Importantly, this result forms a perfect symmetry with the derivation for the elementary charge ($e \propto \mu_0^3$). Together, they paint a complete picture:

- The **fundamental quantum of charge** (e) is a *non-linear function of the vacuum's inertial stiffness* (μ_0).
- The **fundamental quantum of action** ($\hbar$) is a *non-linear function of the vacuum's elastic compliance* (ε_0).

This duality between stiffness/charge and compliance/action represents a deep, foundational symmetry of the vacuum. It suggests that the laws of electromagnetism and the laws of quantum mechanics are two sides of the same coin, both emerging from the fundamental tension between inertia and compliance in the fabric of reality.

This derivation solidifies the framework's central claim: the constants that define our physical laws are not a random assortment of numbers, but are deeply interconnected parameters that describe the single, underlying, elastic quantum vacuum.

J. Derivation of Vacuum Constants and the Speed of Light from a Single Parameter

The ultimate test of the framework's internal consistency lies in its ability to define the fundamental constants of the vacuum, μ_0 and ε_0, and subsequently the speed of light, c, from a single, dimensionless parameter. Using the equivalence between the static and dynamic origins of the gravitational constant G (from Eq. XVIII.1)and the expression for the dynamic origin of permittivity from vacuum power principles (from Eq. XVIII.21), we can substitute Eq. (XVIII.21) into Eq. (XVIII.1) to obtain that:

$$\left(\frac{3}{5}\right)4\pi\left(\mu_0^2 \cdot 2\sqrt{\frac{3}{5}4\pi}\right) \equiv \mu_0\alpha^2 \rightarrow$$

$$\mu_0 \equiv \alpha^2 \cdot \frac{1}{16\pi \cdot \sqrt{\frac{3}{5}4\pi}} \cdot \frac{10}{3} \rightarrow$$

$$\boxed{\mu_0 \equiv \frac{10}{3}\alpha^3} \tag{XVIII.27}$$

where we have used $\alpha \equiv \frac{1}{16\pi \cdot \sqrt{\frac{3}{5}4\pi}}$ (XVIII.15) and the derived result is a first-order approximation. Using this result, we can express ε_0 purely in terms of α:

$$\left(\frac{3}{5}\right)4\pi\varepsilon_0 \equiv \frac{10}{3}\alpha^3\alpha^2 \rightarrow$$

$$\boxed{\varepsilon_0 \equiv \left(\frac{5}{3}\right)^2 \cdot \frac{1}{2\pi} \cdot \alpha^5} \tag{XVIII.28}$$

Finally, with both vacuum constants defined by α and geometry, we derive the speed of light using the fundamental relation $c^2 = 1/(\mu_0\varepsilon_0)$:

$$c^2 \equiv \frac{1}{\left(\frac{10}{3}\alpha^3\right) \cdot \left(\frac{25}{18\pi}\alpha^5\right)} \rightarrow$$

$$c^2 \equiv \frac{1}{\frac{125}{27\pi} \cdot \alpha^8} \rightarrow \boxed{c \equiv \frac{1}{\sqrt{\frac{125}{27\pi}} \cdot \alpha^4}} \tag{XVIII.29}$$

This result represents the culmination of the unified framework. The constants μ_0, ε_0, and c are no longer fundamental in their own right. They are revealed to be interdependent functions of a single, more primary parameter, α. The entire framework of vacuum electrodynamics and spacetime kinematics is determined not just by a number, but by the *scalar norm (α)* of the rank-2 symmetric *structural damping tensor, $\zeta_{\mu\nu}$*, which encodes the vacuum's intrinsic dissipative properties.

Additionally, the relationship $c \propto \alpha^{-4}$ confirms that the product $c \cdot \alpha^4$ is indeed a constant composed of the geometric factors derived throughout this section. As anticipated in Section XI B, this is precisely the normalization constant required to reconcile the first-order, bare geometric expression for $\hbar$ with its physical value, thus closing a crucial loop of internal coherence in the theory.

Moreover, the derived relationship, $c \propto \alpha^{-4}$, offers a profound insight into the nature of causality. The speed of light is not an arbitrary limit but an *emergent property* dictated by the vacuum's inherent *"friction"*. A hypothetical, perfectly frictionless vacuum ($\alpha \rightarrow 0$) would permit an infinite propagation speed, rendering causality instantaneous. It is therefore the small, non-zero damping of spacetime—the inherent viscosity of the quantum oscillator substrate—that establishes a finite causal speed limit, giving the universe its structure in time.

This synthesis elevates the fine-structure constant to the role of the primary architect of physical reality. It is the scalar measure of the vacuum's fundamental dissipative geometry. The value of α dictates the vacuum's inertial resistance (μ_0) and elastic compliance (ε_0), and the interplay between these two properties, governed by α, sets the exact value of the cosmic speed limit, c. Knowing the norm of the vacuum's damping tensor is equivalent to knowing the fundamental operational rules of spacetime.

Remark XVIII.1 (On the operational character of this derivation). *The goal of this section was to provide an operational first-order route by which the vacuum constants (μ_0, ε_0, c) may be expressed directly in terms of the single damping parameter α. As a result, part of the geometric information becomes absorbed into the effective numerical prefactors associated with α, yielding concise single-parameter expressions such as $\mu_0 = \frac{10}{3}\alpha^3$ and $\varepsilon_0 = \left(\frac{5}{3}\right)^2 \frac{1}{2\pi}\alpha^5$. These coefficients are therefore not independent geometric predictions, but first-order parametrizations consistent with the full geometric closures derived in Part V.*

Part V: A Geometric Derivation of Constants of Nature

The previous Part completed the first deductive path of the QEG framework. The preceding Parts have established the network of physical constants based on the physical properties of the substrate (such as modal reciprocity, dimensional collapse, and dissipation). This path successfully determined some *identities* and *interrelations* of the constants (e.g., $G \equiv \mu_0\alpha^2$ and $G \equiv (\frac{3}{5})4\pi\epsilon_0$).

We now present a second, convergent deductive path. The following Part provides the rigorous geometric foundation for these physical identities. We will now set aside the physical postulates of the previous sections and begin anew from a minimal set of foundational *geometric principles* (homogeneity, isotropy, covariance). From these axioms alone, we will formally derive the scaling laws and, crucially, the *precise numerical prefactors* for all constants, thereby validating the entire physical structure as the consequence of a self-consistent geometry, and providing a rigorous, independent validation of the physical model.

XIX. EXPLICIT CONSTRUCTION OF THE GEOMETRIC INVARIANTS

a. Conceptual motivation. We start from a medium in which interactions, propagations, and deformations can occur. In QEG, this medium must obey a minimal set of restrictions dictated by the action and by the intrinsic symmetries of spacetime: homogeneity, isotropy, covariance, Lorentz invariance, and scale freedom. From this standpoint, there must exist *dimensionless invariants* that survive under these symmetries and thus uniquely characterize the medium. Our goal is to derive such invariants directly from the Lagrangian dynamics at minimal (quadratic, scale-free) order, and only thereafter elevate them to physically interpretable composites.

Action and linearized equations

The full QEG action is background-independent, with the dynamical field $G_{\mu\nu}$ defining the spacetime geometry itself. For the analysis of low-energy (long-wavelength) excitations, we follow the standard procedure of linearizing the theory around a flat background. We expand the field as a fluctuation $h_{\mu\nu}$ over the Minkowski metric $\eta_{\mu\nu}$:

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \tag{XIX.1}$$

The action, when expanded to second order in $h_{\mu\nu}$, yields the quadratic Lagrangian for the fluctuations. The kinetic part of this action is given by:

$$S^{(2)}_{\text{kin}} = \int d^4x \left[\frac{\kappa}{2}(\partial_\alpha h_{\mu\nu})(\partial^\alpha h^{\mu\nu}) + \dots\right], \tag{XIX.2}$$

where indices are now raised and lowered with the background Minkowski metric $\eta_{\mu\nu}$, and covariant derivatives reduce to partial derivatives ∂_α. The resulting linearized Euler-Lagrange equations in this infrared (IR) regime are:

$$\begin{aligned} \kappa\,\Box h_{\mu\nu} - \Big(\partial V/\partial G^{\mu\nu}\Big)\Big|_{\bar{G}=\eta} &= J_{\mu\nu}\,, \\ K\,h = J, \qquad K := \kappa\,\Box &\overset{\text{static}}{\longrightarrow} -\kappa\nabla^2\,, \end{aligned} \tag{XIX.3}$$

where $\Box \equiv \partial_\alpha\partial^\alpha$ is the d'Alembertian operator on the flat background. This approximation is sufficient to derive the geometric invariants that characterize the substrate's response.

A. Gauge channel: an explicit projector and unit residue ⇒ $N_g = 1/4$

b. Explicit projector. Define the spatial trace-free divergence of h_{ij} (Latin i, j denote spatial indices) by

$$\mathcal{S}_i[h] \;:=\; \partial^j\Big(h_{ij} - \tfrac{1}{3}\delta_{ij}\,h^k{}_k\Big)$$

Set the 1-form

$$A_i \;:=\; c_\Pi\,\mathcal{S}_i[h]\,, \qquad A_0 := 0, \tag{XIX.4}$$

with c_Π a dimensionless constant to be fixed by residue normalization below. (Other linear, local, rotationally-covariant maps $\Pi[h] \to A$ differ by field redefinitions and surface terms at this order; see Appendix B.) Inserting (XIX.4) back into the quadratic action generated by (XIX.2) and integrating by parts gives an effective gauge-sector Lagrangian

$$\begin{aligned} \mathcal{L}^{(2)}_{\text{gauge}} \;&=\; \frac{c_g}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; A_\mu J^\mu, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned} \tag{XIX.5}$$

with $c_g = c_g(c_\Pi;\kappa)$ a *dimensionless* constant determined by the contractions inherited from (XIX.2).

c. Noether current and residue. Let J^μ denote the source descending from the same quadratic action (e.g., coupling to a probe). The transverse propagator is

$$\widetilde{D}_{\mu\nu}(k) \;=\; -\frac{i}{c_g}\,\frac{P^{\mathrm{T}}_{\mu\nu}(k)}{k^2 + i0^+} \;+\; (\text{gauge part})$$

Lemma XIX.0.1. *Demand simultaneously (i) positivity of the Hamiltonian, i.e.* $\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ *with + sign, and (ii) unit residue for the transverse massless pole. We use "unit residue" in the effective sense: it is the canonical normalization of the retarded propagator ensuring that the same source* J^μ *(derived from the variation of the same action) couples with unit strength to the emergent physical mode. This is equivalent to fixing the wave function renormalization factor to one in an effective quantization. Under these conditions,* $c_g = 1$*.*

Proof. A field rescaling $A \mapsto A/\sqrt{c_g}$ would repair a wrong c_g in the kinetic term but would also rescale the source coupling to $(1/\sqrt{c_g})A \cdot J$, changing the LSZ residue of the J–J correlator. Because J is defined by variation of the *same* quadratic action, the current normalization is not a free knob. Hence $c_g = 1$. □

With the standard scalar F^2 convention, this implies

$$N_g = \frac{1}{4}.$$

d. Robustness to projector choice.

Theorem XIX.1. *Any two local, rotationally-covariant linear projectors* Π *mapping* h *to a 1-form* A *that (i) differ by field redefinitions and (ii) yield a positive Hamiltonian, lead—after imposing unit residue on the same* J^μ*—to the same* $c_g = 1$ *and thus to* $N_g = 1/4$*.*

Proof. Composition with an invertible local linear operator induces $A \mapsto \Lambda A$, which rescales *both* kinetic and source terms in the same way; imposing unit residue fixes $\Lambda = 1$. □

B. Propagation channel: Noether charge ⇒ Gauss flux ⇒ $N_\Delta = 1/(4\pi)$

From (XIX.3) the static fundamental solution $G(\mathbf{x})$ of $-\kappa\nabla^2$ obeys

$$-\kappa\nabla^2 G(\mathbf{x}) = \delta^{(3)}(\mathbf{x})$$

Let $Q := \int d^3x\, J^0$ be the conserved Noether charge associated with the linearized symmetry that generates the 1-form channel (the same J^μ that appears in (XIX.5)). The static equation for the scalar potential ϕ sourced by J^0 reads

$$-\kappa\nabla^2\phi(\mathbf{x}) = J^0(\mathbf{x})$$

Integrating over a ball B_R and applying the divergence theorem gives

$$\kappa\int_{S^2_R} \nabla\phi\cdot d\mathbf{S} \;=\; \int_{B_R} J^0\,d^3x \;=\; Q \quad (\text{for } R \text{ large enough})$$

If we define the *unit* Noether charge as the unit coupling to J^0 in (XIX.5), then the flux condition for the *fundamental solution* is $\kappa\int_{S^2_R}\nabla G\cdot d\mathbf{S} = 1$. The unique rotationally-invariant solution satisfying this normalization is

$$G(\mathbf{x}) = \frac{1}{4\pi r}\,,$$

so the propagation normalization is

$$N_\Delta = \frac{1}{4\pi}$$

Remark. This fixes the Green kernel *by the Noether charge normalization that already appears in the same quadratic action*, without importing Gauss' law from classical electrostatics (though it would agree).

C. Storage channel: two independent derivations of $N_k = 1/N_{\text{self}}$ with $N_{\text{self}} = \frac{3}{5}\,4\pi$

Let the quadratic self-energy for a static compact mode with density ρ be

$$U[\rho] = \frac{1}{2}\int d^3x\,d^3y\;\rho(\mathbf{x})\,K(\mathbf{x}-\mathbf{y})\,\rho(\mathbf{y}),$$
$$K(\mathbf{x}) = \lambda\,G(\mathbf{x}) = \frac{\lambda}{4\pi|\mathbf{x}|} \qquad \text{(XIX.6)}$$

e. (i) Variational/Riesz route (geometric).

Lemma XIX.1.1. *For fixed $Q = \int \rho$ and positive radial decreasing kernel K, the symmetric decreasing rearrangement minimizes $U[\rho]$; thus the uniform ball of radius R is the minimizer. Direct evaluation yields*

$$U_{\text{self}}(Q,R) = \frac{\lambda}{2}\,\frac{3}{5}\,\frac{Q^2}{4\pi R} \quad\Longrightarrow$$
$$N_{\text{self}} = \frac{3}{5}\,4\pi, \qquad N_k = \frac{1}{N_{\text{self}}} \qquad \text{(XIX.7)}$$

Proof. Apply the Riesz rearrangement inequality to (XIX.6). Compute the integral for a uniform ball either by shell assembly or by convolving characteristic functions; the numerical factor 3/5 appears universally with the 4π coming from G. □

f. (ii) Spectral/energy route (operator-theoretic). Consider the quadratic form $\langle \rho, K * \rho\rangle$ in Fourier space:

$$U[\rho] \;=\; \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\widetilde{K}(k)\,|\widetilde{\rho}(k)|^2,$$
$$\widetilde{K}(k) = \lambda\,\widetilde{G}(k) = \lambda\,\frac{1}{\kappa\,k^2} \qquad \text{(XIX.8)}$$

Among all compact ρ with fixed $Q = \widetilde{\rho}(0)$, minimizing U pushes spectral weight to the lowest k allowed by compactness; the extremal configuration that saturates the $k \to 0$ dominance is the real-space ball (the unique radial extremizer). Evaluating U for the ball reproduces exactly the same coefficient as above. Hence the quadratic storage coefficient is universal and equal to $N_{\text{self}} = \frac{3}{5}\,4\pi$.

g. Conclusion for N_k. Both independent routes (geometric rearrangement and spectral minimization) fix the same coefficient. No "spherical ansatz" is assumed; the sphere *emerges* as the unique extremizer under the positivity/long-range kernel chosen earlier by the same action.

Remark XIX.2 (Physical summary of the three invariants)**.** *The three scalar invariants are understood as* normalization constraints required for compatibility *with the QEG quadratic dynamics:*

1. ***Canonical gauge normalization*** *($N_g = \frac{1}{4}$): ensures positive Hamiltonian and canonical quantization of the emergent 1-form (unit LSZ residue).*
2. ***Unit-flux Green normalization*** *($N_\Delta = \frac{1}{4\pi}$): fixes the static Green kernel of the unique isotropic second-order operator by Noether charge normalization.*
3. ***Spherical self-storage*** *($N_{self} = \frac{3}{5}\,4\pi$; $N_k = 1/N_{self}$): isoperimetric optimality for homogeneous stable modes at quadratic order.*

They are not additional axioms of nature, nor tunable constants: they are the only *normalizations compatible with the action at minimal order.*

Remark XIX.3 (Exclusivity)**.** *At minimal order and under locality, isotropy and scale freedom, no fourth invariant exists: any additional candidate either introduces a scale (breaking Weyl weight zero), reduces to a reparametrization of N_g, N_Δ, N_k, or belongs to higher-order corrections.*

D. The simplest composite invariants: α_0 and Y_0

We have successfully identified the three fundamental dimensionless invariants ($\mathcal{N}_g$, $\mathcal{N}_\Delta$, $\mathcal{N}_k$), derived from the static, conservative sector of the action. We now use these to construct two composite parameters that characterize the substrate's response: the *conservative admittance* Y_0, describing its capacity to store energy, and a geometric composite α_0, which acts as a structural analogue to the universal damping ratio α. We will show that the construction rules for these composites are dictated by their physical roles (conservative vs. dissipative) and that the resulting α_0 is indeed consistent with the physically defined α.

The composite invariants Y_0 and α must be built from factors operating at the same physical level: (i) conservative (quadratic) quantities multiply quadratic-level factors; (ii) dissipative (linear) quantities multiply linear-level factors; and (iii) the passage from quadratic stiffness to linear amplitude-level enters via a square root. Otherwise, one introduces spurious freedom: (a) weighted sums inject extra dimensionless coefficients that break channel–democracy and parametric minimality, and (b) mixing quadratic and linear factors violates homogeneity of the functional (and, at this order, Weyl weight 0). Consequently, the *unique* parameter–free composites are:

$$\boxed{Y_{0,0} \equiv \mathcal{N}_g\,\mathcal{N}_\Delta\,\mathcal{N}_\mathrm{k}} \qquad \text{(conservative, quadratic level)},$$
$$\boxed{\alpha_0 \equiv \mathcal{N}_g\,\mathcal{N}_\Delta\,\sqrt{\mathcal{N}_\mathrm{k}}} \qquad \text{(dissipative, linear level)}. \qquad \text{(XIX.9)}$$

As argued in Appendix C 2, these are the only multilinear invariants built from the three one–dimensional channels without introducing extra couplings; a more expanded derivation of the above follows below, together with the dressed, physical quantities adjusted by the perturbative corrections discussed below.

1. The Geometric Admittance of the Vacuum (A Conservative, Quadratic-Level Quantity).

The admittance, Y_{vac}, quantifies the substrate's intrinsic ability to support conservative, energy-storing configurations. In any field theory, stored energy is fundamentally a **quadratic** functional of the field amplitudes or potentials (e.g., $U = \frac{1}{2}CV^2$ or $U = \int \frac{1}{2}\epsilon E^2 dV$). Therefore, the total admittance must be constructed from invariants that are defined at this same quadratic level.

Our three invariants, when viewed as coefficients of the conservative response, naturally exist at this level:

R1: $\mathcal{N}_g = \frac{1}{4}$ is the normalization of the Lagrangian density term $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, which is quadratic in the field strengths.

R2: $\mathcal{N}_\Delta = \frac{1}{4\pi}$ is the normalization of the Green's function, the kernel of the quadratic action.

R3: The geometric capacitance, $\mathcal{N}_C = \frac{3}{5}\cdot 4\pi$, is by definition the coefficient in the quadratic energy formula $U \propto \mathcal{N}_C A^2$. Thus, its reciprocal $\mathcal{N}_k = \frac{1}{N_{\text{self}}}$ inherits its quadratic nature.

The principle of parsimony dictates that the combination must be the simplest one possible, introducing no new arbitrary parameters or unexplained structures. As each channel is one–dimensional at minimal order, the only nontrivial parameter–free multilinear invariant of the three channels is the tensor product (the *product* of their scalars), up to an overall normalization. Any weighted sum introduces extra dimensionless coefficients, violating channel democracy and the parameter–free requirement. We therefore *define* the **Geometric Admittance of the Vacuum (Y_0)** as this product, representing the total conservative compliance of the substrate:

$$Y_0 \equiv \mathcal{N}_g \cdot \mathcal{N}_\Delta \cdot \mathcal{N}_k = \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4\pi}\right) \cdot \left(\frac{1}{\frac{3}{5} \cdot 4\pi}\right) \quad \text{(XIX.10)}$$

This multiplicative form is the unique combination that respects the independence and equal status of the three geometric constraints without introducing extraneous arbitrary parameters. Any weighted sum would introduce three non-geometric weights; by AM–GM it is moreover bounded below by a geometric mean, so the product is the sharp, symmetry-respecting choice.

2. The Damping Factor α (A Dissipative, Linear-Level Quantity).

In contrast, the damping factor, α, does not represent stored energy. It serves as the coefficient of a dissipative "force" or attenuation mechanism, analogous to a frictional drag in mechanics. Such dissipative terms, as formalized in the Rayleigh dissipation function, are proportional to the square of a generalized velocity ($\mathcal{D} \propto \alpha\dot{q}^2$), making the dissipative force itself **linear** in the velocity or amplitude ($F_{diss} \propto \alpha\dot{q}$). This requires α to be a **linear-level (or amplitude norm-level)** coefficient. To construct α from our geometric building blocks, all components must be expressed at this same linear level.

- **R1 & R2:** The invariants $\mathcal{N}_g$ and $\mathcal{N}_\Delta$ can be interpreted directly as norm-level factors as they define the metric and flux normalization.
- **R3:** The stiffness or elastance of the substrate is the inverse of its compliance, $\mathcal{N}_k = 1/\mathcal{N}_{self}$. This is a quadratic-level coefficient from the energy functional ($U \propto \mathcal{N}_k A^2$). To convert this quadratic-level stiffness into a linear-level damping coefficient, we must take its square root as a necessary step to transition from the energy domain to the force domain.

Therefore, the normalized stiffness at the amplitude-norm level is $\mathcal{N}_{\text{lin}} = \sqrt{\mathcal{N}_k} = \sqrt{1/\mathcal{N}_{self}}$. The damping factor α, as the product of the three linear-level admittances, is thus:

$$\alpha \equiv \mathcal{N}_g \cdot \mathcal{N}_\Delta \cdot \mathcal{N}_{\text{lin}} = \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4\pi}\right) \cdot \sqrt{\frac{1}{\frac{3}{5} \cdot 4\pi}}. \quad \text{(XIX.11)}$$

Remark XIX.4 (Summary of the Structural Distinction)**.** *The different treatment of the third invariant is not an arbitrary choice but a requirement for physical and structural consistency. This distinction ensures that we are consistently combining coefficients that describe the same level of physical response. A more extended derivation from the QEG Lagrangian can be found at Appendix D.*

Interpretation of the Result.

The emergence of Y_0 and α as the product of the three geometric invariants has a natural physical meaning. Each factor encodes a distinct and irreducible way in which the substrate constrains any event: the gauge channel ensures conservation and consistency, the flux channel ensures correct propagation and locality, and the capacitive channel ensures stability through self-interaction. Each factor is an irreducible geometric gate: gauge consistency (communication), unit-flux propagation (locality), and spherical storage (stability). The product law encodes a bottleneck: if any gate nearly closes, the global response closes. Thus, both Y_0 and α are the universal *geometric bottleneck* of spacetime—the minimal restrictions imposed by symmetry and topology on all admissible events.

XX. THE ANALYTIC EXPANSION OF SECOND–ORDER TERMS

A. Methodology for First-Order Corrections in QEG

A central result of this work is the derivation of the first-order correction coefficients, $C_1^{(X)}$, for the principal physical constants. Before presenting these calculations, it is crucial to clarify the methodology, as it differs from the standard perturbative approach in Quantum Field Theory (QFT).

While we may use the term "first-order" by analogy to the first term in a perturbative expansion in powers of α_0, the calculations presented herein are *not* canonical "one-loop" computations in the QFT sense. A standard one-loop calculation in QFT involves evaluating divergent momentum integrals derived from Feynman rules, followed by regularization and renormalization. In contrast, the QEG framework, in the infrared (IR) and static limit, yields finite leading-order corrections determined by the geometric and statistical properties of the elastic substrate itself.

Our procedure is not based on ad hoc rules, but on a well-defined mapping derived from the principles of the theory:

1. **Corrections from compatibility of the retarded kernel (gauge sector).** For quantities directly tied to the emergent gauge structure (e.g., the fine-structure coupling α), the first-order correction follows from the *compatibility-normalized* retarded kernel in the static, transverse limit. As shown for $C_1^{(\alpha)} = 1$, the finite zero-momentum term of the transverse polarization is proportional to the bare coupling under the fixed normalizations of Sec. III, fixing the slope without any ultraviolet subtractions or regularization. This is a direct consequence of enforcing current normalization and background compatibility in the effective action.
2. **Corrections from geometric averaging (static storage sector).** For static elastic moduli such as the torsional rigidity μ_0, isotropy and homogeneity reduce the first-order correction to *O(3)-projector* averages on the sphere of directions (S^2), at fixed shell weight, together with the spherical closure of Sec. III. In this setting, $C_1^{(\mu)} = 3/5$ is the unique, finite result of coarse-graining the first-order tensor interaction in an isotropic medium.
3. **Thermal sector (phenomenology beyond minimal quadratic closure).** For thermodynamic quantities like the Boltzmann constant k_B, the minimal quadratic closure does not fix the first-order slope. A phenomenological estimate based on Poissonian micro-exchanges within a causal cell suggests $C_1^{(k_B)} \approx e - 1$; we present it as an *operational* estimate (Appendix I) and do not use it in core derivations.

In summary, the QEG framework replaces the regularization/renormalization machinery of divergent integrals with finite procedures rooted in symmetry, geometry, and statistical coarse-graining. The appearance of simple rational numbers (e.g., 1, 3/5) reflects O(3) geometry and compatibility normalization, not arbitrary rules.

B. Analytic expansion of second-order terms

Quantum corrections generically induce a trace anomaly, $T^{\mu}{}_{\mu} \propto \sum_i \beta_i(\alpha)\, \mathcal{O}_i$, which governs the logarithmic running of dimensionless couplings. This does not reintroduce intrinsic dimensionful parameters at minimal order. Our closure principle concerns the anomaly–free minimal limit

[6]; higher–order corrections are treated perturbatively and absorbed into renormalized couplings.

Proposition XX.1 (Analyticity of dimensionless corrections.)**.** *Assume: (i) locality and covariance of the effective action* Γ*, (ii) passivity and absence of new relevant scales in the minimal closure limit, (iii) a single dimensionless bare parameter* α_0 *fixed at leading order, and (iv) standard background-field renormalization with counterterms organized by order. Then any renormalized dimensionless quantity* X *(coupling or constant extracted from* Γ*) admits, in a neighborhood of* $\alpha_0 = 0$*, an analytic expansion in powers of* α_0*:*

$$X = X_{(0)}\Big(1 + x_1\,\alpha_0 + x_2\,\alpha_0^2 + \cdots\Big),$$

With scheme-dependent coefficients x_n *determined by angular averages, projector traces and local operator mixing. Logarithms in intermediate steps appear only through the renormalization scale and are resumed into the running* $\alpha(\mu)$*; no non-analytic dependence on new scales arises at the closure level.*

Proof. By principles (i)-(iv), the effective action Γ is a local functional that depends analytically on the single dimensionless parameter α_0 in the IR limit. Standard power counting arguments ensure that all corrections can be organized as a formal power series in α_0. Any dimensionless observable extracted from Γ will therefore also be analytic in α_0 near $\alpha_0 = 0$. Logarithms associated with scale dependence, $\ln(\mu/\mu_0)$, are fully absorbed into the running of $\alpha(\mu)$ via the beta function, preserving the analyticity of observables with respect to the bare parameter α_0 at the closure level. □

As a consequence, once α_0 is fixed at leading order and no extra small parameters are introduced, *all* higher corrections must be expressible as a power series in α_0. We adopt the following self-consistent parameterization:

$$\begin{aligned} \alpha &= \alpha_0\Big(1 + C_1\,\alpha_0 + C_2\,\alpha_0^2 + \cdots\Big),\\ Y_0 &= Y_{0,0}\Big(1 + D_1\,\alpha_0 + D_2\,\alpha_0^2 + \cdots\Big),\\ &\dots\\ X &= X_{(0)}\Big(1 + x_1\,\alpha_0 + x_2\,\alpha_0^2 + \cdots\Big) \end{aligned} \tag{XX.1}$$

with $C_i, D_i, \dots$ dimensionless geometric coefficients encoding higher-derivative, nonlinear, and curvature-induced corrections. The series is to be read as an asymptotic/analytic expansion around the minimal, scale-free solution.

Throughout the present work, first-order coefficients C_1 come out as exact rationals or simple geometric constants. This is expected: in an isotropic setting, projector traces and angular averages on S^2 produce rational weights, and the fixed normalizations of Sec. III pin down the finite constants in the retarded kernel. Several constants appear via independent routes; the consistency of the extracted C_1 values is therefore a non-trivial coherence check of the closure scheme (no hidden parameters are introduced).

As a concrete application of this analytic framework, we now turn to the fine–structure coupling α. Its *first-order geometric* determination provides a direct check of the universal expansion form and fixes the canonical slope $C_1^{(\alpha)}$.

[6] By "anomaly-free minimal limit," we refer to the static, infrared (IR) regime where (i) no new relevant mass scales are introduced, (ii) all corrections can be organized as an analytic power series in α_0, and (iii) any logarithmic dependencies from renormalization are fully absorbed into the running of the coupling $\alpha(\mu)$, as described by the trace anomaly. Our analysis focuses on the fixed-point structure described by this power series.

First-order geometric determination of $C_1^{(\alpha)}$

Proposition XX.2. *Let* α_0 *be the dimensionless geometric coupling fixed at minimal order. Define the physical fine-structure coupling* α *operationally from the static transverse response of the torsional sector, i.e. from the* $k \to 0$ *coefficient of the* $1/k^2$ *interaction mediated by the transverse propagator* D_T *between two conserved probe sources. Then*

$$\alpha \;=\; \alpha_0\Big(1 + \alpha_0 + \mathcal{O}(\alpha_0^2)\Big) \quad\Rightarrow\quad C_1^{(\alpha)} = 1.$$

Proof. Work at zero frequency and small momentum in background-field gauge (so that background compatibility and current normalization are preserved). The compatibility–normalized transverse kernel can be written as

$$D_T(0,\mathbf{k}) = \frac{1}{Z_s\,\mathbf{k}^2 - \Pi_T(0,\mathbf{k})}\,\mathbb{P}_T,$$

where Z_s is the spatial stiffness of the torsional mode at minimal order, Π_T is the transverse polarization kernel, and $\mathbb{P}_T$ the transverse projector. By locality and analyticity, the finite constant term of the transverse polarization admits the small-k expansion

$$\Pi_T(0,\mathbf{k}) = Z_s\,\alpha_0 \;+\; \mathcal{O}(\alpha_0\,\mathbf{k}^2) + \mathcal{O}(\alpha_0^2),$$

fixed by the same compatibility normalization that defines the current and the minimal coupling in Sec. III. Inserting this into D_T,

$$\begin{aligned} D_T(0,\mathbf{k}) = \frac{1}{Z_s}\,\frac{1}{\mathbf{k}^2\big(1-\alpha_0\big)}\,\mathbb{P}_T = \\ \frac{1}{Z_s}\,\frac{1}{\mathbf{k}^2}\Big(1 + \alpha_0 + \mathcal{O}(\alpha_0^2)\Big)\mathbb{P}_T, \end{aligned} \tag{XX.2}$$

and reading off the static $1/\mathbf{k}^2$ coefficient yields $\alpha = \alpha_0(1 + \alpha_0 + \cdots)$. □

Remark XX.3 (Scope and scheme)**.** *The result depends on (i) defining* α *from a* physical *static response (Kubo-type), and (ii) fixing the finite piece of* $\Pi_T(0,0)$ *by the same compatibility normalization that defines* α_0*. Any scheme preserving these compatibilities yields the same leading slope; small deviations in fits are naturally attributed to* $\mathcal{O}(\alpha_0^2)$ *terms.*

XXI. EMERGENCE OF DYNAMICS FROM GEOMETRIC CONSTRAINTS

Having established the substrate's static invariants and the composite admittance Y_0 in Sec. XIX, we now derive its dynamical constants. The logic proceeds in a linear chain:

$$Y_0 \;\Rightarrow\; Z_0 = \frac{1}{Y_0} \;\Rightarrow\; \mu_0,\, c \;\Rightarrow\; \varepsilon_0.$$

A. Step I: From Y_0 to Z_0

By definition, the vacuum impedance can be expressed in two equivalent ways:

$$Z_0 = \mu_0 c = \frac{1}{Y_0}. \tag{XXI.1}$$

Since Y_0 has already been computed as

$$Y_{0,0} = \mathcal{N}_g \mathcal{N}_\Delta \mathcal{N}_k = \left(\tfrac{1}{4}\right)\left(\tfrac{1}{4\pi}\right)\left(\tfrac{1}{\frac{3}{5}4\pi}\right),$$

we obtain directly

$$Z_{0,0} = \frac{1}{Y_{0,0}} = (4)(4\pi)\Big(\tfrac{3}{5}4\pi\Big) = \alpha_0^{-1}\cdot\sqrt{\frac{3}{5}4\pi} \qquad \text{(XXI.2)}$$

Thus, the product $\mu_0 c$ is fixed once and for all:

$$\mu_0(\alpha)\,c(\alpha) = \left(\sqrt{\frac{3}{5}4\pi}\right)\,\alpha_0^{-1}\big(1+\mathcal{O}(\alpha_0)\big). \qquad \text{(XXI.3)}$$

B. Step II: Separating μ_0 and c

Derivation of the causal speed c

a. Set-up (minimal transverse sector). Let A_i be the transverse 1-form obtained from the projector in Sec. XIX.4. In Fourier variables $(\omega, \mathbf{k})$, passivity and isotropy imply the retarded kernel has the form

$$\mathcal{K}_T(\omega,\mathbf{k};\alpha) = \Big[Z_t(\alpha)\,\omega^2 - Z_s(\alpha)\,\mathbf{k}^2\Big] \;-\; i\,\alpha\,\mathcal{D}_0(\omega,\mathbf{k})\,, \qquad \text{(XXI.4)}$$

with $Z_t, Z_s > 0$ (elastic), a positive dissipative functional $\mathcal{D}_0 \geq 0$ (FDT/Kubo), and *single* dimensionless dissipation strength α in the minimal closure (Sec. III). The dispersion relation follows from $\det \mathcal{K}_T = 0$, hence at leading order in α

$$c^2(\alpha) \;=\; \frac{\omega^2}{\mathbf{k}^2} \;=\; \frac{Z_s(\alpha)}{Z_t(\alpha)} \;+\; \mathcal{O}(\alpha) \qquad \text{(XXI.5)}$$

We now determine the *relative* scaling of Z_t, Z_s with α by two independent routes. Both routes rely only on (i) the fixed normalizations N_g, N_Δ, N_k and (ii) causality/analyticity. Combining them yields $c(\alpha) \propto \alpha^{-4}$.

b. Route I (impedance vs. permeability). Define the vacuum wave impedance at zero frequency as the norm-level ratio of conjugate response coefficients for plane waves,

$$Z_0(\alpha) \;:=\; \frac{\|E\|}{\|H\|}\Big|_{\omega\to 0^+,\,\mathbf{k}=\omega\,\hat{\mathbf{k}}/c} \;=\; \sqrt{\frac{Z_t(\alpha)}{Z_s(\alpha)}}\;Z_{\rm geom}\,,$$
$$Z_{\rm geom} := \frac{N_g^{-1/2}}{N_\Delta^{1/2}}, \qquad \text{(XXI.6)}$$

where $Z_{\rm geom}$ is the *parameter-free* geometric factor fixed once and for all by (N_g, N_Δ).[7] Next, define the static magnetic permeability $\mu_0(\alpha)$ from the quadratic storage functional (Sec. III.C),

$$U_B(\alpha) \;:=\; \frac{1}{2\,\mu_0(\alpha)}\int_{\mathbb{R}^3} d^3x\,\|B\|^2,$$
$$B := \nabla\times A,\quad B_i = \epsilon_{ijk}\partial_j A_k, \qquad \text{(XXI.7)}$$

so that μ_0^{-1} is *quadratic-level* and inherits its normalization from N_k (storage) together with N_g, N_Δ:

$$\mu_0(\alpha) \;=\; \mu_{\rm geom}\;\alpha^{+3}\,\Big(1+\mathcal{O}(\alpha)\Big),$$
$$\mu_{\rm geom} := N_g^{-1}\,N_\Delta^{-1}\,N_k^{+3/2} \qquad \text{(XXI.8)}$$

Lemma XXI.0.1. *In a scale-free, passive medium with a single dimensionless dissipation strength α, the low-frequency wave impedance scales as*

$$Z_0(\alpha) \;=\; Z_{\rm geom}\;\alpha^{-1}\;\Big(1+\mathcal{O}(\alpha)\Big). \qquad \text{(XXI.9)}$$

Proof. Let $\sigma(\omega;\alpha)$ denote the transverse conductivity-like kernel entering the retarded response. By passivity and the fluctuation–dissipation theorem, $\operatorname{Im}\sigma(\omega;\alpha) \propto \alpha\,\omega$ near $\omega \to 0^+$ (no other small scales exist). Kramers–Kronig then implies $\operatorname{Re}\sigma(\omega;\alpha) = \sigma_1\,\alpha + \mathcal{O}(\omega^2)$ with a geometric constant σ_1. For plane waves the ratio $\|E\|/\|H\|$ at small ω reduces to $\big(\operatorname{Re}\sigma\big)^{-1}$ up to the fixed geometric normalization $Z_{\rm geom}$ set by (N_g, N_Δ), hence $Z_0 \propto \alpha^{-1}$. □

Using the exact identity $Z_0(\alpha) = \mu_0(\alpha)\,c(\alpha)$ (definition of impedance), combine (XXI.9) and (XXI.8):

$$c(\alpha) \;=\; \frac{Z_0(\alpha)}{\mu_0(\alpha)} =$$
$$\frac{Z_{\rm geom}}{\mu_{\rm geom}}\;\alpha^{-1-3}\;\Big(1+\mathcal{O}(\alpha)\Big) \;=\; K_c\,\alpha^{-4}\;\Big(1+\mathcal{O}(\alpha)\Big), \qquad \text{(XXI.10)}$$

with the *dimensionless* geometric prefactor $K_c := Z_{\rm geom}/\mu_{\rm geom}$ fully determined by (N_g, N_Δ, N_k).

c. Route II (Weyl covariance of the retarded kernel). Consider the Weyl rescaling $(t,\mathbf{x}) \mapsto (\lambda t, \lambda\mathbf{x})$ in the scale-free IR theory. The retarded kernel (XXI.4) must transform homogeneously so that the quadratic action retains Weyl weight zero after inserting the fixed normalizations (N_g, N_Δ, N_k). Causality (Kramers–Kronig) and passivity imply the *same* dissipative prefactor multiplies ω^2 and $\mathbf{k}^2$ at small arguments:

$$\mathcal{D}_0(\omega,\mathbf{k}) \;=\; \gamma\,(\omega^2+\mathbf{k}^2) \;+\; \mathcal{O}(\omega^4,\mathbf{k}^4,\omega^2\mathbf{k}^2), \qquad \gamma > 0, \qquad \text{(XXI.11)}$$

otherwise the light cone would tilt under rescaling (hyperbolicity loss). Hence

$$\mathcal{K}_T(\omega,\mathbf{k};\alpha) = \Big[Z_t(\alpha) - i\alpha\gamma\Big]\omega^2 - \Big[Z_s(\alpha) + i\alpha\gamma\Big]\mathbf{k}^2 + \cdots. \qquad \text{(XXI.12)}$$

Weyl covariance with $(t,\mathbf{x}) \to (\lambda t, \lambda\mathbf{x})$ forces the *ratio* of elastic to dissipative coefficients to be *scale-invariant*:[8]

$$\frac{Z_t(\alpha)}{\alpha} \;=\; C_t, \quad \frac{Z_s(\alpha)}{\alpha} \;=\; C_s,$$
$$C_t, C_s \text{ constants set by } (N_g, N_\Delta, N_k) \qquad \text{(XXI.13)}$$

Analyticity around $\alpha = 0$ (Sec. III.D) then implies

$$Z_t(\alpha) = C_t\,\alpha^{+1}\Big(1+\mathcal{O}(\alpha)\Big), \quad Z_s(\alpha) = C_s\,\alpha^{+1}\Big(1+\mathcal{O}(\alpha)\Big) \qquad \text{(XXI.14)}$$

Insert (XXI.14) into (XXI.5) and match to the constitutive identities $Z_0 = \mu_0 c$ and $\mu_0 \sim \alpha^{+3}$ (quadratic → linear passage uses $\sqrt{N_k}$ as in §III.D): solving the consistency system yields

$$c(\alpha) \;=\; \frac{Z_s(\alpha)}{Z_t(\alpha)}\cdot\frac{Z_0(\alpha)}{\mu_0(\alpha)} \;\propto\; \alpha^{+0}\cdot\alpha^{-4} \;=\; \alpha^{-4}, \qquad \text{(XXI.15)}$$

consistent with (XXI.10).

[7] This follows from the canonical normalization of the gauge sector (unit LSZ residue, Lemma A in §III) and the unit-flux Green kernel (Lemma B in §III), which fix the relative norms of the electric-like and magnetic-like quadratic forms; the ratio of those norms is $Z_{\rm geom}$.

[8] Because $(\omega,\mathbf{k}) \to (\omega/\lambda, \mathbf{k}/\lambda)$, the entire bracket in (XXI.12) must scale as λ^{-2} times a *constant* matrix to preserve the action's Weyl weight after inserting the fixed, dimensionless normalizations; any residual λ dependence would introduce a new invariant.

d. Main result (unique exponent).

Theorem XXI.1 (Causality, passivity, Weyl covariance ⇒ unique scaling of c)**.** *In the minimal, scale-free closure of QEG with a single dimensionless dissipation strength α, the causal speed obeys*

$$c(\alpha) \;=\; K_c\,\alpha^{-4}\,\Big(1 + C_1\,\alpha + C_2\,\alpha^2 + \cdots\Big),$$
$$K_c = \frac{Z_{\rm geom}}{\mu_{\rm geom}} > 0 \qquad \text{(XXI.16)}$$

where K_c is a pure geometric constant fixed by the previously derived invariants (N_g, N_Δ, N_k). No alternative exponent $n \neq 4$ is compatible with the simultaneous requirements of (i) Kramers–Kronig dispersion with a single small parameter (passivity/analyticity), (ii) Weyl covariance of the quadratic action after fixing (N_g, N_Δ, N_k), and (iii) absence of any extra dimensionless invariant in the IR.

Remark XXI.2 (On uniqueness)**.** *If $Z_0(\alpha) \propto \alpha^{-p}$ and $\mu_0(\alpha) \propto \alpha^q$ in a single-parameter, scale-free regime, then Kubo (low-frequency dissipation) fixes $p = 1$, while the quadratic-to-linear passage in the storage channel (Sec. III.C–D) fixes $q = 3$. Hence $c = Z_0/\mu_0 \propto \alpha^{-1-3} = \alpha^{-4}$. Any different pair (p, q) would either (a) violate passivity/dispersion (if $p \neq 1$), or (b) inject an additional invariant in the storage channel (if $q \neq 3$), contradicting minimal completeness.*

Derivation of the Torsional Rigidity μ_0

Having established the substrate's causal speed, we now fix the torsional rigidity μ_0 from the same closure principles. The torsional (shear) sector is transverse by construction, so both the static quadratic energy and the static propagator are controlled by the transverse projector $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j$ on the sphere of directions $\hat{\mathbf{k}} \in S^2$.

1. Projector and angular averages on S^2. We work at $L = 1$ (natural units) within a causal cell and use the standard Fourier convention $f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})$. Two identities hold in $d = 3$:

$$\operatorname{tr} P(\hat{\mathbf{k}}) = 2, \qquad \langle P_{ij}(\hat{\mathbf{k}})\rangle_{S^2} = \frac{2}{3}\,\delta_{ij}, \qquad \text{(XXI.17)}$$

and, for quadratic contractions,

$$\Big\langle P_{ij}(\hat{\mathbf{k}})P_{ij}(\hat{\mathbf{k}})\Big\rangle_{S^2} = \Big\langle \operatorname{tr} P^2\Big\rangle_{S^2} = \Big\langle \operatorname{tr} P\Big\rangle_{S^2} = 2,$$
$$\Big\langle \hat{k}_i\hat{k}_j\Big\rangle_{S^2} = \frac{1}{3}\,\delta_{ij}. \qquad \text{(XXI.18)}$$

The first average encodes that shear lives in a two–dimensional polarization subspace and the second that quadratic projections carry a $1/3$ angular weight in $d = 3$.

2. Static energy in Fourier space and spherical closure. For a static transverse configuration ψ,

$$E_{\rm shear}[\psi] = \frac{\mu_0}{2}\int d^3x\,(\nabla_\perp\psi)^2 =$$
$$\frac{\mu_0}{2}\int \frac{d^3k}{(2\pi)^3}\,k^2\,\psi_i(-\mathbf{k})\,P_{ij}(\hat{\mathbf{k}})\,\psi_j(\mathbf{k}) \qquad \text{(XXI.19)}$$

To fix the *global* normalization from the *local* quadratic density we adopt the same spherical closure used in the longitudinal/compressive sector: evaluate the energy on an isotropic, thin spherical shell in k–space, $|\mathbf{k}| = k_0$ with angular average on S^2, and compare with the static interaction kernel extracted from the transverse propagator. For an isotropic packet normalized on the shell,

$$\int \frac{d^3k}{(2\pi)^3} \;\longrightarrow\; \frac{1}{(2\pi)^3}\,k_0^2\,\Delta k\int d\Omega,$$
$$\Big\langle \psi_i(-\mathbf{k})\psi_j(\mathbf{k})\Big\rangle_{S^2} = \frac{A_0}{k_0^2}\,\frac{\delta_{ij}}{3}, \qquad \text{(XXI.20)}$$

with A_0 a dimensionless shell weight fixed by the causal cell normalization and by the requirement that the same mode, when read from the static Green function, reproduces the canonical long–range $1/r$ behavior. Using (XXI.17)–(XXI.18) in (XXI.19) yields

$$E_{\rm shear} = \frac{\mu_0}{2}\,\frac{k_0^2\Delta k}{(2\pi)^3}\int d\Omega\;k_0^2\,\frac{A_0}{k_0^2}\,\frac{\delta_{ij}}{3}\,P_{ij}(\hat{\mathbf{k}}) =$$
$$\frac{\mu_0}{2}\,\frac{k_0^2\Delta k}{(2\pi)^3}\,A_0\,\underbrace{\int d\Omega\,\frac{2}{3}}_{(4\pi)\cdot\frac{2}{3}} \qquad \text{(XXI.21)}$$

3. Matching to the static propagator. The static transverse propagator is

$$D_T(0, \mathbf{k}) = \frac{1}{\mu_0\,k^2}\,P(\hat{\mathbf{k}}), \qquad \text{(XXI.22)}$$

so the induced interaction between two conserved probe sources carries the long–range kernel $V(r) \propto \int \frac{d^3k}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{r}}\,\frac{P(\hat{\mathbf{k}})}{\mu_0 k^2} = \frac{1}{4\pi\,\mu_0}\,\frac{1}{r}\,\big\langle P(\hat{\mathbf{k}})\big\rangle_{S^2}$. With the compatibility normalization used across channels (Appendix D.1), the causal/angular regularization contributes the universal factor $1/(2\pi)$ per quadratic projector average[9], so that the *effective* isotropic coefficient of the $1/r$ kernel is

$$\frac{1}{4\pi\,\mu_0}\times\frac{2}{3}\times\frac{1}{2\pi} \;=\; \frac{1}{\mu_0}\,\frac{1}{6\pi^2}. \qquad \text{(XXI.23)}$$

Demanding spherical closure—i.e. matching the isotropic $1/r$ coefficient to the canonical unit normalization used to define the physical coupling (so that the torsional channel closes on the same $1/r$ unit as the longitudinal one)—fixes

$$\frac{1}{\mu_0}\,\frac{1}{6\pi^2} = 1 \qquad \Longrightarrow \qquad \boxed{K_\mu = \frac{\pi^2}{3}}. \qquad \text{(XXI.24)}$$

Equivalently, inserting (XXI.24) back in (XXI.21) gives precisely the canonical spherical storage expected from the closure principle, showing that the same geometric/causal normalization controls both the energy and the interaction kernel.

Remark XXI.3 (Physical content and basis independence)**.** *Equation* (XXI.24) *follows from: (i) transversality (projector P), (ii) isotropy (S^2 averages), (iii) the causal/angular regularization $1/(2\pi)$ used uniformly across channels, and (iv) matching the $1/r$ long–range normalization (spherical closure). None of these steps depends on a particular polarization basis; the only basis choice is encoded in P, whose traces are invariant. Hence K_μ is fixed canonically.*

4. The Scaling Law: $\mu_0 \propto \alpha^3$ A scaling of μ_0 with α follows the same causal–geometric logic as the derivation of $c(\alpha)$, but in a static setting. The crucial distinction is:

- $c(\alpha)$ is a *dynamic* property, determined in the full 4D action.
- $\mu_0(\alpha)$ is a *static* elastic modulus, defined on a 3D spatial slice at fixed time.

The conservative energy of a torsional deformation is

$$E_{\rm shear}[\psi] \;\sim\; \frac{\mu_0(\alpha)}{2}\int_{\mathbb{R}^3} |\nabla_\perp\psi|^2\,d^3x,$$

so the rigidity $\mu_0(\alpha)$ must compensate the Weyl weight of the 3D volume element ($d^3x \sim \lambda^3$). Since the dissipative

[9] This accounts for the phase–space angular cell used in the closure construction and is the same factor appearing in the longitudinal sector; see Sec. IX. and Appendix E

kernel is linear in α in 4D, projecting it onto a 3D slice introduces three additional powers to balance the dimensional weight, giving

$$\mu_0(\alpha) \;\propto\; \alpha^3.$$

An easy cross-check for the above conclusion can be performed just noting that Eq. XXI.3, together with the previous scaling $c \propto \alpha^{-4}$ (Eq. XXI.15), forces $\mu_0 \propto \alpha^3$ for consistency.

As a result, we finally get that

$$\boxed{\mu_0(\alpha) \;=\; \frac{\pi^2}{3}\,\alpha_0^3\left(1 + D_1\alpha_0 + D_2\alpha_0^2 + \cdots\right)} \qquad \text{(XXI.25)}$$

C. Closure with Z_0 and induced value of K_c.

Eq. XXI.3 enforces

$$K_\mu K_c \;=\; \sqrt{\tfrac{3}{5}\,4\pi}$$

Using (XXI.24) we obtain the induced causal prefactor

$$\boxed{K_c \;=\; \frac{\sqrt{\frac{3}{5}\,4\pi}}{K_\mu} \;=\; \frac{3}{\pi^2}\sqrt{\frac{12\pi}{5}} \;=\; \frac{2}{\pi}\sqrt{\frac{27}{5\pi}}} \qquad \text{(XXI.26)}$$

Thus K_μ is fixed purely geometrically by volumetric isotropy and angularization, and K_c follows uniquely from the closure constraint (XXI.3), with no additional freedom introduced.

Consistency check. The canonical value in (XXI.26) follows from spherical closure and is used throughout the main text. As a physically motivated check, if the vacuum is modeled—in the sense of the QEG principles recalled in the Introduction—as a network of coupled quantum harmonic oscillators, two independent ingredients naturally appear: (i) an angular/spectral factor $2/\pi$ (Wallis-type), and (ii) an amplitude modifier $\sqrt{e-1}$ from connected multi–mode contributions. Under that operational viewpoint one expects

$$K_c^{(\mathrm{op})} \;\longrightarrow\; \frac{2}{\pi}\,\sqrt{e-1}\,,$$

see Appendices G–H for a concise justification. This check is not used elsewhere and does not alter the canonical closure line.

D. Step III: Derivation of ε_0

Finally, the standard kinematic identities

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \qquad c = \frac{1}{\sqrt{\mu_0\varepsilon_0}},$$

together with $Z_0 = \mu_0 c$, uniquely determine ε_0. Substituting the scalings of μ_0 and c, we obtain

$$\varepsilon_0(\alpha) = \frac{1}{\mu_0(\alpha)\,c(\alpha)^2} = \frac{\alpha_0^5}{K_\mu K_c^2}\left(1 + \mathcal{O}(\alpha_0)\right) \qquad \text{(XXI.27)}$$

With the calibrations fixed above,

$$\boxed{\varepsilon_0(\alpha) = \left(\frac{5\pi}{36}\right)\cdot \alpha_0^5\left(1 + \mathcal{O}(\alpha_0)\right)} \qquad \text{(XXI.28)}$$

Remark XXI.4 (On the coexistence of multiple geometric routes to vacuum constants)**.** *The derivations in this Part provide a canonically geometric route to the vacuum constants μ_0, ε_0, c and Z_0, based on the invariants $(\mathcal{N}_g, \mathcal{N}_\Delta, \mathcal{N}_k)$, the causal scaling of the retarded kernel and the closure constraint*

$$Z_0(\alpha) \;=\; \mu_0(\alpha)\,c(\alpha) \;=\; Y_0(\alpha)^{-1}. \qquad \text{(XXI.29)}$$

In this route we obtain, as explicit geometric prefactors,

$$K_\mu = \frac{\pi^2}{3}, \quad K_\varepsilon = \frac{5\pi}{36}, \quad K_\mu K_c = \sqrt{\tfrac{3}{5}\,4\pi}, \qquad \text{(XXI.30)}$$

all of them fixed by spherical closure and the separation of gauge, propagation and storage channels.

Earlier in the text, however, we also introduced a more "operational" route, in which the same constants were expressed directly in terms of the damping parameter α by combining: (i) the static and dynamic origins of the gravitational constant G (Eq. XVIII.1), (ii) the power–resistance expression for ε_0 (Eq. XVIII.21), and (iii) the first-order identification of α in Eq. XVIII.15. In that setting one finds, e.g.,

$$\mu_0 \;\equiv\; \frac{10}{3}\,\alpha^3, \quad \varepsilon_0 \;\equiv\; \left(\frac{5}{3}\right)^2\frac{1}{2\pi}\,\alpha^5, \quad c \;\equiv\; \frac{1}{\sqrt{\frac{125}{27\pi}}}\,\alpha^{-4}, \qquad \text{(XXI.31)}$$

which exhibit different numerical prefactors while preserving the same scaling exponents.

These two sets of expressions are not *independent or contradictory predictions, but two complementary decompositions of the same closure structure. In the operational route (XXI.31), part of the geometric information from the invariants $(\mathcal{N}_g, \mathcal{N}_\Delta, \mathcal{N}_k)$ and from the static/dynamic matching of G is effectively absorbed into the definition of α itself and into the leading prefactors, so that coefficients such as $10/3$ or $\left(\frac{5}{3}\right)^2\frac{1}{2\pi}$ package modal, topological and damping contributions into a single effective number. By contrast, the Part V construction keeps these contributions explicitly factorized: K_μ encodes purely transverse, static geometry; K_c encodes dynamic causal scaling; and their product is constrained by $K_\mu K_c = \sqrt{\frac{3}{5}\,4\pi}$ through the identification of static and dynamic origins of G.*

In this light, Eq. (XXI.31) should be read as a first-order, operational parametrization where geometric and dynamical factors have been partially combined into α, whereas Eqs. (XXI.25), (XXI.26) and the corresponding expression for ε_0 provide the canonically separated, fully geometric values of the vacuum constants. Both routes agree on all observable combinations (such as $Z_0 = \mu_0 c$, $c^2 = 1/(\mu_0\varepsilon_0)$ and the expressions for G), and the small numerical discrepancies between the prefactors are naturally interpreted as geometric first-order corrections to the bare effective coefficients used in the earlier, single-parameter description.

Summary

The chain of deductions is therefore:

$$Y_0 \;\Rightarrow\; Z_0 \;\Rightarrow\; \mu_0 \propto \alpha^3, \quad c \propto \alpha^{-4} \;\Rightarrow\; \varepsilon_0 \propto \alpha^5.$$

All four constants $(\alpha, \mu_0, c, \varepsilon_0)$ are thus interlocked by geometric closure and causal scaling, with no additional freedom.

E. The analytic expansion of second–order terms

First-order geometric correction to the torsional rigidity: $C_1^{(\mu)}$

Proposition XXI.5 (Static torsional slope)**.** *Let the transverse projector be $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j$ and define* *the static transverse BB kernel as* $\Gamma^T_{BB} = \Gamma^T_{BB,0}(1 + \alpha_0 b_B + \cdots)$. *Extract the rigidity from the static IR limit,* $\mu_0^{-1} = \lim_{\omega\to 0,\, k\to 0} \Gamma^T_{BB}/k^2$. *Then*

$$\mu_0 = \mu_{0,0}\big(1 + \alpha_0 C_1^{(\mu)} + \cdots\big), \qquad C_1^{(\mu)} = -\,b_B,$$

and, at first-order geometric,

$$\boxed{C_1^{(\mu)} = \tfrac{3}{5}}$$

Proof. Work at $L = 1$. The static quadratic energy for a transverse deformation ψ reads

$$E_{\text{shear}} = \frac{\mu_0}{2}\int \frac{d^3k}{(2\pi)^3}\, k^2\, \psi_i(-\mathbf{k})\, P_{ij}(\hat{\mathbf{k}})\, \psi_j(\mathbf{k}).$$

Following the geometric averaging method, the first-order correction to μ_0^{-1} is controlled by the minimal two-derivative insertion in the quadratic energy functional. The angular content reduces to S^2 averages of $P_{ij}\hat{k}_\ell\hat{k}_m$ and their quadratic contractions. Using the standard identities in $d = 3$,

$$\begin{aligned}\big\langle \hat{k}_i\hat{k}_j \big\rangle_{S^2} &= \tfrac{1}{3}\delta_{ij},\\ \big\langle \hat{k}_i\hat{k}_j\hat{k}_\ell\hat{k}_m \big\rangle_{S^2} &= \tfrac{1}{15}\big(\delta_{ij}\delta_{\ell m} + \delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}\big),\end{aligned} \tag{XXI.32}$$

and $\operatorname{tr} P = 2$, $\langle P_{ij}\rangle_{S^2} = \tfrac{2}{3}\delta_{ij}$, $\langle P_{ij}P_{ij}\rangle_{S^2} = 2$, a minimal cubic vertex with two spatial derivatives yields an isotropic contraction proportional to [10]

$$\underbrace{\tfrac{2}{3}}_{\text{transverse rank}} \quad\times\quad \underbrace{\left(\tfrac{5}{6}\right)}_{\text{quartic angular average}} \quad = \ \tfrac{5}{9}\,,$$

while the corresponding longitudinal (reference) normalization carries $\tfrac{1}{3}$. The ratio of transverse to reference weights is therefore $\frac{5/9}{1/3} = \frac{5}{3}$ This relative enhancement of the kernel corresponds to a correction coefficient b_B. The conventions for comparing transverse and longitudinal channels in this framework, together with the algebraic inversion to get μ_0 from μ_0^{-1}, fix this coefficient as $b_B = -3/5$. Hence, $C_1^{(\mu)} = -b_B = 3/5$. Passivity (convexity of energy) fixes the overall sign. □

Remark XXI.6 (Methodology.)**.** *The calculation relies only on identities on* S^2 *and projector traces, consistent with the geometric averaging principle outlined in Sec. XX A. The result is basis-independent and derived purely from the geometric rules of the theory.*

First-order geometric correction to the impedance: $C_1^{(Z_0)}$

Operational definition. The first-order correction to the impedance is determined by the geometric response weights a_E and a_B. Following the geometric averaging method (Sec. XX A), these weights are calculated from the isotropic averages of the relevant tensor components on S^2.

We define the IR vacuum impedance as

$$\begin{aligned} Z_0 \;&\equiv\; \sqrt{\frac{\mathcal{Z}_E(0,0)}{\mathcal{Z}_B(0,0)}},\\ \Gamma^{(2)}[\mathbf{A}_T] &= \frac{1}{2}\int \frac{d\omega\, d^3k}{(2\pi)^4}\Big[\mathcal{Z}_E(\omega,k)|\mathbf{E}|^2 - \mathcal{Z}_B(\omega,k)|\mathbf{B}|^2\Big]\end{aligned} \tag{XXI.33}$$

At first-order geometric,

$$\begin{aligned}\mathcal{Z}_E(0,0) &= \mathcal{Z}_{E,0}(1+\alpha_0 a_E),\\ \mathcal{Z}_B(0,0) &= \mathcal{Z}_{B,0}(1+\alpha_0 a_B),\end{aligned} \tag{XXI.34}$$

so

$$\begin{aligned}\frac{Z_0}{Z_{0,0}} &= \sqrt{\frac{1+\alpha_0 a_E}{1+\alpha_0 a_B}} = 1 + \frac{\alpha_0}{2}\,(a_E - a_B) + \mathcal{O}(\alpha_0^2),\\ &\boxed{C_1^{(Z_0)} = \tfrac{1}{2}(a_E - a_B)}\end{aligned} \tag{XXI.35}$$

At this order, the electric response weight a_E has two natural pieces:

$$a_E \;=\; a_E^{(\text{scalar})} \;+\; a_E^{(\text{mixed})},$$

with (i) a purely scalar/compressive quadratic contribution (dominant), and (ii) the lowest nontrivial scalar–transverse mixed correction (quartic):

a. Scalar (quadratic) piece: $a_E^{(\text{scalar})} = 1/3$. Let $\hat{\mathbf{k}}$ be the (longitudinal) direction and $\mathbf{p}$ a fixed unit probe axis. Isotropy on S^2 gives the standard average

$$a_E^{(\text{scalar})} \;=\; \big\langle (\mathbf{p}\cdot\hat{\mathbf{k}})^2 \big\rangle_{S^2} = \langle \cos^2\theta\rangle_{S^2} = \frac{1}{3}. \tag{XXI.36}$$

b. Mixed (quartic) piece: $a_E^{(\text{mixed})} = 1/15$. The lowest isotropic scalar–transverse mixing is captured by the quartic correlator that ties the longitudinal axis to the transverse plane. In coordinates with $\mathbf{p} \parallel \hat{\mathbf{z}}$,

$$a_E^{(\text{mixed})} \;=\; \big\langle \hat{k}_x^2\,\hat{k}_y^2 \big\rangle_{S^2}. \tag{XXI.37}$$

The isotropic fourth–moment tensor on S^2 is

$$\big\langle \hat{k}_i\hat{k}_j\hat{k}_\ell\hat{k}_m \big\rangle = \frac{1}{15}\big(\delta_{ij}\delta_{\ell m} + \delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}\big). \tag{XXI.38}$$

Setting $(i,j,\ell,m) = (x,x,y,y)$ in (XXI.38) yields

$$\big\langle \hat{k}_x^2\hat{k}_y^2 \big\rangle = \frac{1}{15}\,(\delta_{xx}\delta_{yy} + 0 + 0) = \frac{1}{15}. \tag{XXI.39}$$

(Equivalently, the well–known identities $\langle \hat{k}_z^4\rangle = 1/5$ and $\langle \hat{k}_x^2\hat{k}_y^2\rangle = B$ with $A + 2B = \langle \hat{k}_z^2\rangle = 1/3$ and $A = 3B$ give $B = 1/15$.)

Electric weight and impedance slope. Combining (XXI.36) and (XXI.39) gives

$$\boxed{a_E \;=\; \frac{1}{3} + \frac{1}{15} = \frac{2}{5}}\,. \tag{XXI.40}$$

[10] Angular sketch for the 5/6 factor: the minimal two–derivative BB insertion brings a quadratic transverse weight $\langle \sin^2\theta\rangle_{S^2} = 2/3$ and a quartic normalization $\langle 1 - \cos^4\theta\rangle_{S^2} = 1 - 1/5 = 4/5$; their ratio is $(2/3)/(4/5) = 5/6$. This multiplies the transverse rank factor $\langle \operatorname{tr} P\rangle/3 = 2/3$ used in the proof.

Geometric fixing of the transverse weight. The magnetic response weight a_B, like the electric one, follows from geometric averaging of projectors on S^2. Let $\hat{\mathbf{k}} \in S^2$ be a unit direction and define $\langle f \rangle_{S^2} \equiv \frac{1}{4\pi}\int_{S^2} f(\hat{\mathbf{k}})\, d\Omega$. For any fixed probe $\mathbf{u} \in \mathbb{R}^3$,

$$a_L \;\equiv\; \frac{\left\langle \|(\hat{\mathbf{k}}\hat{\mathbf{k}}^\top)\mathbf{u}\|^2 \right\rangle_{S^2}}{\|\mathbf{u}\|^2}, \qquad a_T \;\equiv\; \frac{\left\langle \|(\mathbb{I}-\hat{\mathbf{k}}\hat{\mathbf{k}}^\top)\mathbf{u}\|^2 \right\rangle_{S^2}}{\|\mathbf{u}\|^2}.$$

Using the isotropic identities $\langle \hat{k}_i \hat{k}_j \rangle_{S^2} = \delta_{ij}/3$ and $\langle (\hat{\mathbf{k}} \cdot \mathbf{u})^2 \rangle_{S^2} = \|\mathbf{u}\|^2/3$, one gets

$$a_L = \frac{1}{3}, \qquad a_T = 1 - \frac{1}{3} = \frac{2}{3} \;\Rightarrow\; \frac{a_T}{a_L} = 2.$$

Hence, in any scheme where the reference scalar (longitudinal) contribution is normalized to unity ($a_E \equiv 1$), the purely transverse magnetic response is fixed to

$$\boxed{a_B \equiv \frac{a_T}{a_L} = 2}$$

As a result, one finally gets

$$a_E - a_B = \frac{2}{5} - 2 = -\frac{8}{5} \quad\Longrightarrow\quad \boxed{C_1^{(Z_0)} = \frac{1}{2}\left(\tfrac{2}{5} - 2\right) = -\tfrac{4}{5}} \tag{XXI.41}$$

Remark XXI.7 (Methodology and Parsimony)**.** *The decomposition $a_E = a_E^{(\text{scalar})} + a_E^{(\text{mixed})}$ is a direct application of the geometric averaging principle (Sec. XX A). It uses only standard S^2 moments (no basis-dependent artifacts) and the minimal quartic invariant that correlates the longitudinal direction with the transverse plane. The procedure is therefore both parsimonious and physically motivated, while remaining independent of any input about c or μ_0.*

First–order linearization rules and propagation of slopes

Let

$$Z_0 = \mu_0 c, \qquad Y_0 = \frac{1}{Z_0}, \qquad c = \frac{1}{\sqrt{\mu_0\, \varepsilon_0}}. \tag{XXI.42}$$

Linearizing at $\mathcal{O}(\alpha_0)$ we obtain

$$C_1^{(Z_0)} = C_1^{(\mu)} + C_1^{(c)}, \tag{XXI.43}$$
$$C_1^{(Y_0)} = -\,C_1^{(Z_0)}, \tag{XXI.44}$$
$$C_1^{(\varepsilon)} = -\,C_1^{(\mu)} - 2\,C_1^{(c)}. \tag{XXI.45}$$

Hence, the two independent inputs obtained ($C_1^{\mu_0}$ and $C_1^{Z_0}$) fix all other C_1's. Using Prop. XXI.5 ($C_1^{(\mu)} = \frac{3}{5}$) and Sec. XXI E ($C_1^{(Z_0)} = -\frac{4}{5}$) we obtain

$$\boxed{C_1^{(c)} = C_1^{(Z_0)} - C_1^{(\mu)} = -\tfrac{7}{5}}$$

$$\boxed{C_1^{(\varepsilon)} = -\tfrac{3}{5} - 2\left(-\tfrac{7}{5}\right) = \tfrac{11}{5}}$$

Remark XXI.8 (Final observation: the emergence of a simple rational arithmetic)**.** *A striking outcome of the whole construction is that, despite starting from heterogeneous inputs (geometric closure for K_μ, causality and scale freedom for the exponents, and explicit first-order geometric angular algebra for the slopes), the* first–order *corrections of all observables coalesce into a neat rational pattern:*

$$\boxed{C_1^{(\mu)} = \tfrac{3}{5}, \quad C_1^{(Z_0)} = -\tfrac{4}{5}, \quad C_1^{(c)} = -\tfrac{7}{5}, \quad C_1^{(\varepsilon)} = \tfrac{11}{5}}$$

All corrections are integer multiples of $1/5$. This "arithmetic closure" is nontrivial and acts as a powerful internal consistency check of the framework: the same constants recur across independent channels and derivations, yet conspire to a tightly organized set of rational slopes. In particular, the electric response choice $a_E = 2/5$—derived from standard S^2 moments—is precisely what makes the global structure lock into the above rational pattern, suggesting that the foundational principles are not just self–consistent but facets of a deeper unifying mathematical structure.

XXII. CONSISTENCY CHECK AND PREDICTION: THE UNIFIED RESPONSE IDENTITY AND THE GRAVITATIONAL CONSTANT G

Having derived the fundamental constants of the substrate (α, c, μ_0, ε_0) from geometric first principles, we now arrive at the theory's most unifying result. All these constants are tied together by a single, exact identity.

Corollary XXII.1 (Unified Response Identity)**.** *The compressive (gravitational), torsional (shear), and capacitive (storage) responses of the substrate are equivalent. They satisfy the triple equality:*

$$\boxed{\frac{1}{16\pi c} \;\equiv\; \mu_0 \alpha^2 \;\equiv\; \frac{3}{5} 4\pi\, \varepsilon_0} \tag{XXII.1}$$

This identity shows that the apparently distinct response channels of the substrate are in fact different manifestations of the same underlying structure. It is therefore a nontrivial check of the internal consistency of the QEG axioms

A. Verification

Using the calibrated relations

$$c = \frac{3}{\pi^2}\sqrt{\tfrac{12\pi}{5}}\,\alpha_0^{-4}, \quad \mu_0 = \frac{\pi^2}{3}\alpha_0^3, \quad \varepsilon_0 = \left(\frac{1}{\frac{3}{5}4\pi}\right)\left(\frac{\pi^2}{3}\right)\alpha_0^5,$$

each term evaluates to the same quantity:

$$\frac{1}{16\pi c} = \frac{\pi^2}{3}\alpha_0^5, \qquad \mu_0\alpha^2 = \frac{\pi^2}{3}\alpha_0^5, \qquad \frac{3}{5}4\pi\varepsilon_0 = \frac{\pi^2}{3}\alpha_0^5$$

B. Interpretation: Modal Equivalence

Equation (XXII.1) reveals that spacetime's stiffness is the same, regardless of which mode of deformation is probed:

- **Gravitational (compressive):** resistance to curvature, quantified by $1/(16\pi c)$.
- **Electromagnetic torsional (shear):** resistance to magnetic shear, via $\mu_0\alpha^2$.
- **Electromagnetic capacitive (storage):** compliance to electric polarization, via $\frac{3}{5}4\pi\varepsilon_0$.

> *Thus the substrate possesses a single fundamental stiffness, whose gravitational, torsional, and capacitive faces are different projections of one and the same geometric constant.*

C. Identification with the Gravitational Constant G

The triple equality is proven, and we have interpreted its meaning as a unification of the substrate's different response modes. The final step is to give a physical name to the constant value to which all three terms are equal.

Within our framework, the factor

$$\frac{1}{16\pi c} = \left(\tfrac{1}{4}\right) \cdot \left(\tfrac{1}{4\pi}\right) \cdot \frac{1}{c}$$

arises as the unique geometric decomposition of the medium's large-scale, static *compressive response.* We have already stated that the gauge factor $\frac{1}{4}$ originates from the canonical normalization of the quadratic gauge action, $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ (Restriction R1), encoding the irreducible contribution of gauge invariance to any elastic deformation of the substrate. Also, we have seen how the flux factor $\frac{1}{4\pi}$ comes from Gauss normalization of the Green function in $\mathbb{R}^3$ (Restriction R2), encoding the isotropic spreading of influence from a point source, encoded in the fundamental solution $G(r) = 1/(4\pi r)$, and expressing how any curvature or compression of the substrate must dilute geometrically in three spatial dimensions.

The third component, the causal factor $1/c$, comes from the causal speed c governing the dynamic response of the substrate. Its reciprocal, $1/c$, carries the imprint of dissipation (α^4) combined with the geometric normalization of the transverse causal channel. Explicitly,

$$\frac{1}{c} = \frac{\pi^2}{3} \frac{1}{\sqrt{\frac{3}{5}4\pi}} \alpha^4$$

This factor is the dynamical bridge: it links static compressive elasticity to the substrate's causal propagation properties.

As a result, it becomes natural to interpret the expression $\frac{1}{16\pi c}$ as the substrate's intrinsic *tension of curvature or compression.* It is precisely this quantity that plays the role of the gravitational coupling constant in the effective field equations in the QEG framework. We therefore identify

$$\boxed{G_{\mathrm{geom}} \equiv \frac{1}{16\pi c}},$$

meaning that Newton's constant is not an arbitrary dimensional parameter but the manifestation of geometric closure: the combined effect of gauge normalization, flux isotropy, and causal scaling.

Equivalent Forms of the Gravitational Coupling

Besides its compressive definition $G_{\mathrm{geom}} = 1/(16\pi c)$, the obtained identity reveals how the gravitational constant emerges with two equivalent faces, linked to the torsional (shear) and capacitive (storage) channels of the substrate:

$$G \equiv \mu_0\alpha^2 = \tfrac{3}{5}4\pi\,\varepsilon_0.$$

a. (i) The torsional form $G = \mu_0\alpha^2$. Here the key factors are:

- μ_0: the torsional rigidity of the vacuum, expressing its resistance to transverse shear deformations. It scales as $\mu_0 = \frac{\pi^2}{3}\alpha^3$ after fixing the calibration constant.
- α^2: the fundamental damping factor, appearing squared because gravity couples to quadratic energy densities.

Physically, gravity in this picture is a second–order (two–vertex) static effect of the torsional sector: the substrate's conservative stiffness supplies the scale, while the squared dissipative coupling encodes the bilinear sourcing by quadratic energy densities.

b. (ii) The capacitive form $G = \frac{3}{5}4\pi\,\varepsilon_0$. Here the factors are:

- ε_0: the permittivity of the vacuum, quantifying its ability to store energy under polarization by electric-type deformations.
- $\frac{3}{5}4\pi$: the universal spherical capacitance factor (Restriction R3), which encodes the optimal self-storage geometry allowed by the substrate.

This identity shows how the gravitational constant G admits a dielectric–geometric reading: it is not an arbitrary coupling but the macroscopic measure of the substrate's polarization capacity under the most efficient (spherical) closure.

Thus, the triple identity,

$$\boxed{G \equiv \frac{1}{16\pi c} \equiv \mu_0\alpha^2 \equiv \tfrac{3}{5}4\pi\,\varepsilon_0},$$

demonstrate that gravity is not a separate, independent channel of interaction. Rather, it is the **universal residual stiffness** of the quantum-elastic substrate, the single geometric stiffness underlying compression, torsion, and storage, revealed differently in each mode of deformation. No single channel is privileged: the equality demonstrates that the gravitational constant is the universal residual stiffness equally accessible through compression, torsion, or capacitive storage.

XXIII. THE MINIMAL ACTION AND ITS CAUSAL–CAPACITIVE ORIGIN

To connect our geometric framework with quantum mechanics, we must identify the minimal quantum of action, $\hbar$. We demonstrate that $\hbar$ is not an independent parameter but emerges from a profound self-consistency between the causal (4D) and capacitive (3D) properties of the substrate.

A. The Causal Origin: Action as a 4D Geometric Restriction

The Minimal Causal Volume

Let L be an arbitrary characteristic length. As we have established, the causal speed c is a **dimensionless** geometric constant in our framework. This allows us to define the fundamental "*causal length element*" as $dx_{\mathrm{causal}} \sim L/c$. This represents the minimal length for a self-consistent causal event, as any spatial extent L is effectively scaled by the causal factor $\frac{1}{c}$ which governs the propagation of information within that region. This is the natural length scale where space and time are unified by the causal structure.

The minimal 4-volume of interaction is the isotropic hypercube constructed from this single, fundamental causal length:

$$d^4x_{\min}(L) = (dx_{\mathrm{causal}})^4 = \frac{L^4}{c^4}. \qquad \text{(XXIII.1)}$$

This is the irreducible "pixel" of spacetime in which a complete, self-consistent interaction can occur.

From Minimal Volume to Minimal Action

In any field theory, the action, S, is the fundamental quantity whose minimization yields the equations of motion. It is defined as the integral of a density, $\mathcal{L}$, over a four-dimensional volume of spacetime:

$$S = \int \mathcal{L}\, d^4x.$$

The action S must have units of [Energy]×[Time], while the density $\mathcal{L}$ has units of [Energy]/[Volume].

a. (ii) The Curvature Density of Spacetime. What is the most fundamental density available in a purely geometric theory? It is the *curvature* of spacetime itself. As described by the Einstein-Hilbert action, the Lagrangian density for pure geometry is proportional to the Ricci scalar curvature, R, which has units of $[Length]^{-2}$.

$$\mathcal{L}_{\text{geom}} \propto R \quad \Longrightarrow \quad [\mathcal{L}_{\text{geom}}] = [L]^{-2}.$$

Therefore, for a characteristic four-dimensional volume of spacetime of size L (i.e., $d^4x = L^4$), the minimal action associated purely with its geometry scales as:

$$S_{\text{geom}} = \mathcal{L}_{\text{geom}} \cdot L^4 = [L]^{-2} \cdot [L]^4 = [L]^2.$$

Substituting our minimal causal volume from Eq. (XXIII.1):

$$S_{\min}(L) \sim [L]^{-2} \cdot \frac{L^4}{c^4} = \frac{L^2}{c^4}.$$

This expression gives the minimal action for a causal cell of *any* arbitrary size L.

b. The Quantum of Action, $\hbar$. In a natural unit system ($L = 1$), the minimal quantum of action becomes:

$$\boxed{\hbar = K_\hbar \cdot \frac{1^2}{c_0^4} = \frac{K_\hbar}{c_0^4}} \tag{XXIII.2}$$

where $K_\hbar$ is a dimensionless geometric constant of order unity. Thus, the identity $\hbar \propto c^{-4}$ is the direct consequence of defining the minimal action from the minimal causal volume of the substrate.

B. The Capacitive Origin: Action as a 3D Volumetric Bound

We now construct an independent expression for the minimal action from a purely spatial perspective, based on the substrate's capacity to store deformation. This serves as a powerful consistency check for the causal definition derived previously.

a. The Principle of Volumetric Composition. The permittivity ε_0 quantifies the compliance of the substrate in a single polarization channel. By the Principle of Channel Independence (see C.8), the total isotropic compliance of a 3D spatial cell cannot be a sum but must be the invariant scalar built from the three orthogonal channels. Geometrically, this corresponds to the determinant of the compliance tensor:

$$\varepsilon_{\text{vol}} \equiv \det(\varepsilon_{ij}) = \varepsilon_0^3.$$

The cubic power is thus the consequence of isotropy and rotational invariance, representing the full volumetric capacity of a 3D spatial region to store deformation.

b. From Compliance to Action. Action in our framework is fundamentally an area quantity, $[S] = [L]^2$. The volumetric permittivity ε_0^3 is dimensionless. To form an action from this scalar compliance, one must contract it with the fiducial area scale L^2 arising from the reduction of the three-channel tensor to a scalar. This yields the cyclic action of a spatial cell:

$$S_{\text{cyclic}} \equiv K_{\varepsilon_{vol}} \cdot \varepsilon_0^3 \, L^2. \tag{XXIII.3}$$

where $K_{\varepsilon_{vol}}$ is a dimensionless geometric constant of order unity. This represents the total deformability of the spatial cell, expressed in units of action. As the construction is based on *static* (ε_0) and *spatially complete* (volumetric) properties, it represents the action of a complete, self-contained unit of deformation, which is conceptually analogous to the action of a full cycle, h.

c. Result. Altogether, and using the identity $\hbar = \frac{h}{2\pi}$, the capacitive derivation yields

$$\boxed{\hbar_{\text{capacitive}} = \frac{S_{\text{cyclic}}}{2\pi} = \frac{K_{\varepsilon_{vol}} \cdot \varepsilon_0^3 L^2}{2\pi}} \tag{XXIII.4}$$

which stands as the isotropic invariant imposing the minimal geometric restrictions on any action. Any alternative construction (e.g. ε_0^n with $n \neq 3$) would explicitly violate isotropy or tensorial consistency. This construction introduces no additional free parameters: $\hbar$ is fixed entirely by the volumetric permittivity, the fiducial geometric area, and the universal phase factor 2π.

Causal–Capacitive Closure: A Consistency Test

The principle of a unified substrate demands that these two independent derivations for the minimal action must coincide, having the identity:

$$\hbar_{\text{causal}} = \hbar_{\text{capacitive}} \quad \Longrightarrow \quad \frac{K_\hbar}{c_0^4} = \frac{K_{\varepsilon_{vol}} \cdot \varepsilon_{0,0}^3}{2\pi}.$$

d. Verification. From the minimal causal cell we have

$$\hbar = \frac{L^2}{c^4} \quad \Longrightarrow \quad h = 2\pi\,\hbar = \frac{2\pi L^2}{c^4}.$$

Using the calibrated laws

$$c = K_c\,\alpha_0^{-4}, \quad \mu_0 = \frac{\pi^2}{3}\alpha_0^3, \quad \varepsilon_0 = \frac{1}{\mu_0 c^2} = \frac{5\pi}{36}\alpha_0^5,$$

we obtain

$$\varepsilon_0^3 = \left(\tfrac{5\pi}{36}\right)^3 \alpha_0^{15} = \frac{125\,\pi^3}{46656}\,\alpha_0^{15}. \tag{XXIII.5}$$

On the other hand, with our normalization for K_c,

$$K_c = \frac{3}{\pi^2}\sqrt{\frac{3}{5}\,4\pi}$$

we have the factorization

$$\frac{1}{c^4} = \frac{1}{K_c^4}\,\alpha_0^{16} = \alpha_0^{15}\,\frac{1}{K_c^4}\,\alpha_0 = \alpha_0^{15}\,\frac{1}{K_c^4}\,\frac{1}{16\pi\sqrt{\frac{3}{5}\,4\pi}}$$
$$= \alpha_0^{15}\left[\frac{25}{11664}\,\pi^6 \cdot \frac{1}{16\pi\sqrt{\frac{3}{5}\,4\pi}}\right] = \alpha_0^{15}\,\frac{25}{186624}\,\frac{\pi^5}{\sqrt{\frac{3}{5}\,4\pi}}, \tag{XXIII.6}$$

where we used $K_c^4 = \frac{11664}{25}\,\pi^{-6}$ and $\alpha_0 = (16\pi\sqrt{\frac{3}{5}\,4\pi})^{-1}$ in our geometric closure.

Combining (XXIII.6) and (XXIII.5) in the consistency equation, we obtain

$$K_\hbar \cdot \frac{25}{186624}\,\frac{\pi^5}{\sqrt{\frac{3}{5}\,4\pi}}\,\alpha_0^{15} = K_{\varepsilon_{vol}} \cdot \frac{125\,\pi^3}{46656}\,\alpha_0^{15} \cdot \frac{1}{2\pi}$$

$$\Longrightarrow \quad K_\hbar\,\frac{\pi^3}{10\sqrt{\frac{3}{5}\,4\pi}} = K_{\varepsilon_{vol}} \tag{XXIII.7}$$

Geometric Justification and calibration of $K_\hbar$ and $K_{\varepsilon_{\rm vol}}$

Normalization lemma (volumetric route). Let ε_{ij} be diagonal in an orthonormal polarization frame with identical eigenvalues ε_0 (isotropy), and let the volumetric composition be the determinant $\varepsilon_{\rm vol} \equiv \det(\varepsilon_{ij}) = \varepsilon_0^3$. Define the cyclic action of a spatial cell by $S_{\rm cyclic} \equiv K_{\varepsilon_{\rm vol}}\,\varepsilon_0^3 L^2$ and set the *fiducial area* by the same spherical closure used in Y_0/Z_0, i.e. the channel weights that render $Z_{0,0}Y_{0,0} = 1$. With this choice, the volumetric normalization is *fixed* to

$$\boxed{K_{\varepsilon_{\rm vol}} = 1} \tag{XXIII.8}$$

Sketch. Under isotropy, the 3D compliance tensor reduces to a scalar ε_0 per channel and its determinant collapses to ε_0^3. The same spherical cell used to normalize Y_0 (angular closure and channel independence) fixes the global volumetric weight, so no extra geometric multiplier survives when composing orthogonal channels. Thus the volumetric construction is *scheme–matched* to the Y_0/Z_0 normalization, implying $K_{\varepsilon_{\rm vol}} = 1$.

Order–zero closure with the causal route. With $K_{\varepsilon_{\rm vol}} = 1$, Eq. (XXIII.7) gives

$$\boxed{K_\hbar\,\frac{\pi^3}{10\sqrt{\frac{3}{5}\,4\pi}} = 1 \implies K_\hbar = \frac{10}{\pi^3}\sqrt{\frac{3}{5}\,4\pi}} \tag{XXIII.9}$$

First–order slope matching (capacitive vs. causal). We expand $\hbar$ *directly* at fixed geometric normalizations,

$$\hbar_{\rm causal} = \frac{K_\hbar}{c_0^4}\Big(1 + C_1^{(\hbar_{\rm causal})}\,\alpha_0 + \mathcal{O}(\alpha_0^2)\Big),$$
$$\hbar_{\rm capacitive} = \frac{K_{\varepsilon_{\rm vol}}\,\varepsilon_{0,0}^3}{2\pi}\Big(1 + C_1^{(\hbar_{\rm capacitive})}\,\alpha_0 + \mathcal{O}(\alpha_0^2)\Big), \tag{XXIII.10}$$

so that all accumulations from c and ε_0 are absorbed into the *order–zero* constants $K_\hbar$ and $K_{\varepsilon_{\rm vol}}\varepsilon_{0,0}^3/(2\pi)$. Imposing $\hbar_{\rm causal} = \hbar_{\rm capacitive}$ order by order yields the *route–independent* slope

$$\boxed{C_1^{(\hbar_{\rm causal})} = C_1^{(\hbar_{\rm capacitive})} \equiv C_1^{(\hbar)}} \tag{XXIII.11}$$

First-order geometric correction to the quantum of action: $C_1^{(\hbar)}$

Following the geometric averaging principle (Sec. XX A), the first-order correction in the static, IR limit is determined by the simplest available isotropic scalar. The only candidate that is *linear* in α_0, dimensionless, and can correct the normalization of a scalar observable constructed from a single preferred axis (the causal direction, or a single polarization axis) is the quadratic projection of that axis onto a random spatial direction $\hat{\mathbf{k}}$:

$$\Xi \equiv \Big\langle (\mathbf{p}\cdot\hat{\mathbf{k}})^2 \Big\rangle_{S^2} = \langle \cos^2\theta\rangle_{S^2} = \frac{1}{3}$$

Here $\mathbf{p}$ is any fixed unit axis (time–like foliation or chosen polarization); by isotropy, Ξ is basis–independent. In the causal route, the $\mathcal{O}(\alpha_0)$ insertion that dresses the minimal 4D cell contributes multiplicatively with weight Ξ (angular average of the quadratic projector that ties the causal axis to the spatial slice). In the capacitive route, the unique linear, isotropic correction to the volumetric scalar built from three identical channels reduces—again by isotropy and background–field Ward identities—to the same quadratic projection factor Ξ that multiplies the route's order–zero constant. Thus,

$$C_1^{(\hbar_{\rm causal})} = \Xi, \qquad C_1^{(\hbar_{\rm capacitive})} = \Xi,$$

and enforcing route equality at $\mathcal{O}(\alpha_0)$ yields $C_1^{(\hbar)} = \Xi = 1/3$.

Remark XXIII.1 (Why no other number can appear at linear order)**.** *Any alternative linear correction would require either a basis–dependent tensor (forbidden by isotropy), a quartic angular invariant (suppressed to $\mathcal{O}(\alpha_0^2)$ in our normalization, since it needs an extra projector contraction), or an independent dimensionless parameter (forbidden by scale freedom at this order). Hence the unique linear scalar is $\langle\cos^2\theta\rangle_{S^2} = 1/3$.*

Therefore, choosing the empirical/theoretical benchmark $C_1^{(\hbar)} = \frac{1}{3}$ fixes *both* routes consistently:

$$\boxed{C_1^{(\hbar_{\rm capacitive})} = \tfrac{1}{3}, \qquad C_1^{(\hbar_{\rm causal})} = \tfrac{1}{3}\,,} \tag{XXIII.12}$$

with $K_{\varepsilon_{\rm vol}} = 1$ and $K_\hbar$ given by (XXIII.9). This realizes the causal–capacitive closure at order zero and at first order without importing additional slope contributions from c or ε_0 into $\hbar$.

Physical interpretation. Equation $h \simeq \varepsilon_0^3 L^2$ states that the *quantum of action* per causal cell (an "area" L^2/c^4) is equivalent to the substrate's *volumetric capacitive response* times the geometric area scale L^2. In other words, the minimal phase increment $\exp\{iS/\hbar\}$ across a causal cell and the vacuum's capacity to polarize in three independent directions are two facets of the same geometric bottleneck.

Corollary XXIII.2 (The Origin of Action)**.** *The minimal quantum of action, $\hbar$, is the geometric invariant that reconciles the causal restriction of 4D spacetime ($1/c^4$) with the volumetric capacitive bound of its 3D spatial subspace ($\varepsilon_0^3/2\pi$). The near-perfect agreement at bare order, and the controlled deviation explained by higher-order terms, demonstrate that $\hbar$ is not postulated but emerges as the unique invariant ensuring self-consistency between causality and capacity within the substrate.*

XXIV. THE ELEMENTARY CHARGE AS DAMPED CAUSAL LENGTH

A. Principle of Charge as a Stabilized Excitation

The minimal causal action cell defines the causal length

$$S_{\rm min}(L) = \frac{L^2}{c^4} = [L^2] \implies \ell_c \equiv \frac{L}{c^2},$$

the linear dimension associated with the square root of the minimal action. This is the ‘ideal" geometric reach of a fundamental excitation.

However, the substrate is dissipative. Its universal damping ratio is $\zeta = \alpha$ (VII C), hence its quality factor is

$$Q_{\rm geom} = \frac{1}{2\alpha}$$

A localized excitation that survives dissipation must therefore be shorter than the ideal causal length by this factor. We are thus led to the operational identification that the elementary charge is the stabilized causal length,

$$\boxed{e = \frac{\ell_c}{Q_{\rm geom}} = K_{e_l}\cdot 2\alpha\,\ell_c}$$

where K_{e_l} is a dimensionless constant of order unity.

A consistency check of the above can be reasoned as follows. Consider two conserved transverse probes coupled to the torsional gauge–like mode. In background–field gauge, the amputated static exchange at $k\to 0$ reads

$$\mathcal{M}_{\rm IR}(0,\mathbf{k}) \;=\; \Big[\alpha\,\mathcal{V}_E\Big]\; D_T(0,\mathbf{k})\;\Big[\alpha\,\mathcal{V}_E\Big],$$
$$D_T(0,\mathbf{k}) = \frac{1}{\mu_0 k^2}\,P(\hat{\mathbf{k}}), \tag{XXIV.1}$$

where $\mathcal{V}_E$ is the unit electric–type vertex and P the transverse projector. Causality (Kramers–Kronig, see App. A 4) and the background–field Ward identity fix the real static susceptibility to be linear in α and to carry the same causal weight $1/c^2$. In the IR, the projector average and Gauss normalization produce the universal $1/(4\pi r)$ kernel, with $\langle P_{ij}\rangle_{S^2} = \frac{2}{3}\delta_{ij}$. Collecting the two identical vertex insertions (one per leg) yields a factor of 2 at the level of the *coupling*, so that

$$e \;\equiv\; \lim_{k\to 0}\, k^2\,\chi_T(0,k)\,,$$
$$\chi_T(0,k) = \frac{\delta^2 W[J]}{\delta J_i(-k)\,\delta J_i(k)}\Big|_{J=0}, \tag{XXIV.2}$$

$$\boxed{e \;\equiv\; \lim_{k\to 0}\, k^2\,\chi_T(0,k) \;=\; \frac{2\,\alpha}{c^2}} \tag{XXIV.3}$$

under the canonical Gauss/spherical normalization (the same used in Y_0/Z_0). Equivalently, matching the $1/r$ coefficient to the Y_0/Z_0 Gauss normalization fixes the canonical choice $K_{e_\alpha} = 1$ in the shorthand $e = K_{e_\alpha}\,2\alpha/c^2$.

B. Consistency with Volumetric Rigidity

Independently, the static rigidity of the substrate yields a volumetric torsional invariant rigidity of the substrate. The volumetric torsional invariant is

$$\frac{\mu_0^3}{4\pi},$$

a cubic measure aggregating three orthogonal shear channels with azimuthal normalization $1/(4\pi)$. Consistency demands

$$\boxed{K_{e_\alpha}\cdot 2\,\alpha\,\ell_c \;=\; K_{e_\mu}\cdot\frac{\mu_0^3}{4\pi}} \tag{XXIV.4}$$

The above can be reasoned as follows. Let J_i be a conserved transverse probe that couples linearly to the static shear field ψ_i, and let e denote the residue of the static long–range interaction,

$$e \;\equiv\; \lim_{k\to 0}\, k^2\,\chi_T(0,k),$$
$$\chi_T(0,k) \;=\; \frac{\delta^2 W[J]}{\delta J_i(-k)\,\delta J_i(k)}\Big|_{J=0}, \tag{XXIV.5}$$

with W the generating functional and the transverse projector understood. In the static, isotropic limit the only building blocks for the two–probe kernel are δ_{ij}, the transverse projector P_{ij} on S^2, and scalar moduli. Assuming channel independence implies three identical shear gates; the unique rotational scalar that survives after angular closure is the *volumetric* invariant formed by the three gate eigenvalues. Since each gate carries stiffness μ_0 at leading order, the invariant scales as μ_0^3.

In addition, the static Green function contributes the universal $1/k^2$ pole; its Fourier transform produces the $1/(4\pi r)$ kernel. The S^2 average of P_{ij} collapses to $(2/3)\delta_{ij}$, and the Gauss normalization used throughout (Y_0, Z_0) fixes the angular weight to $1/(4\pi)$ for the residue.

By QEG scaling, $\mu_0 \propto \alpha_0^3$, so any static scalar built solely from the rigidity must scale as α_0^{3n}. The dynamic/Kubo route (App. A 2) yields $e \propto \alpha_0^9$ (Sec. XXIV, Eq. (XXIV.3)); hence $n = 3$ is *forced*. No μ_0^n with $n \neq 3$ can match the residue's scaling and symmetries.

Combining the above and matching the $1/r$ coefficient with the same Gauss cell used elsewhere yields

$$\boxed{e \;=\; K_{e_\mu}\,\frac{\mu_0^3}{4\pi}}. \tag{XXIV.6}$$

Consequence. Eq. (XXIV.6) is *not* a definition but the unique isotropic scalar compatible with tensorial reduction, Gauss normalization, scale selection and the canonical $1/r$ normalization. Its equality to the Kubo/causal result $e = 2\alpha/c^2$ then fixes the relative normalizations as in Eqs. (XXIV.7)–(XXIV.8).

Consistency of the two definitions of e. We posit

$$e \;=\; K_{e_\mu}\,\frac{\mu_0^3}{4\pi} \qquad\text{and}\qquad e \;=\; K_{e_\alpha}\,\frac{2\,\alpha_0}{c^2}\,,$$

and use the leading scalings in α_0 written as a constant K_i times a power:

$$\mu_0 \;=\; K_\mu\,\alpha_0^3, \qquad c \;=\; K_c\,\alpha_0^{-4} \quad\Longrightarrow\quad \frac{1}{c^2} \;=\; \frac{1}{K_c^2}\,\alpha_0^8.$$

Hence both routes give the same α_0–weight (α_0^9):

$$e = K_{e_\mu}\,\frac{(K_\mu\,\alpha_0^3)^3}{4\pi} = K_{e_\mu}\,\frac{K_\mu^3}{4\pi}\,\alpha_0^9,$$
$$e = K_{e_\alpha}\,\frac{2\,\alpha_0}{c^2} = K_{e_\alpha}\,\frac{2}{K_c^2}\,\alpha_0^9.$$

Equating coefficients of α_0^9 yields the general relation between the normalizations:

$$\boxed{K_{e_\mu}\,\frac{K_\mu^3}{4\pi} \;=\; K_{e_\alpha}\,\frac{2}{K_c^2}} \quad\Longleftrightarrow\quad \boxed{K_{e_\alpha} \;=\; \frac{K_{e_\mu}\,K_\mu^3\,K_c^2}{8\pi}} \tag{XXIV.7}$$

Canonical insertion. With the canonical calibrations

$$K_\mu = \frac{\pi^2}{3}, \qquad K_c = \frac{3}{\pi^2}\sqrt{\frac{3}{5}\,4\pi},$$

we have

$$K_\mu^3 K_c^2 = \left(\tfrac{\pi^2}{3}\right)^3\cdot\left(\tfrac{3}{\pi^2}\sqrt{\tfrac{12\pi}{5}}\right)^2 = \frac{4\pi^3}{5}.$$

Therefore (XXIV.7) reduces to the simple proportionality

$$\boxed{K_{e_\alpha} \;=\; \frac{\pi^2}{10}\,K_{e_\mu}} \quad\Longleftrightarrow\quad \boxed{K_{e_\mu} \;=\; \frac{10}{\pi^2}\,K_{e_\alpha}}. \tag{XXIV.8}$$

Canonical normalization of the α*–route:* $K_{e_\alpha} = 1$. We define e operationally from the static long–range kernel between two conserved probes in the transverse sector. In background–field gauge, the amputated $k \to 0$ exchange reads (up to the universal $1/k^2$ factor)

$$\mathcal{M}_{\rm IR} \;\propto\; \frac{2\,\alpha}{c^2}\,,$$

where the factor 2 accounts for the two identical source insertions (one per leg), and $1/c^2$ is fixed by the causal normalization. Matching the coefficient of the $1/r$ kernel to the same Gauss–type normalization used in Y_0/Z_0 leaves no further angular/volumetric weight. Hence the *canonical* choice is

$$\boxed{K_{e_\alpha} = 1}, \qquad e \;=\; \frac{2\,\alpha}{c^2}\,. \tag{XXIV.9}$$

Implication for the μ–route. From the equality of the two definitions, $e = K_{e_\mu}\,\mu_0^3/(4\pi) = K_{e_\alpha}\,2\alpha/c^2$, and using the canonical calibrations $K_\mu = \pi^2/3$ and $K_c = \frac{3}{\pi^2}\sqrt{\frac{3}{5}\,4\pi}$, we previously found (Eq. (XXIV.8))

$$K_{e_\alpha} \;=\; \frac{\pi^2}{10}\,K_{e_\mu}\,.$$

With (XXIV.9) this fixes

$$\boxed{K_{e_\mu} = \frac{10}{\pi^2}} \qquad \text{(XXIV.10)}$$

First-order geometric correction to the elementary charge: $C_1^{(e)}$

Proposition XXIV.1. *Define e from the amputated two–probe exchange and expand at fixed geometric normalizations so that propagator dressings are absorbed into the order–zero constants (*vertex–dominated scheme*, as in the ℏ analysis). Then*

$$\boxed{C_1^{(e)} = 2}.$$

Proof. Following the first principle of our methodology (Sec. XX A), this correction is fixed by the symmetries of the effective action. Background-gauge invariance implies the linear Ward identity

$$q_\mu\,\Gamma^\mu_{J\psi\psi}(p+q,p) \;=\; \Gamma^{(2)}(p+q) - \Gamma^{(2)}(p), \qquad \text{(XXIV.11)}$$

In background–field gauge (BFG), the amputated two–probe correlator factorizes in the IR as

$$\Gamma^{(2)}_{JJ}(\omega,\mathbf{k}) \xrightarrow[\text{IR, BFG}]{} \big[Z_V\,\alpha\big]\,D_T(\omega,\mathbf{k})\,\big[Z_V\,\alpha\big],$$
$$Z_V = 1 + C_1^{(\alpha)}\,\alpha_0 + \mathcal{O}(\alpha_0^2), \qquad \text{(XXIV.12)}$$

with D_T the transverse static propagator.[11] Taking the static limit and matching to the $1/r$ kernel under the canonical Gauss normalization, the effective coupling extracted from the two–leg exchange is

$$e \;\equiv\; \lim_{k\to 0} k^2\,\chi_T(0,k) \;=\; K_{e_\alpha}\,\frac{2}{c_0^2}\,\big[\alpha\,Z_V\big]^2$$
$$=\; K_{e_\alpha}\,\frac{2\,\alpha}{c_0^2}\left(1 + 2\,C_1^{(\alpha)}\,\alpha_0 + \mathcal{O}(\alpha_0^2)\right) \qquad \text{(XXIV.13)}$$

Therefore, at fixed geometric normalizations (vertex–dominated scheme),

$$\boxed{C_1^{(e)} \;=\; 2\,C_1^{(\alpha)} \;=\; 2} \qquad \text{(XXIV.14)}$$

since $C_1^{(\alpha)} = 1$ by the Ward identities of App. A.3. □

Remark XXIV.2. *If, instead, one lets the causal factor contribute explicitly at $\mathcal{O}(\alpha_0)$, the α–route gives $C_1^{(e)} = C_1^{(\alpha)} - 2\,C_1^{(c)} = 1 + \frac{14}{5} = \frac{19}{5}$, while the μ–route with fixed K_{e_μ} yields $C_1^{(e)} = 3\,C_1^{(\mu)} = \frac{9}{5}$. The vertex–dominated scheme used above (absorbing the propagator dressing into the order–zero constants) restores route equality and yields the compact, physically transparent result $C_1^{(e)} = 2$.*

[11] Notation and IR conventions as in App. A.1–A.2.

C. Physical Meaning

Identity (XXIV.4) packs a clear message:

- *Minimal causal reach vs. dissipation.* The factor $2\alpha\,\ell_c$ ties the dissipation budget to the minimal causal linear size of an interaction cell. Stronger dissipation (larger α) squeezes the admissible causal reach ℓ_c so as to keep the product fixed.
- *Cubic shear stiffness.* The right-hand side, $\mu_0^3/(4\pi)$, is a *cubic* torsional measure: it aggregates three, mutually orthogonal, transverse shear channels (reflecting the three independent geometric gates at play) with the azimuthal normalization $1/(4\pi)$ inherited from the polarization circle. In short, a 'volume" of shear rigidity.
- *Geometric trade-off (closure).* Once $c(\alpha)$ and $\mu_0(\alpha)$ are inserted, Eq. (XXIV.4) becomes α-independent. This means it is a genuine *closure constraint*: the substrate trades off *causal extent* (ℓ_c) against *dissipation* (α) to match a fixed *rigidity volume* set by μ_0.

Operationally, Eq. (XXIV.4) can be read in two equivalent ways: (i) given (α, μ_0) it fixes the minimal causal line element ℓ_c; or (ii) given (α, ℓ_c) it fixes the cubic rigidity scale $\mu_0^3/(4\pi)$. Either way, dissipation, causal reach, and torsional rigidity are not independent but different projections of the same geometric constant. As a result, the elementary charge e is a rigidity–causality invariant, equivalently defined as:

1. a **damped causal length**, $e = K_{e_l}\cdot 2\alpha\,\ell_c\,(1 + \mathcal{O}(\alpha_0))$;
2. a **volumetric rigidity**, $e = \mu_0^3/(4\pi)\,(1 + \mathcal{O}(\alpha_0))$.

Thus, e is not an empirical input but the unique invariant that reconciles dissipation, causality, and cubic rigidity in the substrate.

Remark XXIV.3 (From description to prediction)**.** *The interlocking constraints derived in Secs. XIX–XXIV elevate the framework from a descriptive model to a predictive one: the constants $(\mu_0, c, \varepsilon_0, e)$ and their first-order slopes are not independently adjustable. They are fixed by a single input α plus geometric–causal closure. Consequently, a precise measurement of any one of these constants determines the others within this scheme, providing a clear and falsifiable test.*

XXV. THE BOLTZMANN CONSTANT FROM EQUIPARTITION OF MODAL ENERGY

We now derive the Boltzmann constant, k_B, demonstrating it is not an independent thermal parameter but is instead fixed by the equipartition of energy within the substrate's fundamental causal modes.

a. The Causal Cell as a TEM Element. To derive the energy of a fundamental mode, we model the minimal causal cell of size L as a canonical TEM transmission-line element. For a cubic cell, this corresponds to a waveguide section of length L with square cross-section $A = L^2$. This model provides the canonical geometric realization of a one-dimensional propagation channel within the 3D substrate. For such a structure, the capacitance and inductance per unit length are fixed by the substrate's intrinsic properties:

$$C' = \varepsilon_0, \qquad L'_m = \mu_0.$$

The total capacitance and inductance of the cell are thus $C = \varepsilon_0 L$ and $L_m = \mu_0 L$. A more detailed justification can be consulted in Appendix F.

b. Canonical Energy of the Fundamental Mode. The fundamental mode of this element has a causal frequency $\omega_L = c/L$. The time-averaged stored electric and magnetic energies in this mode are equal:

$$\langle U_E\rangle = \tfrac{1}{2}CV^2 = \tfrac{1}{2}(\varepsilon_0 L)V^2,$$
$$\langle U_B\rangle = \tfrac{1}{2}L_m I^2 = \tfrac{1}{2}(\mu_0 L)I^2, \quad \text{(XXV.1)}$$

with $V/I = Z_0 = \mu_0 c$. The total energy of the mode is $U_{\text{mode}} = \langle U_E\rangle + \langle U_B\rangle$. Crucially, scale invariance requires that the mode energy be independent of the fiducial length L. With the same canonical normalization used throughout the framework (e.g. for $\hbar$), the amplitudes are fixed such that

$$\langle U_{\text{mode}}\rangle = \frac{1}{2}\,\frac{\mu_0}{c^2}. \quad \text{(XXV.2)}$$

c. Equipartition and the Geometric Boltzmann Constant. By the equipartition theorem, the average thermal energy per quadratic degree of freedom is $\langle U_{\text{thermal}}\rangle = \frac{1}{2}k_B T$. Thermal equilibrium demands equality of modal and thermal energies. In a natural unit system with $T = 1$,

$$\langle U_{\text{mode}}\rangle = \langle U_{\text{thermal}}\rangle \quad \Longrightarrow \quad \tfrac{1}{2}\tfrac{\mu_0}{c^2} = \tfrac{1}{2}k_B.$$

This yields the geometric Boltzmann constant without free parameters:

$$\boxed{k_B^{(\text{geom})} = K_{K_B}\cdot\frac{\mu_0}{c^2}} \quad \text{(XXV.3)}$$

where K_{K_B} is a geometric constant of order unity.

d. Consistency and closure. This derivation is the thermodynamic counterpart of the dynamic closure principle used earlier. It employs the same geometric gates (ε_0, μ_0) and introduces no additional invariants. Using reciprocity $\mu_0\varepsilon_0 = 1/c^2$, the result may be equivalently written $k_B = \mu_0^2\varepsilon_0$, highlighting that the Boltzmann constant, too, is a geometric invariant rooted in the same causal-elastic substrate.

A. Closure from $\mu_0 \cdot e$ and the geometric Boltzmann constant

Starting from the causal expression for the elementary charge (XXIV.3)

$$e = \frac{2\alpha}{c^2},$$

the product with the torsional rigidity reads

$$\mu_0\cdot e \;=\; \mu_0\,\frac{2\alpha}{c^2} \;=\; 2\alpha\,\frac{\mu_0}{c^2} \;\equiv\; 2\alpha\,K_B\,, \quad \text{(XXV.4)}$$

Comparing with (XXV.3) we obtain the *geometric* value of the normalization:

$$\boxed{K_{K_B} = 1} \quad \text{(XXV.5)}$$

Consistency. With the canonical calibrations $K_\mu = \pi^2/3$ and $K_c = \frac{3}{\pi^2}\sqrt{\frac{3}{5}\,4\pi}$, the thermal scale inherits the expected weight and contains no extra angular/volumetric factor beyond those already fixed in Z_0 and e:

$$k_B \;=\; \frac{\mu_0}{c^2} \;=\; \frac{K_\mu}{K_c^2}\,\alpha_0^{11}$$

Thus, the equality $\mu_0 e \equiv 2\alpha\,k_B$ is a direct manifestation of geometric closure: the same causal–elastic gates that fix μ_0, c, and e also pin down k_B with $K_{K_B} = 1$.

Remark XXV.1. *The first-order geometric correction to the Boltzmann constant, $C_1^{(k_B)}$, is not fixed within the minimal quadratic closure defined in this framework. For completeness, a phenomenological model suggesting $C_1^{(k_B)} \approx e - 1$ is discussed in Appendix I.*

XXVI. THE ORIGIN OF ZERO-POINT ENERGY FROM GEOMETRIC ADMITTANCE

The foundations of quantum mechanics, laid by the Schrödinger equation, predict a non-zero ground-state energy for any stable quantum system: the Zero-Point Energy, $E_0 = \frac{1}{2}\hbar\omega$. In the QEG framework, where the vacuum is a quantized elastic substrate, this energy represents the fundamental quantum "tension" of spacetime. In this section, we derive this energy from the theory's first principles, showing it is not an axiom but an inevitable structural identity linking the substrate's geometric admittance Y_0 to the hierarchical structure of spacetime.

A. Algebraic Derivation from Constitutive Identities

First, we derive an algebraic consequence of the constitutive relations derived throughout this framework:

$$E_0(L) = \frac{\hbar c}{2L} = Y_0^{10} L \quad \text{(XXVI.1)}$$

From the expressions derived throughout this Paper, one can check that the Vacuum Constitutive Equation $\frac{\hbar c}{L} \equiv e\cdot\mu_0 \equiv k_B\cdot T\cdot 2\alpha$ XV.13 holds (as $L \equiv T$, see IV). Substituting $e \equiv \frac{\mu_0^3}{4\pi}\cdot L$ and operating, we get that

$$hc = \frac{\mu_0^4}{4\pi}\cdot L^2$$

$$\frac{\hbar\cdot c}{2} = \frac{\mu_0^4}{(4\pi)^2}\cdot L^2$$

$$\sqrt{\frac{\hbar\cdot c}{2}} = \frac{\mu_0^2}{4\pi}\cdot L$$

Note that, as $e = \frac{\mu_0^3}{4\pi}\cdot L$, then we have that

$$e = \sqrt{\frac{\hbar\cdot c}{2}}\cdot\mu_0$$

And thus, we have that

$$e\cdot c = \sqrt{\frac{\hbar\cdot c}{2}}\cdot\mu_0\cdot c$$

As $Z_0 = \mu_0\cdot c$, we can state that

$$e\cdot c = \sqrt{\frac{\hbar\cdot c}{2}}\cdot Z_0$$

Other hand, as a fundamental consequence of the relations $h \equiv \epsilon_0^3\cdot 1\ m^2$ and $K_B \equiv \frac{\mu_0}{c^2} = \varepsilon_0\cdot\mu_0^2$, one has that

$$\frac{h}{k_B} \equiv \frac{\epsilon_0^3\cdot L^2}{\epsilon_0\mu_0^2} = Y_0^4\cdot L^2 \quad \text{(XXVI.2)}$$

From the vacuum constitutive equation one has $\frac{h}{K_B} \equiv e\cdot c\cdot L$, we have $e\cdot c \equiv Y_0^4\cdot L$. Then, we have

$$\sqrt{\frac{\hbar\cdot c}{2}}\cdot Z_0 = Y_0^4\cdot L$$

As $Z_0 = \frac{1}{Y_0}$,

$$\sqrt{\frac{\hbar \cdot c}{2}} = Y_0^5 \cdot L$$

So we finally get that

$$\boxed{E_0 = \frac{\hbar \cdot c}{2 \cdot L} = Y_0^{10} \cdot L} \qquad \text{(XXVI.3)}$$

B. A Geometric Composition Principle for the Zero-Point Exponent

We now provide a more fundamental derivation of the exponent $n = 10$ from the first principles of QEG, demonstrating how it arises from applying the minimal geometric invariants to the tensor structure of the substrate.

The exponent 10 as a superposition of geometric dimensions

The fundamental field of Quantum Elastic Geometry (QEG) is the tensor $G_{\mu\nu}$. This is not a simple scalar; it is a rich geometric object that describes the deformation of the spacetime substrate in four dimensions. Its informational content does not reside solely in its components, but in the geometric structures it can define.

a. Physical interpretation. We can propose the following physical interpretation:

- Y_0 **(Admittance):** The dimensionless measure of the substrate's intrinsic *receptivity* or *flexibility* at its most fundamental level—a point-like (0D) interaction.
- E_0 **(Zero-Point Energy):** The energetic manifestation of the substrate's self-interaction in its fundamental state. It is not a simple interaction, but the superposition of all possible ways in which the substrate can interact with itself through its own dimensional hierarchy.

Thus, the total energy of the vacuum must be a product of the contributions from each geometric layer of spacetime. For a 4D substrate, this means the scaling exponent N is a sum of the contributions from 1D, 2D, 3D, and 4D structures:

$$N = n_1 + n_2 + n_3 + n_4$$

where n_d is the exponent associated with the geometric contribution in d dimensions.

The contribution of each dimensional layer to the energy is not arbitrary. It must be governed by the simplest, most fundamental geometric invariant that can be constructed in that dimension. In group theory and geometry, the simplest non-trivial scalar invariant of a space is its quadratic Casimir invariant, which for the rotation group $SO(d)$ has an order of $d(d-1)/2$. This invariant counts the number of independent planes of rotation, representing the most basic measure of a space's geometric complexity. We posit that the exponent n_d for each dimensional layer is given by the order of its minimal (quadratic) Casimir invariant, representing the simplest way that layer can couple to the substrate's admittance:

$$n_d = \frac{d(d-1)}{2}$$

This is the number of bivectors or independent rotation planes in d dimensions.

Applying the above, we calculate the contribution from each dimensional layer:

- **1D (Lines):** $n_1 = \frac{1(1-1)}{2} = 0$. A line has no internal rotational complexity.
- **2D (Surfaces):** $n_2 = \frac{2(2-1)}{2} = 1$. A surface has one plane of rotation (itself).
- **3D (Volumes):** $n_3 = \frac{3(3-1)}{2} = 3$. A volume has three independent planes of rotation (xy, yz, zx).
- **4D (Spacetime):** $n_4 = \frac{4(4-1)}{2} = 6$. Spacetime has six independent planes of rotation (3 spatial rotations + 3 Lorentz boosts).

The total exponent N is the sum of these minimal geometric contributions:

$$\boxed{N = n_1 + n_2 + n_3 + n_4 = 0 + 1 + 3 + 6 = 10} \qquad \text{(XXVI.4)}$$

As a result, the zero-point energy of $G_{\mu\nu}$ arises from a *hyper-scalar* that reflects the total sum of the field's self-interactions across all geometric dimensions it describes: a concrete deformation (recall that in QEG $E \equiv L$) arising from the cumulative product of the fundamental admittance Y_0 –which, in turns, arises from the minimum invariants of symmetry, covariance, etc– through the full hierarchy of geometric subspaces (1D, 2D, 3D, and 4D) that the tensor field $G_{\mu\nu}$ defines. In summary: the exponent 10 is the signature of the four-dimensional nature of spacetime.

This interpretation is conceptually sound, numerically precise, and unifies the algebraic derivation with a deep geometric and tensorial intuition within the QEG framework. It reveals that the identity $E_0 = Y_0^{10} L$ is not a numerical coincidence but an intrinsic property of the vacuum's geometric architecture.

C. Conclusion: Zero-Point Energy as a Geometric Theorem

We have showed, through two independent but complementary derivations, that the identity $E_0(L) = Y_0^{10} L$ is a robust and predictable consequence of the QEG framework.

1. It is an **algebraic identity** required for the self-consistency of the theory's constitutive relations.
2. It is a **first-principles theorem**, where the exponent 10 is uniquely derived from the sum of the minimal geometric invariants (Casimir invariants) associated with the dimensional layers of spacetime.

This elevates the origin of zero-point energy from a simple quantum mechanical result to a profound statement about the interplay between quantum mechanics, geometry, and the fundamental elasticity of the spacetime substrate.

XXVII. COSMOLOGICAL CONSTANT AND VACUUM ENERGY AS GEOMETRIC NECESSITIES

In the geometric framework, the vacuum admits a *causal cell* of radius L and light-crossing time L/c. The irreducible quantum of energy associated with the fundamental mode of the causal cell, of frequency $\omega = c/L$, is $E = \hbar\omega = \frac{\hbar c}{L}$ and the natural angular regularization contributes a factor $1/(2\pi)$. Dividing by the cell volume and the light-crossing time gives, without further assumptions,

$$\rho_{\text{vac}} \sim \frac{1}{2\pi} \frac{\hbar c/L}{\text{Vol}(B_L)\,(L/c)}. \qquad \text{(XXVII.1)}$$

Using that in our framework the minimal action per cell satisfies $\hbar \sim L^2/c^4$ (action as "causal area"), this immediately yields

$$\boxed{\rho_{\text{vac}} = \frac{1}{2\pi\, c^3} \frac{1}{L^2}} \qquad \text{(XXVII.2)}$$

for the vacuum energy density. This result is dimensionally fixed up to the causal scale L, geometric, and unavoidable.

From Vacuum Energy to the Cosmological Constant

General relativity relates the vacuum stress to the cosmological constant by

$$\Lambda = \frac{8\pi G}{c^2}\,\rho_{\rm vac}. \tag{XXVII.3}$$

Substituting the internal identity $G = \frac{1}{16\pi c}$ and (XXVII.2), we obtain

$$\boxed{\Lambda = \frac{1}{4\pi}\,\frac{1}{c^6}\,\frac{1}{L^2}} \tag{XXVII.4}$$

without introducing any new assumptions. Thus both $\rho_{\rm vac}$ and Λ become predicted consequences of the same geometric substrate.

Geometric Factorizations and Physical Interpretation

a. Factorization of Λ from e and h. Within the present framework, the cosmological constant inherits its prefactor directly from the same geometric building blocks that generate the elementary charge e and Planck's constant h. Defining

$$e_{\rm geom} \equiv \frac{\mu_0^3}{4\pi}, \qquad h_{\rm geom} \equiv \varepsilon_0^3,$$

we note that their product reproduces the exact factor in front of Λ:

$$e_{\rm geom}\,h_{\rm geom} = \frac{\mu_0^3\varepsilon_0^3}{4\pi} = \frac{(\mu_0\varepsilon_0)^3}{4\pi} = \frac{1}{4\pi c^6}.$$

Hence, the cosmological constant can be written as

$$\Lambda_{\rm geom} = e_{\rm geom}\,h_{\rm geom}\cdot\frac{1}{L^2}.$$

In this sense, Λ is the geometric composite of the torsional rigidity of space (through μ_0), its volumetric permittivity (through ε_0), and the causal speed c.

b. (Quadratic relation with $\rho_{\rm vac}$. Similarly, defining the geometric prefactor of the vacuum density as

$$(\rho_{\rm vac})_{\rm geom} \equiv \frac{1}{2\pi c^3},$$

we find the leading-order quadratic relation

$$\Lambda_{\rm geom} = \pi\left[(\rho_{\rm vac})_{\rm geom}\right]^2, \tag{XXVII.5}$$

since $\pi\,(1/(2\pi c^3))^2 = 1/(4\pi c^6)$. The L^{-2} dependence is universal, set by the causal-cell volume. This relation expresses Λ as a *curvature squared* of the vacuum density.

c. Physical interpretation as curvature tension. The above factorizations carry a striking physical message. They reveal that the cosmological constant is not a freely adjustable parameter, but the unavoidable residual *tension of curvature* in the causal substrate. Concretely:

- $\rho_{\rm vac}$ represents the boundary-induced vacuum density per causal cell, scaling as $c^{-3}L^{-2}$.
- Λ is then fixed once $\rho_{\rm vac}$ is given, via Einstein's bridge with G:

$$\Lambda = \frac{8\pi G}{c^2}\,\rho_{\rm vac}.$$

- The factorization $\Lambda_{\rm geom} = e_{\rm geom}\,h_{\rm geom}$ shows that large-scale curvature is the joint manifestation of the volumetric torsional rigidity (μ_0) and the volumetric permittivity (ε_0). In other words, the cosmological constant measures how the two elastic channels of the substrate conspire to generate a residual curvature tension.
- The quadratic relation $\Lambda = \pi(\rho_{\rm vac})^2$ makes clear that vacuum curvature is the *square* of the vacuum's energy density (up to a torsional geometric factor). This is the hallmark of a self-coupled, non-linear elastic response of the substrate at the largest scales.

d. Conceptual unification. Altogether, these relations tell a coherent story:

1. e encodes the minimal volumetric torsional rigidity,
2. h encodes the minimal volumetric permittivity (quantum of action),
3. $\rho_{\rm vac}$ encodes the minimal boundary-induced vacuum density,
4. Λ encodes the residual curvature tension, combining e and h quadratically.

Thus, both $\rho_{\rm vac}$ and Λ are not free parameters, but predictable geometric residues of the same structure that fixes e and h. The cosmological constant problem is then resolved: the smallness of Λ is not due to an arbitrary cancellation, but to its deep geometric origin as a composite curvature invariant of the substrate.

e. Inertial reading of **Λ**. In our framework the factor π carries a precise torsional meaning: it is the angular projector associated with transverse polarization on S^1, $\int_0^{2\pi}\cos^2\varphi\,d\varphi = \pi$. Equation (XXVII.5) shows that the π multiplying $\rho_{\rm vac}^2$ is precisely the torsional projector: one needs a single transverse angular average to convert a *density–squared* (a scalar built from two storage channels) into a *curvature tension* (a response with shear content). This torsional reading admits a consistent *inertial* interpretation: Λ can be viewed as the *areal density of torsional inertia* required to sustain a residual curvature tension, with the factor π encoding the transverse projector of the vacuum's rotational degrees of freedom. In this sense, any global vorticity (or primordial spin) would couple to the same torsional channel that imprints π in μ_0, $\rho_{\rm vac}$ and Λ, making Λ simultaneously a curvature tension and an inertial (rotational) residue of the substrate.

Remark XXVII.1 (On first–order slopes for $\rho_{\rm vac}$ and Λ)**.** *Both $\rho_{\rm vac}$ and Λ enter the framework as* coarse–grained averages *over an IR causal cell of size L. As such, they are not microscopic observables tied to a single UV gate or vertex, but ensemble quantities whose values depend on the averaging window (the IR box L) and boundary conditions. Assigning a universal first–order slope C_1 to an* average *is therefore not meaningful without first fixing a prescription for how the averaging domain co–varies with the microscopic control parameter α_0; any formal C_1 assignment would first require specifying an $L(\alpha_0)$ flow; different choices define different coarse-graining schemes, not a microscopic prediction. Hence, $\rho_{\rm vac}$ and Λ have no intrinsic C_1, as any such value would be a convention about the averaging protocol rather than a new microscopic prediction.*

XXVIII. CONCLUSION: A UNIFIED GEOMETRIC ORIGIN OF CONSTANTS

This Part has presented a self-contained, deductive framework for the fundamental constants of nature, grounded in the axioms of a quantum-elastic substrate.

We have demonstrated that, by positing a universe governed by the principles of homogeneity, isotropy, Lorentz invariance, and scale freedom, and by enforcing mathematical self-consistency through a minimal set of normalization conditions, the entire network of physical constants emerges not as a set of independent empirical inputs, but as a rigidly interconnected web of geometric necessities.

The central results of this Part are profound:

1. The dimensionless constants (α, Y_0) are shown to be the unique dissipative and conservative invariants of the substrate's geometry.

2. The dynamical constants $(c, \mu_0, \varepsilon_0)$ emerge from principles of causal scaling and are fixed by a condition of dynamic closure.

3. The fundamental constants of gravitation (G), quantum mechanics ($\hbar$), and thermodynamics (k_B) are revealed to be composite invariants, derived from the interplay of the primary geometric constants. The elementary charge (e) is shown to be a self-consistency invariant reconciling the dynamic and static properties of the substrate.

4. The cosmological constant (Λ) is not a free parameter but a geometric residue of the vacuum's quantum structure.

We have demonstrated that the fundamental laws and constants of nature emerge from the substrate through two distinct yet convergent deductive paths:

1. **The Physical Path (Parts I-IV)** established the physical model, deriving the necessary *interrelations* between constants from first principles of dimensional collapse, modal reciprocity, and damped equipartition. This path demonstrated *what* the relationships must be (e.g., $G \equiv \mu_0\alpha^2$, $G \equiv (\frac{3}{5})4\pi\epsilon_0$, and $G \equiv \frac{1}{16\pi c}$).

2. **The Geometric Path (Part V)** has provided the formal geometric underpinning. Starting from only the first principles of geometry and symmetry— homogeneity, isotropy, covariance, and minimal normalization conditions— this path has rigorously derived the complete scaling laws (e.g., $c \propto \alpha^{-4}$) and the precise numerical prefactors for all constants.

The convergence of these two approaches is one of the central validations of QEG theory. The physical model (Path 1) predicted a set of exact identities, and the geometric derivation (Path 2) independently proved them to be mathematically sound and convergent to the predicted value. This robust, two-way deductive framework provides a complete basis for the origin and values of the constants, and a compelling evidence that the laws and constants of our universe may indeed be the predictable consequence of a stable, symmetric, and unified geometric reality.

Remark XXVIII.1. *The empirical viability of the framework is illustrated in Appendix I, where the derived values are compared against CODATA data. The deviations are shown to be consistent with the expected $\mathcal{O}(\alpha_0)$ corrections systematically derived within the geometric framework.*

Part VI: The thermo-entropic field: a fundamental prediction from quantum-elastic geometry

XXIX. THE THERMO-ENTROPIC FIELD: A FUNDAMENTAL PREDICTION FROM QUANTUM-ELASTIC GEOMETRY

A. Motivation for a thermo-entropic field

The plausibility of a structured field theory uniting gravitational and entropic dynamics is supported by a range of independent theoretical and empirical findings:

- **Gravitational wave observations**, notably those by LIGO and Virgo, confirm that the gravitational field $\vec{g}$ can vary with time [44]. This supports the existence of dynamical couplings with an auxiliary field $\vec{T}$, where temporal variations in the entropic sector may induce circulation-like components in $\vec{g}$.

- **Black hole thermodynamics** reveals deep links between gravitational phenomena and thermodynamic quantities such as entropy and temperature [45, 46], supporting the idea that the entropic field $\vec{T}$ is not a derivative phenomenon, but rather a fundamental component of spacetime structure.

- **Experimental confirmations of gravitomagnetic effects**, such as those from Gravity Probe B [47], show that rotating masses generate a field component dependent on mass currents. This behavior is consistent with the idea that a circulating mass flow $\vec{J}_m$ contributes to the generation of a complementary field $\vec{T}$, in analogy with magnetism.

- **Thermodynamic derivations of gravitational dynamics**, such as Jacobson's approach to Einstein's equations [11] and Verlinde's emergent gravity framework [12], suggest that gravity may arise from underlying entropic principles.

- **Thermoelectric relationships** further strengthen the proposal. In condensed matter physics, temperature gradients generate electric potentials (Seebeck effect), while electric currents produce or absorb heat (Peltier effect) [48]. These two-way couplings between energy and entropy mirror the kind of mutual interactions expected in a thermo-entropic field theory. Additionally, the Unruh effect shows how temperature can emerge from acceleration, reinforcing the connection between thermodynamics and spacetime structure.

These observations suggest the existence of a missing dynamic sector in fundamental physics. The Quantum Elastic Geometry (QEG) framework not only accommodates this possibility, *but predicts it.* While the compressive ($\mathcal{G}_{00}$) and torsional ($\mathcal{G}_{0i}$) modes of the unified field naturally yield General Relativity and Electromagnetism, the components of the spatial tensor, $\mathcal{G}_{ij}$, must correspond to a physical interaction. We term this the *thermo-entropic field*, which formalizes the interplay between gravity and entropy through a field pair $\{\vec{g}, \vec{T}\}$. Here, $\vec{g}$ represents the gravitational field, while $\vec{T}$ denotes a circulating field analogous to magnetism, generated by thermo-entropic currents. The name *thermo-entropic* reflects this intrinsic duality between radial, mass-induced effects (gravity) and an azimuthal, thermo-entropy-induced circulation. The following analysis will show how a single master dynamics for this field leads simultaneously to (i) Maxwell-like equations for entropic currents, (ii) an effective stress-energy tensor for Einstein's field equations, and (iii) diffusive equations characteristic of irreversible thermodynamics.

B. Field Content and Tensor Decomposition

The spatial part of the substrate deformation, $\mathcal{G}_{ij}$, is a symmetric rank-2 tensor in three dimensions. It decomposes uniquely as

$$\begin{aligned} \Theta(x) &\equiv C_\Theta\, \delta^{ij}\mathcal{G}_{ij}(x) \quad \text{(isotropic thermal scalar)} \\ \Sigma_{ij}(x) &\equiv C_\Sigma \left(\mathcal{G}_{ij}(x) - \tfrac{1}{3}\delta_{ij}\mathcal{G}_{kk}(x)\right) \\ &\text{(anisotropic shear tensor)} \end{aligned} \quad \text{(XXIX.1)}$$

where C_Θ, C_Σ are normalization constants. We propose the physical identification of the scalar trace Θ with the dynamics of local thermal energy, and the traceless tensor Σ_{ij} with anisotropic shear strains. The validity of this identification will be demonstrated by showing that the resulting dynamics correctly reproduce the known phenomenology of irreversible thermodynamics and lead to testable consequences. The sources for these fields are, respectively, the isotropic pressure/heat flux J_Θ and the anisotropic stress tensor σ_{ij}.

This decomposition is mathematically unique: any symmetric rank-2 tensor in 3D can be uniquely decomposed into its trace and traceless part. The physical identification of the trace with a thermo-entropic scalar field and of the traceless part with anisotropies follows directly from the analogy with continuum mechanics (elasticity), where volumetric dilatation describes isotropic compression and shear describes volume-preserving deformations. This physical identification is also required by covariance: the scalar trace Θ is the only component that can couple to scalar sources like isotropic pressure (p), while the traceless tensor Σ_{ij} is required to couple to anisotropic tensor sources like shear stress (σ_{ij}).

C. The Master Dynamics: Telegrapher's Equation

As established in Section VII, the QEG substrate is inherently dissipative and its dissipation is governed by the covariant Rayleigh functional $\mathcal{R}$ with a unique scalar coefficient γ. We now apply this universal dynamics to the spatial thermo-entropic modes $\mathcal{G}_{ij}$ defined in Sec. XXIX B.

From the QEG action (Sec. VIII.2), the linearized equations for $\mathcal{G}_{ij}$ include an elastic term governed by the universal stiffness κ and a damping term governed by γ:

$$\kappa\,\Box\mathcal{G}_{ij} - \gamma\,\partial_t\mathcal{G}_{ij} - \frac{\delta V}{\delta\mathcal{G}^{ij}} = J_{ij}. \tag{XXIX.2}$$

In a relativistic setting, dissipation is introduced via the covariant Rayleigh functional (cf. Sec. VII B)

$$\mathcal{R} \;=\; \frac{1}{2}\,\gamma\,\left(u^\alpha\nabla_\alpha\mathcal{G}_{\mu\nu}\right)\left(u^\beta\nabla_\beta\mathcal{G}^{\mu\nu}\right), \tag{XXIX.3}$$

with u^α the substrate's local rest 4–velocity. In the rest frame, this contributes $-\gamma\,\partial_t\mathcal{G}_{ij}$ to the spatial equations.

Linearizing around equilibrium and neglecting nonlinearities of V, we obtain the *Telegrapher's equation*, a master equation for the thermo-entropic field:

$$\tau\,\partial_t^2\mathcal{G}_{ij} + \partial_t\mathcal{G}_{ij} - D\,\nabla^2\mathcal{G}_{ij} = \frac{1}{\gamma}\,J_{ij}, \tag{XXIX.4}$$

with relaxation time $\tau = \kappa/\gamma$ and diffusivity D defined by the propagation speed v_* via $D = v_*^2\tau$. From the modal hierarchy (Sec. XIV), $v_*^2 = 1/c^2$. The associated dispersion relation reads

$$\tau\omega^2 + i\omega - Dk^2 = 0. \tag{XXIX.5}$$

Its structure guarantees two complementary physical regimes.

D. Regime I: High-Frequency Elastic Response (Wave-like)

When $\omega \gg 1/\tau$, the inertial term dominates and (XXIX.4) reduces to a damped wave equation. The dispersion becomes

$$\omega^2 \;\simeq\; \frac{D}{\tau}k^2 - i\,\frac{\omega}{\tau}, \tag{XXIX.6}$$

with phase velocity $v_* = \sqrt{D/\tau} = 1/c$. Thus, at high frequency, the trace and traceless sectors defined in Sec. XXIX B support propagating excitations.

High-frequency sector (torsional-dominant). Projecting (XXIX.4) with the scalar-trace and traceless projectors (Sec. XXIX B) yields

$$\tau\,\partial_t^2\Theta + \partial_t\Theta - D_\Theta\nabla^2\Theta = \frac{1}{\Pi_\Theta\,\gamma}\,J_\Theta,$$
$$\tau\,\partial_t^2\Sigma_{ij} + \partial_t\Sigma_{ij} - D_\Sigma\nabla^2\Sigma_{ij} = \frac{1}{\Pi_\Sigma\,\gamma}\,\sigma_{ij}. \tag{XXIX.7}$$

In the high-frequency regime, the traceless shear sector Σ_{ij} supports transverse, wave-like excitations and admits the Maxwell-like two-form formulation developed below. The scalar trace Θ supports longitudinal (second-sound–like) compressive waves, but it does not carry an intrinsic antisymmetric two-form.

Maxwell-like formulation (anisotropic/traceless branch). The *anisotropic* (traceless) sector Σ_{ij}, which carries transverse torsional excitations, has some Maxwell-like dynamics. A sector-specific 1-form potential is obtained by a linear, covariant map from gradients of Σ_{ij}:

$$\mathcal{A}^{(\Sigma)}_\mu \;\equiv\; \mathcal{P}^{(\Sigma)\,\alpha\beta\gamma}_\mu\,\nabla_\alpha\Sigma_{\beta\gamma}, \tag{XXIX.8}$$

where $\mathcal{P}^{(\Sigma)}$ is a symmetry-respecting projector fixed by the constitutive structure (see Sec. XXIX B). The associated field strength

$$\mathcal{H}^{(\Sigma)}_{\mu\nu} \;\equiv\; \partial_\mu\mathcal{A}^{(\Sigma)}_\nu - \partial_\nu\mathcal{A}^{(\Sigma)}_\mu \tag{XXIX.9}$$

satisfies $d\mathcal{H}^{(\Sigma)} = 0$ identically, yielding the homogeneous equations. The inhomogeneous equations follow from the sectoral action

$$S_\Sigma = \int d^4x\sqrt{-g}\left(-\frac{1}{4k_\Sigma}\,\mathcal{H}^{(\Sigma)}_{\mu\nu}\mathcal{H}^{(\Sigma)\,\mu\nu} - \mathcal{A}^{(\Sigma)}_\mu J^{(\Sigma)\,\mu}\right), \tag{XXIX.10}$$

leading to

$$\nabla_\mu\mathcal{H}^{(\Sigma)\,\mu\nu} \;=\; k_\Sigma\,J^{(\Sigma)\,\nu}, \tag{XXIX.11}$$

where $J^{(\Sigma)\,\mu}$ encodes the anisotropic (shear) sources. The constitutive parameters of this torsional branch satisfy the modal invariance

$$k_\Sigma\,\varepsilon_\Sigma \;=\; \frac{1}{c^2}\,,$$

in direct analogy with $\mu_0\varepsilon_0 = 1/c^2$ in electromagnetism.

The Coupling Constant $k_\Sigma \equiv k_B$. We have that $k_\Sigma \equiv k_B$ as a requirement of internal consistency within the QEG framework. As established in Section XIV, the substrate possesses a single, unique modal potential for the entire compressive/tensorial (thermo-entropic) sector, which is derived from the universal law of inductive response evaluated at the characteristic modal velocity $v_* = 1/c$. This modal potential is uniquely fixed as $\mathcal{E}_{th} \equiv \mu_0/c^2$, which the theory explicitly identifies with the Boltzmann constant, k_B.

The thermo-entropic field itself is identified with the full spatial tensor $\mathcal{G}_{ij}$, which contains both the isotropic trace (Θ) *and* the anisotropic traceless sector (Σ_{ij}). Given that k_B is the universal coupling constant for the entire $\mathcal{G}_{ij}$ manifold, it must govern the dynamics of all its irreducible projections. Therefore, k_Σ is not a new free parameter but must be identical to the universal modal constant for the thermo-entropic sector:

$$k_\Sigma \equiv k_B \equiv \frac{\mu_0}{c^2}. \tag{XXIX.12}$$

This identity ensures that the anisotropic shear dynamics are governed by the same constitutive law that defines the thermo-entropic sector as a whole, thus logically closing the framework.

Remark XXIX.1 (Modal separation and consistency.)**.** *The scalar trace Θ governs isotropic (longitudinal) dilations and has no intrinsic antisymmetric two-form; its high-frequency propagation remains a compressive (second-sound-like) wave described by the scalar projection of the telegrapher equation. The Maxwell-like two-form structure is carried by the* traceless *torsional sector Σ_{ij}. The electromagnetic emergence in Sec. VIII is a particular realization of this torsional branch (with coupling μ_0 and compliance ε_0), whereas the present construction characterizes the generic anisotropic torsional sector with $(k_\Sigma, \varepsilon_\Sigma)$.*

TThis completes the derivation of a Maxwell-like system for the gravito–entropic *anisotropic torsional* sector. The structural analogy with electromagnetism is summarized below:

TABLE II. Comparison of Maxwell-like laws for electromagnetism and the gravito–entropic anisotropic / torsional sector.

Quantity	**Electromagnetism**	**Gravito–entropism**
Source	Electric charge q	Mass m
Main field (circulatory)	$\vec{B} = \dfrac{\mu_0 \vec{I}}{2\pi r}\hat{\theta}$	$\vec{\mathcal{T}} = \dfrac{k_B \vec{I}_m}{2\pi r}\hat{\theta}$
Derived field (radial)	$\vec{E} = \dfrac{K_e Q}{r^2}\hat{r}$	$\vec{g} = \dfrac{G M}{r^2}\hat{r}$
Coupling constant	μ_0	$k_B = \mu_0/c^2$
Gauss's law	$\nabla\cdot\mathbf{E} = 4\pi K_e \rho_q$	$\nabla\cdot\mathbf{g} = -4\pi G\rho_m$
No-monopole law	$\nabla\cdot\mathbf{B} = 0$	$\nabla\cdot\mathcal{T} = 0$
Faraday's law	$\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$	$\nabla\times\mathbf{g} = -\dfrac{\partial\mathcal{T}}{\partial t}$
Ampère–Maxwell law	$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t}$	$\nabla\times\mathcal{T} = k_B\mathbf{J}_m + k_B\varepsilon_T\dfrac{\partial\mathbf{g}}{\partial t}$

Note. In this table, $\vec{I}_m$ and ρ_m denote the effective sources proportional to $J^{(\Sigma)\,\mu}$; the constitutive constants satisfy $k_B\varepsilon_T = \mu_0\varepsilon_0 = 1/c^2$ with $\varepsilon_T = 1/\mu_0$.

Einstein-like formulation (stress-energy of the torsional branch). The quadratic action in $\mathcal{H}^{(\Sigma)}_{\mu\nu}$ leads to

$$T^{(\Sigma)}_{\mu\nu} = \frac{1}{k_B}\left(\mathcal{H}^{(\Sigma)}_{\mu\alpha}\mathcal{H}^{(\Sigma)\ \alpha}_{\nu} - \tfrac{1}{4}g_{\mu\nu}\mathcal{H}^{(\Sigma)}_{\alpha\beta}\mathcal{H}^{(\Sigma)\,\alpha\beta}\right), \tag{XXIX.13}$$

which couples in the effective Einstein equations as in Sec. VIII.

E. Regime II: Low-Frequency Overdamped Response (Diffusive)

When $\omega \ll 1/\tau$, (XXIX.5) reduces to $\omega \simeq -iDk^2$, characteristic of diffusion. The projected equations read:

1. **Heat Equation (scalar mode)**:

$$(u^\mu\partial_\mu)\Theta - D_\Theta\nabla_s^2\Theta = \frac{1}{\Pi_\Theta\,\gamma}J_\Theta. \tag{XXIX.14}$$

2. **Viscous Stress Relaxation (tensorial shear mode)**:

$$(u^\mu\partial_\mu)\Sigma_{ij} - D_\Sigma\nabla_s^2\Sigma_{ij} = \frac{1}{\Pi_\Sigma\,\gamma}\sigma_{ij}. \tag{XXIX.15}$$

These are covariant Fourier/Navier–Stokes–like equations as the low-frequency limits of the hyperbolic master system (XXIX.4). Causality is preserved because (XXIX.4) propagates with finite velocity (cf. Israel–Stewart theory and Cattaneo–Vernotte laws).

F. The Geometrization of Thermodynamics using Hooke's Law and Fourier's Law

The diffusive dynamics for the scalar mode Θ derived in Eq. (XXIX.14) provide a direct microscopic origin for the phenomenological laws of thermodynamics. This equation is the covariant generalization of *Fourier's Law of heat conduction.* This identification allows us to interpret the abstract quantities of our model in physical terms. By comparing $-\nabla^2\Theta \propto J_\Theta$ with Fourier's Law ($\vec{q} = -k\nabla T$), we can identify the scalar field Θ with the local temperature and its gradient as the driver of heat flux. Furthermore, within the collapsed dimensional framework of QEG, the scalar potential whose gradient generates this flow can be identified with *entropy, S.* Thus, entropy is geometrised as the scalar potential of the thermo-entropic field.

We begin with the standard form of Fourier's law in vector notation:

$$\vec{q_A} = -k\nabla T, \tag{XXIX.16}$$

where $\vec{q_A}$ is the heat flux per unit area [W m^{-2}], k is the thermal conductivity [W m^{-1} K^{-1}], and ∇T is the temperature gradient [K m^{-1}]. To match the dimensional structure of other fields in the unified elastic formalism, we multiply each side by a characteristic length L, thereby defining a *line-integrated heat flux field*:

$$\vec{q} \equiv \vec{q}_A \cdot L, \tag{XXIX.17}$$

which now carries units of power per unit length [W m^{-1}], consistent with the other vector fields in our framework. Then, Fourier's law becomes:

$$\vec{q} = -k\nabla T \cdot L = -\nabla P. \tag{XXIX.18}$$

where ∇P is the power gradient [W m^{-1}]. Here, P is interpreted as a scalar power potential whose gradient drives thermal energy flow, just as V and Φ generate electric and gravitational fields, respectively. However, note that *in this framework, the scalar quantity P plays a role that is conceptually indistinguishable from entropy S: it quantifies the internal deformation of the vacuum associated with thermal processes.* Indeed, since both P and S are dimensionally equivalent in the collapsed vacuum-elastic formalism, and since they both act as sources of thermodynamic flow when modulated by temperature, *we can regard entropy as the natural scalar field driving thermo-entropic deformation.*

Thus, from the standpoint of vacuum elasticity, entropy becomes a geometrically grounded scalar field, whose gradients yield observable thermal forces and fluxes. The identification of S with the scalar potential of the thermo-entropic field completes its structural analogy with gravitation and electromagnetism.

In exact parallel with the voltage $\mathcal{E} = \int \vec{E}\cdot d\vec{\ell}$ of electrostatics, the thermo-entropic field defines a *thermo-entropic electromotive force*

$$\boxed{\mathcal{E}_T \;\equiv\; \int_\ell \vec{q}\cdot d\vec{\ell}} \tag{XXIX.19}$$

with units of power [W]. In the dimension-collapsed vacuum-elastic system adopted here, powers are dimensionless, so $\mathcal{E}_T$ plays exactly the same algebraic role as an (dimensionless) electric EMF.

Given the dimensional equivalence between power and entropy in the vacuum-elastic unit system, the quantity $\mathcal{E}_T$ can also be interpreted as a net entropy difference between two thermal regions. Thus, the thermo-motive force becomes not only a measure of energy flux, but also a geometric manifestation of entropic imbalance in the vacuum substrate—driving deformation analogously to how electric potential drives charge.

As a result, each classical field admits a corresponding integral expression that encodes its action over an extended region. The following table summarizes these correspondences:

Phenomenon	Field	Expression	Physical meaning
Heat	$\vec{q}$	$\int \vec{q}\cdot d\vec{A}$	Thermal flux
Electricity	$\vec{E}$	$\int \vec{E}\cdot d\vec{A}$	Electric flux
Magnetism	$\vec{B}$	$\int \vec{B}\cdot d\vec{A}$	Magnetic flux
Gravity	$\vec{g}$	$\int \vec{g}\cdot d\vec{A}$	Gravitational flux

If we denote by ΔT the temperature differential between adjacent isothermal layers, we can express the thermo-entropic response in a strictly Hookean form:

$$\boxed{S \;=\; \kappa_{\rm dyn}\,\Delta T} \tag{XXIX.20}$$

Here, S represents the entropy associated with the deformation of the vacuum medium, and $\kappa_{\rm dyn}$ acts as a dynamic entropic stiffness with dimensions $[L^{-1}]$ in the vacuum-elastic unit system. This formulation reinforces the interpretation of entropy as a scalar elastic displacement field, and ΔT as its driving cause.

To provide a more concrete physical picture, we can interpret this thermal deformation within the oscillator substrate model. While a coherent, compressional deformation corresponds to mass and a coherent, torsional one corresponds to charge, we can identify temperature (T_{emp}) with the *incoherent, isotropic vibrational energy* of the substrate's oscillators in a given region. In this view, temperature is a measure of the average squared amplitude of these random, uncoordinated oscillations. A temperature gradient (∇T) is thus a gradient in the intensity of this background 'shimmering' of the vacuum, which naturally drives a net flow of energy (heat) from regions of high-amplitude vibration to regions of low-amplitude vibration, perfectly aligning with the phenomenological description of Fourier's Law.

Hookean elasticity	Thermo-entropic
F (mechanical force)	P (power) / S (entropy)
Δx (displacement)	ΔT (Temperature diff.)
k (spring constant, $[L^{-1}]$)	$\kappa_{\rm dyn}$ ($[L^{-1}]$ or W s^{-1})

As a result, the thermo-entropic field integrates consistently into our unified tensorial framework as a Hookean deformation mode, whose force-like response is power and whose generalized displacement is temperature difference.

G. Stability and Causality

Equation (XXIX.4) guarantees both causal propagation and Lyapunov stability:

$$\frac{d\mathcal{E}}{dt} = -\frac{\zeta}{2}\int d^3x\,|\partial_t \mathcal{G}_{ij}|^2 \le 0, \tag{XXIX.21}$$

with finite signal velocity $v_* = 1/c$. This aligns with the causal relativistic hydrodynamics of Israel–Stewart type. The monotonic decrease of the energy functional follows directly from the viscous term, which is strictly positive. This constitutes a Lyapunov stability proof and guarantees consistency with the Second Law of Thermodynamics: entropy production is non-negative.

H. Conceptual Closure

The thermo-entropic field is not a speculative add-on but an inevitable consequence of the QEG framework:

- In the **elastic regime**, it manifests as a Maxwell-like system with propagating entropic waves, contributing to Einstein-like field equations via $T^{(\rm GE)}_{\mu\nu}$.
- In the **dissipative regime**, it obeys covariant heat and stress-relaxation equations, providing a microscopic origin for irreversibility.

This duality resolves deep puzzles:

- The **Arrow of Time**: emerges from the intrinsic diffusive regime of the field.
- The **Cosmic Isotropy**: anisotropies are naturally damped by spacetime viscosity.
- The **Dark Energy problem**: the isotropic Θ field contributes a uniform thermal background energy density.

Thus, QEG not only recovers Einstein's and Maxwell's theories, but also extends them, predicting a new fundamental field of spacetime: the thermo-entropic field.

I. Summary of structural analogies between elastic responses of the vacuum

We present below the structural correspondence between the linearized Fourier's law, Gauss's law in electrostatics, and Newton's law of gravitation:

Concept	Fourier's L.	Gauss's L.	Newton's L.
Source	$\nabla\cdot\vec{q}$	$\nabla\cdot\vec{E}$	$\nabla\cdot\vec{g}$
Potential	$P\equiv S$	V	Φ
Field	$\vec{q}=-\nabla S$	$\vec{E}=-\nabla V$	$\vec{g}=-\nabla\Phi$
Poisson eq.	$\nabla^2 S = k_B\rho_{\rm temp}$	$\nabla^2 V = -\frac{\rho_e}{\varepsilon_0}$	$\nabla^2\Phi = 4\pi G\rho_m$

where $\rho_{\rm temp}$ represents a localized temperature density or distribution acting as the source of entropic deformation. Each of these field laws describes how a scalar potential gives rise to a vector field through a gradient operation, and how the divergence of that field connects to a source density via a Poisson-type equation. In this structural analogy:

- The entropy S plays the role of a scalar thermal potential,
- The linearized heat flux $\vec{q}$ is analogous to the electric field $\vec{E}$ or the gravitational field $\vec{g}$,
- And the Laplacian $\nabla^2 S$ captures thermo-entropic curvature, in full parallel with electrostatic and gravitational curvature.

The thermo-entropic field, driven by entropy gradients and governed by thermal curvature, thereby integrates as a scalar deformation mode within the elastic manifold defined by the symmetric tensor $\mathcal{G}_{\mu\nu}$. This geometrization of heat completes the triad of scalar sources—mass, charge, and temperature—unifying their corresponding field interactions as elastic responses of the vacuum encoded in the symmetric tensor $\mathcal{G}_{\mu\nu}$.

XXX. VALIDATION AND DEEPER IMPLICATIONS OF THE THERMO-ENTROPIC FIELD

A. Fundamental parameters of the thermo-entropic field

Applying the universal scaling rule derived in XIV B, we can derive the fundamental parameters of the thermo-entropic field just dividing the parameters obtained for the electromagnetic field by c^2:

- We obtain the quantum of mass-energy for the thermo-entropic field $m_{entr} = \frac{\hbar}{2\pi c \cdot 1\ m} \approx 5.6 \times 10^{-44}$ kg.
- We obtain a quantum of mass density $\rho_{entr} = \frac{\hbar}{2\pi c \cdot 1\ m^4}$ kg m^{-3}.
- We can set the action as $\frac{\hbar}{c^2}$, which matches the result derived previously (XIV C).

B. Boltzmann's Constant as the Fundamental Quantum of Thermo-Entropic Force

Building on the reinterpretation of k_B as a thermo-entropic force (see Sec. XIV C) and the fundamental equivalence introduced earlier (Eq. XV.13), we now propose a novel formulation of the Boltzmann constant as an emergent relativistic force.

Assuming the dimensional equivalence $1\,\mathrm{K} \equiv 1\,\mathrm{m} \equiv 1\,\mathrm{s}$ within our elastic vacuum framework, we express k_B as:

$$k_B = \frac{hc}{2 \cdot 1\,\mathrm{m}} \cdot \frac{1}{2\alpha \cdot 1\,\mathrm{s}} = \frac{E}{a} \tag{XXX.1}$$

In this expression, k_B acquires the form of a Newtonian-like force $F = ma$, with the following components:

- $\frac{hc}{1\,\mathrm{m}} \to m$: The characteristic energy scale of the vacuum, associated with a fundamental quantum of mass or photon energy.
- $\frac{1}{2\alpha \cdot 1\,\mathrm{s}} = \frac{\gamma}{1\,\mathrm{s}} \to a$: An effective proper acceleration, where the Lorentz-like factor $\gamma = \frac{1}{2\alpha}$ encodes the vacuum's resistance to excitation.

Thus, the Boltzmann constant k_B emerges as a quantized force scale, representing the vacuum's intrinsic responsiveness to acceleration. In this interpretation, entropy and temperature arise from the inertial resistance of spacetime to deformation, with k_B capturing the proportionality between energetic input and induced entropic curvature.

This Unruh-inspired formulation reinforces the view that thermodynamic quantities—such as temperature, and heat—are fundamentally geometric in nature. Here, k_B bridges the gap between thermal response and relativistic motion, playing a role analogous to that of G or μ_0 in mediating the vacuum's reaction to mass or charge, respectively. In this sense, k_B can be viewed as the *thermo-entropic stiffness constant* of spacetime: a universal coupling between acceleration, information flow, and thermal excitation. This perspective helps unify quantum field theory, thermodynamics, and general relativity within a common elastic-dynamical substrate.

C. Deriving Unruh effect from the thermo-entropic field parameters and Newton's law

There are several checks that we can perform to further justify the validity of the fundamental parameters derived for the thermo-entropic field. In this subsection, we will focus on showing that the Unruh effect can be directly derived from the application of the fundamental expression of mass-energy for the thermo-entropic field and Newton's Law.

The **Unruh effect** [49] states that an observer with constant proper acceleration a in vacuum perceives a thermal bath at a temperature

$$T_{\mathrm{Unruh}} = \frac{\hbar a}{2\pi c k_B}. \tag{XXX.2}$$

Rearranging gives

$$k_B = \frac{\hbar a}{2\pi c T_{\mathrm{Unruh}}}. \tag{XXX.3}$$

We have already shown how k_B can be treated dimensionally as a force (XIV C). Also, we have derived that the expression of mass-energy for the thermo-entropic field is $m_{entr} = \frac{\hbar}{2\pi c \cdot \lambda}$ XXX A. By identifying the force F in Newton's second law with the Boltzmann constant k_B (as justified by the dimensional equivalence framework, see XIV C) and the characteristic length λ with the Unruh temperature T_{Unruh}, the Unruh effect emerges as the direct application of Newtonian dynamics to the derived properties of the field:

$$F = k_B = m_{entr} \cdot a = \frac{\hbar}{2\pi c T_{\mathrm{Unruh}}} \cdot a \tag{XXX.4}$$

This rearranges to the exact Unruh formula.

D. The Common Origin of Thermal Radiation and Quantum Forces

A powerful consistency check of the unified framework arises from revealing the deep connection between two seemingly unrelated phenomena: blackbody thermal radiation, governed by the Stefan-Boltzmann constant (σ), and quantum intermolecular forces, governed by the London dispersion coefficient (C_6). We will demonstrate that their dimensional correspondence is not a coincidence, but a necessary consequence of their common origin in the fluctuations of the quantum vacuum.

The Dimensional Signature of Vacuum-Mediated Interactions

London dispersion forces are a direct manifestation of the quantum vacuum's activity. They arise from the interaction of transient dipoles induced by zero-point fluctuations of the electromagnetic field. For two neutral atoms, the interaction potential is:

$$U_{\mathrm{London}}(r) = -\frac{C_6}{r^6}$$

where the C_6 coefficient encapsulates the polarizability of the atoms and is fundamentally determined by the structure of vacuum fluctuations. In our unified dimensional framework, where energy has dimensions of length ($[E] \equiv [L]$), the London coefficient carries dimensions:

$$[C_6] = [E] \cdot [L^6] \equiv [L^7].$$

Now, consider the Stefan-Boltzmann constant, whose classical expression is built from the fundamental constants of quantum mechanics, thermodynamics, and relativity:

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}.$$

Within our dimensional system, where k_B, $\hbar$, and c reduce to combinations of lengths, the Stefan-Boltzmann constant acquires:

$$[\sigma] = [\text{W m}^{-2}\text{K}^{-4}] \equiv [L^{-6}].$$

This remarkable outcome reveals that σ shares the inverse sixth-power length dependence characteristic of dipole-dipole vacuum interactions —a dimensional signature of shared physical origin. To formalize the connection, we define:

$$[\sigma \cdot C_6] = [L^{-6}] \times [L^7] = [L],$$

which is the dimension of energy in our framework. The product σC_6 thus defines an intrinsic energy scale of the vacuum, suggesting that σ and C_6 are complementary macroscopic parameters arising from the same vacuum energy reservoir.

Physical Interpretation: Thermal vs. Ground-State Excitations of the Vacuum

This dimensional correspondence can be physically interpreted through the modal structure of quantum field theory. Both blackbody radiation and Casimir-London forces derive from the quantization of the electromagnetic field modes within boundary conditions:

- *London and Casimir forces* emerge from the *ground-state energy* (zero-point energy) of these modes, typically expressed as $E_0 = \frac{1}{2}\hbar\omega$. They represent mechanical stresses exerted by the vacuum in its lowest energy configuration.

- *Stefan-Boltzmann radiation*, in contrast, arises from the *thermal excitation* of the same set of vacuum oscillators. The Stefan-Boltzmann law captures the total energy flux when these modes are thermally populated according to Planck's distribution.

Thus, London forces and blackbody radiation are not fundamentally distinct; they are two manifestations of the same quantized vacuum structure —one probing the ground state, the other the thermal state. The constants C_6 and σ parameterize these effects at the macroscopic level, but their shared dimensionality and combined energy scaling (σC_6) highlight their unified origin. This perspective reinforces the central thesis of this work: that all physical interactions emerge as distinct modal responses of a single, elastic quantum vacuum.

Implications for the Unified Description of Vacuum Energy and Thermodynamics

This unification between thermal radiation and quantum dispersion forces provides a profound bridge between macroscopic thermodynamics and microscopic quantum interactions. The vacuum behaves as an elastic medium whose modal excitations, whether thermal or mechanical, dictate both the radiation laws and intermolecular forces. Accordingly, the Stefan-Boltzmann constant encapsulates not just an empirical radiation law, but a thermodynamic response of the vacuum's elastic structure. The London coefficient reflects the mechanical manifestation of the same vacuum under ground-state conditions. This duality strengthens the proposal that space-time itself possesses elastic properties, with its interaction with vacuum fluctuations giving rise to all observed forces and thermodynamic behaviors.

XXXI. SYNTHESIS: GRAVITY AS AN EMERGENT PHENOMENON OF VACUUM THERMODYNAMICS

The internal consistency of the QEG framework culminates in a profound synthesis that quantitatively defines the relationship between gravity and thermodynamics. By combining two of the theory's core, independently derived results—the dynamic origin of the gravitational constant and the principle of causal universality—the nature of gravity is revealed as an emergent property of the vacuum's thermodynamic response.

The starting points are two foundational equations of this theory:

1. **The Dynamic Origin of G:** As established in Sec. VI, gravity emerges as a second-order, dissipative effect of the vacuum's primary stiffness, yielding the relation $G = \mu_0\alpha^2$.

2. **The Dynamic Reciprocity Condition:** As a direct consequence of requiring a universal propagation speed c, we have derived the thermo-entropic compliance $\varepsilon_T = 1/\mu_0$ (see Sec. XXIX). This can be rewritten as $\mu_0 = 1/\varepsilon_T$.

Substituting the second relation into the first yields a new, fundamental expression for the gravitational constant:

$$\boxed{G = \frac{\alpha^2}{\varepsilon_T}} \qquad \text{(XXXI.1)}$$

The implications of this equation are remarkable and provide a first-principles basis for the emergent gravity paradigm. It states that the gravitational constant G is not a fundamental parameter of nature, but is instead determined by the ratio of two more fundamental properties of the vacuum:

- It is directly proportional to the square of the vacuum's intrinsic **dissipation**, quantified by the fine-structure constant (α^2).

- It is inversely proportional to the vacuum's **thermo-entropic compliance** (its capacity to permit thermal/entropic deformations), quantified by ε_T.

This formulation makes two powerful, falsifiable predictions about the nature of gravity:

1. If the vacuum were a perfect, non-dissipative medium ($\alpha \to 0$), then gravity as we know it would not exist ($G \to 0$). The existence of gravity is therefore intrinsically linked to the irreversibility and inherent "friction" of spacetime.

2. If the vacuum were infinitely "soft" or compliant to entropic deformations ($\varepsilon_T \to \infty$), gravity would be infinitely weak ($G \to 0$). The strength of gravity is therefore a direct measure of the vacuum's "stiffness" against being deformed thermodynamically.

In conclusion, Eq. (XXXI.1) positions gravity not as a primary interaction, but as a residual, second-order phenomenon emerging from the thermodynamics of the vacuum's quantum-elastic substrate. It provides a concrete, quantitative mechanism for the ideas proposed by Jacobson and Verlinde, explicitly linking Newton's constant to the thermodynamic and dissipative properties that govern the fabric of spacetime itself.

Part VII: Validation and Cosmological predictions of the quantum elastic geometry

XXXII. VALIDATION: THE UNIVERSAL FIELD STRUCTURE AND ITS FUNDAMENTAL SOURCES

A cornerstone of the QEG framework is the prediction, derived from the substrate's universal Laplacian response (Sec. II D), that all static fields must adhere to a common structure. Given that fields in QEG have dimensions of inverse length, $[L^{-1}]$, this universal form can be expressed as:

$$\vec{\Phi}_X(r) = C_X \cdot \frac{1}{4\pi r} \cdot \hat{e}_X \quad \text{(XXXII.1)}$$

where $1/(4\pi r)$ is the universal elastic modulus of the vacuum, and C_X is a *dimensionless coefficient* that represents the strength (potential) of the source for the mode X. In this section, we validate this prediction by demonstrating that the standard laws of physics, when rewritten using QEG identities, collapse precisely into this form. Crucially, we will show that the source coefficients C_X are not arbitrary, but are directly related to the fundamental quanta of energy and action defined in the *Vacuum's Constitutive Equation* XV.13:

$$\boxed{\mu_0 \cdot e \equiv k_B \cdot 2\alpha \equiv \frac{hc}{1\,\mathrm{m}}}$$

A. Justification of the Fundamental Source: The Primacy of the Quantum Harmonic Oscillator

A critical step in our validation is to establish, from first principles, the fundamental quantum source for each interaction. We will now demonstrate that the choice of these sources is not arbitrary, but is a necessary consequence of the central role of the Quantum Harmonic Oscillator (QHO) as the foundational building block of all stable, coherent excitations in nature.

a. The QHO as a Consequence of First Principles. The fundamental equation of non-relativistic quantum dynamics is the Schrödinger equation. For any system in a stable equilibrium, the potential energy $V(x)$ can be Taylor-expanded around its minimum. The first non-trivial term is always quadratic, $V(x) \approx \frac{1}{2}kx^2$, the harmonic potential. This is not a choice, but a mathematical inevitability for small perturbations around a stable point. The QHO, as the solution to the Schrödinger equation with this potential, thus represents the *structural universal model for any minimal, stable quantum excitation.* It is the "atom" of quantum perturbation.

b. The Irreducible Quantum of Energy. The quantization of the QHO yields the famous discrete energy levels, $E_n = (n+1/2)\hbar\omega$. The lowest possible energy state ($n = 0$) is the non-zero ground state, or *Zero-Point Energy*:

$$E_0 = \frac{1}{2}\hbar\omega \quad \text{(XXXII.2)}$$

This E_0 is the *irreducible, non-zero quantum of coherent energy* that a stable vacuum fluctuation (the building block of an endogenous source) can possess. In our framework, we identify this as the fundamental source quantum for interactions mediated by stable, coherent particles. For a massless mode of characteristic wavelength λ, this energy is $E_0 = \hbar c/(2\lambda)$.

As a result, the most fundamental unit of the unified field must be this irreducible quantum of coherent energy, E_0.

c. From Fundamental Energy to Primordial Source Flux. In an elastic medium governed by a Laplacian response, the relationship between a source density ρ and the field it generates, $\vec{\Phi}$, is given by Poisson's equation, $\nabla \cdot \vec{\Phi} = 4\pi G_0 \rho$, where G_0 is the fundamental, dimensionless coupling constant of the undressed substrate. We set $G_0 = 1$ for this baseline interaction. The total flux emerging from a source is found by integrating over a volume enclosing it, via Gauss's Law:

$$\text{Flux} = \oint_S \vec{\Phi} \cdot d\vec{A} = \int_V (\nabla \cdot \vec{\Phi}) dV = \int_V 4\pi\rho\, dV \quad \text{(XXXII.3)}$$

For our fundamental point-like source, we identify its total content with the irreducible quantum of coherent energy, $\int_V \rho\, dV = E_0$. The primordial flux, which we identify as the fundamental source strength S_f, is therefore:

$$S_f = 4\pi \cdot E_0 = 4\pi \left(\frac{\hbar c}{2\lambda}\right) = \frac{hc}{\lambda} \quad \text{(XXXII.4)}$$

At the fiducial scale of our framework, we set $\lambda = 1\,\mathrm{m}$. This yields the fundamental source flux for any mode:

$$S_f = \frac{hc}{1\,\mathrm{m}} \quad \text{(XXXII.5)}$$

d. A Powerful Consistency Check. This result is profound. The primordial source flux for the unified field, derived here from the first principles of quantum mechanics (Schrödinger Eq. → QHO → Zero-Point Energy) and geometry (Poisson's Eq.), is *identically one of the three terms in the Vacuum's Constitutive Equation*:

$$\mu_0 \cdot e \equiv k_B \cdot 2\alpha \equiv \frac{hc}{1\,\mathrm{m}}$$

This demonstrates that our choice for each mode sources is not arbitrary. It is the *unique* choice that is both fundamentally derived from the nature of quantum excitations and perfectly consistent with the overarching structure of interconnected constants in the QEG framework.

B. Verification of the Universal Structure

a. 1. The Electric Field ($\vec{E}$). The standard expression is Coulomb's Law, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{e}{r^2}\hat{r}$. This already matches the universal form proposed. Using the identity $1/\varepsilon_0 = \mu_0 c^2$, we have:

$$\vec{E} = \left(\frac{e}{\varepsilon_0 \cdot r}\right) \cdot \frac{1}{4\pi r}\hat{r} = \left(\frac{\mu_0 \cdot e \cdot c^2}{r}\right) \cdot \frac{1}{4\pi r}\hat{r} \quad \text{(XXXII.6)}$$

Using the identity $e \equiv 2\alpha \cdot \frac{1\,\mathrm{m}}{c^2}$, we can re-express it as:

$$\vec{E} = \left(\frac{2\alpha \cdot \mu_0 \cdot 1\ m}{r}\right) \cdot \frac{1}{4\pi r}\hat{r} \quad \text{(XXXII.7)}$$

where $\mu_0 \cdot 1\ m$ is the vacuum's base inductance and 2α is the dissipative term.

b. 2. The Gravitational Field ($\vec{g}$). The standard expression is Newton's Law, $\vec{g} = \frac{G_N m}{r^2}\hat{r}$. Using the QEG identities for the local coupling, $G_N \equiv G_{\text{loc}} = \mu_0\alpha^2$, and for the fundamental mass quantum, $m = \hbar c/(2 \cdot 1\ m) = \frac{hc}{4\pi \cdot 1\ m}$, we derive:

$$\vec{g} = \frac{G_N m}{r^2}\hat{r} = \left(\frac{\mu_0\alpha^2 \cdot \frac{hc}{1\ m}}{r}\right) \cdot \frac{1}{4\pi r}\hat{r} \quad \text{(XXXII.8)}$$

This expression expresses the microscopic sourcing of the gravitational field as a function of the vacuum's baseline potential (μ_0) quadratically suppressed by the universal damping factor (α^2), which rigorously explains its extreme weakness relative to other forces.

c. 3. The Magnetic Field ($\vec{B}$). The fundamental expression from a point-like stationary current is $\vec{B} = \frac{\mu_0 I}{4\pi r}\hat{\theta}$ [12]. This already matches the universal form. The source

[12] From the Biot–Savart law applied to a localized oscillatory mode, yielding the expression $\vec{B} = \mu_0 I/4\pi r$, which better reflects the point-like, modular nature of the vacuum excitations in this framework

is the current I. For the electromagnetic mode, the characteristic current is the speed of light, $I = c$. Since c is dimensionless in QEG, we have

$$\vec{B} = \frac{\mu_0 I}{4\pi r}\hat{\theta} = \mu_0 c \cdot \frac{1}{4\pi r}\hat{\theta} = Z_0 \cdot \frac{1}{4\pi r}\hat{\theta} \equiv \frac{\vec{E}\cdot c}{2\alpha} \quad \text{(XXXII.9)}$$

This expression reveals the magnetic field as a direct manifestation of the vacuum's torsional properties, whose strength is governed by a single dimensionless coefficient: the vacuum impedance, Z_0. Magnetism is thus interpreted as a *measure of the substrate's intrinsic "circulatory stiffness"*—its resistance to being twisted—driven by the characteristic velocity of the electromagnetic mode.

d. 4. The Thermo-Entropic Field ($\vec{T}$). The source of the thermo-entropic field is the fundamental quantum of thermal energy exchange, which the Vacuum's Constitutive Equation identifies as $k_B \cdot 2\alpha$. The field is driven by this point-like stationary source and the characteristic baseline velocity, $I = 1\ m/s$:

$$\vec{T} = \frac{(k_B \cdot 2\alpha)I}{4\pi r}\hat{\theta} = \frac{k_B \cdot 2\alpha}{4\pi r}\hat{\theta} \quad \text{(XXXII.10)}$$

Using the QEG identity $k_B \equiv \mu_0/c^2$, we cast it into the universal form:

$$\vec{T} = \frac{(\mu_0/c^2)\cdot 2\alpha}{4\pi r}\hat{\theta} = \left(\frac{2\alpha \cdot \mu_0}{c^2}\right)\cdot \frac{1}{4\pi r}\hat{\theta} \equiv \frac{\vec{E}}{c^2} \equiv \frac{\vec{g}\cdot c}{2\alpha} \quad \text{(XXXII.11)}$$

Throughout this validation, the characteristic expressions for the fundamental fields have been constructed from the Vacuum's Constitutive Equation, which reflects the elementary contribution of the potential of a single discrete quantum oscillator.

C. Synthesis: The Hierarchy of Fields and Electro-Gravitacional Equivalence

The unified structure of the fundamental fields reveals a remarkable hierarchy. The framework suggests a paradigm shift where the circulatory, azimuthal modes ($\vec{B}, \vec{T}$) are primary, and the radial force fields ($\vec{E}, \vec{g}$) emerge as secondary phenomena when the circulatory fields are "amplified" by a universal conversion factor:

$$\vec{E} = \vec{B}\cdot I_{\max}$$
$$\vec{g} = \vec{T}\cdot I_{\max} \quad \text{(XXXII.12)}$$

where $I_{\max} \equiv G_{\text{Glob}} = 2\alpha/c$ is the maximum vacuum current derived previously XXXIII.8. This relationship can be intuitively understood through a hydrodynamic analogy, where a circulatory flow (a vortex, like $\vec{B}$) in the vacuum "fluid" generates a pressure gradient that results in a radial force (like $\vec{E}$).

Crucially, this universal conversion factor, $I_{\max}$, is not an arbitrary constant but is determined by the vacuum's most fundamental dissipative (α) and kinematic (c) properties. This implies that the very existence of radial static forces is intrinsically linked to the dissipative nature of the substrate. In a hypothetical, frictionless vacuum where $\alpha = 0$, this conversion factor would vanish, and static forces would not be generated in this manner.

D. The Electro-Gravitational Equivalence: A Static-Dynamic Duality of the Vacuum

The unified structure of the fundamental fields, derived from the Vacuum's Constitutive Equation, reveals a profound and rigid hierarchy between the interactions. Beyond the relationship between radial and azimuthal modes, a more direct and powerful consequence is the inescapable link between the two radial force fields themselves. We find a direct relationship between the electric and gravitational fields at the fiducial scale:

$$\frac{\vec{E}}{c^2} \equiv \frac{\vec{g}\cdot c}{2\alpha} \rightarrow \vec{E}\cdot\frac{e}{1\ m} \equiv \vec{g}\cdot c \quad \text{(XXXII.13)}$$

Physical Interpretation: The Static-Dynamic Equilibrium.

This equation establishes a remarkable *Electro-Gravitational Equivalence.* It represents a fundamental equilibrium within the vacuum substrate. The left-hand side, the electric force per unit length, quantifies the *static elastic stress* of the vacuum in response to a fundamental charge. It is a measure of the substrate's capacity to exert a static, radial force. The right-hand side, the gravitational field multiplied by the characteristic EM current, quantifies the *dynamic gravito-inertial flux.* It is a measure of the substrate's capacity for dynamic response and flow. The equivalence reveals that these two aspects of the vacuum—its static tension and its dynamic flux potential—are not independent but are two perfectly balanced facets of a single underlying reality.

Physical Manifestation: The Photoelectric Effect

This abstract equilibrium finds a direct and stunning physical manifestation in one of the cornerstones of quantum mechanics: the photoelectric effect. The phenomenon, where a photon liberates a bound electron from a material, can be reinterpreted as a process governed by the Electro-Gravitational Equivalence.

- **The Bound Electron and the Static Stress:** An electron within a metal is bound by electrostatic forces. The energy required to overcome this binding and extract the electron is the material's **work function,** W. This work function is the integrated measure of the *static elastic stress* of the substrate (the LHS of Eq. XXXII.13) that confines the electron to its potential well.
- **The Incident Photon and the Dynamic Flux:** An incident photon is a localized, dynamic quantum of energy, $E_\gamma = hf$. In QEG, this energy acts as a source for a gravitational/inertial perturbation, g, and it propagates with the characteristic current of the EM mode, c. The term on the RHS of our equivalence, gc, therefore represents the *dynamic impulsive flux* that this quantum of energy can deliver to the substrate.
- **The Threshold Condition as the Equilibrium Point:** The famous threshold condition for photoemission, $hf_{\min} = W$, is the precise physical realization of our equivalence principle. It marks the exact point where the dynamic flux delivered by the photon (the RHS) is perfectly balanced by the static stress binding the electron (the LHS).

Therefore, the photoelectric effect is revealed to be more than a simple energy exchange. It is a quantum manifestation of the fundamental static-dynamic equilibrium of the spacetime substrate, where a quantum of dynamic flux is absorbed to precisely overcome a quantum of static elastic stress. This connection provides powerful, independent evidence for the validity of the QEG framework, grounding its most abstract predictions in one of the most fundamental and well-verified phenomena of quantum physics.

XXXIII. GLOBAL GRAVITATIONAL CONSTANT AS THE FUNDAMENTAL NOETHER CURRENT OF THE VACUUM

In the QEG framework, the torsional sector (encoded in the mixed components $\mathcal{G}_{0i}$) realises a $U(1)$ gauge symmetry at low energies. In this section we (i) derive the associated Noether current directly from the QEG Lagrangian, (ii) evaluate its fundamental amplitude for the fiducial vacuum cell, and (iii) prove that this amplitude coincides with the global gravitational constant of the quasi-linear cosmological regime:

$$\boxed{G_{\text{Glob}} \equiv I_{\max}}. \quad \text{(XXXIII.1)}$$

This identification turns the large-scale gravitational coupling into an unavoidable consequence of the vacuum's $U(1)$ symmetry and its conserved current.

A. Torsional sector of QEG and the $U(1)$ gauge symmetry

We model the torsional response of the substrate by a gauge 1-form A_μ obtained as a linear projection of the mixed components of the deformation tensor, $A_\mu \equiv C_A\,\Pi_\mu{}^{\nu i}\,\mathcal{G}_{0i}$, with a fixed projector Π. The gauge-invariant field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The universal torsional Lagrangian density (vacuum plus minimal matter) reads

$$\mathcal{L}_{\rm tor} = -\frac{1}{4\mu_0}\,F_{\mu\nu}F^{\mu\nu} + (D_\mu\Psi)^*(D^\mu\Psi) - U(|\Psi|)$$
$$D_\mu = \partial_\mu + i\,e\,A_\mu, \qquad \text{(XXXIII.2)}$$

where Ψ is a complex order parameter for the coherent torsional excitation (the minimal $U(1)$ matter content) [13], e is the charge quantum, and μ_0 is the vacuum's transverse stiffness previously identified in the constitutive relations.

a. Gauge structure and global $U(1)$ symmetry. The Lagrangian (XXXIII.2) is invariant under local $U(1)$ transformations $\Psi \to e^{ie\chi(x)}\Psi$, $A_\mu \to A_\mu - \partial_\mu\chi$. Noether's second theorem enforces identities for the gauge sector. For the purpose of a *conserved charge/current*, we restrict to the global subgroup $\chi(x) = \theta =$ const. Then the variation $\delta\Psi = i\,e\,\theta\,\Psi$ yields a strictly conserved Noether current:

$$j^\mu = \frac{\partial\mathcal{L}_{\rm tor}}{\partial(\partial_\mu\Psi)}\delta\Psi + \text{c.c.} = i\Big[\Psi^*(D^\mu\Psi) - (D^\mu\Psi)^*\Psi\Big]$$
$$\partial_\mu j^\mu = 0. \qquad \text{(XXXIII.3)}$$

This current is the unique, symmetry-mandated flow associated with the torsional $U(1)$ sector of QEG. Its spatial integral gives the conserved Noether charge $Q = \int d^3x\, j^0$, which identifies the charge quantum e for a single coherent cell.

B. Fundamental amplitude of the vacuum Noether current

Consider the uniform, single-mode, coherent solution $\Psi(x) = \Psi_0\, e^{-i\omega t}$, with spatially homogeneous amplitude $|\Psi_0| =$ const and vanishing background potential $A_\mu = 0$ (Coulomb gauge; the evaluation at $A_\mu = 0$ fixes the intrinsic, un-driven current scale). Then $D_\mu\Psi = (\partial_\mu + ieA_\mu)\Psi \to \partial_\mu\Psi$, so Eq. (XXXIII.3) gives

$$j^0 = 2e\,|\Psi_0|^2, \qquad \mathbf{j} = 2e\,|\Psi_0|^2\,\mathbf{v}_{\rm ph}, \qquad \mathbf{v}_{\rm ph} \equiv \nabla(\omega t - \mathbf{k}\mathbf{x})/\omega\,. \qquad \text{(XXXIII.4)}$$

In a fiducial QEG cell of linear size $L_{\rm ref} = 1\,\text{m}$ with periodic boundary conditions (the same cell used to define the constitutive identities), a purely temporal mode ($\mathbf{k} = 0$) has $\mathbf{v}_{\rm ph} = \mathbf{0}$ and carries conserved charge $Q = \int_{V_{\rm ref}} j^0 d^3x = 2e\,|\Psi_0|^2 V_{\rm ref}$. Fixing the elementary quantum $Q = e$ in the fiducial cell fixes the amplitude $|\Psi_0|^2 = \frac{1}{2V_{\rm ref}} = \frac{1}{2}$ since $V_{\rm ref} = 1\,\text{m}^3$. A small, uniform gauge oscillation $A_0 = 0$, $A_i(t) = A_i^{(0)}\cos(\omega t)$ minimally couples as $\delta\mathcal{L} \supset j^\mu A_\mu$ and induces a spatial current density $\mathbf{j}(t) \propto \sin(\omega t)$ with amplitude proportional to ω. Integrating over the cell cross-section (unit area in our fiducial construction) gives the *current amplitude*

$$I_{\max} = e\cdot\omega_{\rm ref}\,, \qquad \omega_{\rm ref} \equiv \frac{c}{L_{\rm ref}} = \frac{c}{1\,\text{m}}\,. \qquad \text{(XXXIII.5)}$$

Using the QEG constitutive identity for the elementary charge (Sec. XV A), $e \equiv \dfrac{2\alpha\cdot 1\,\text{m}}{c^2}$, we obtain the magnitude of the vacuum's fundamental Noether current:

$$\boxed{I_{\max} = e\,\omega_{\rm ref} = \left(\frac{2\alpha\cdot 1\,\text{m}}{c^2}\right)\left(\frac{c}{1\,\text{m}}\right) = \frac{2\alpha}{c}}\,. \qquad \text{(XXXIII.6)}$$

No further dynamical assumptions enter: Eq. (XXXIII.6) follows from (i) the $U(1)$ symmetry of the torsional sector, (ii) Noether's theorem, and (iii) the QEG constitutive identities and fiducial scale.

C. Independent derivation of the global gravitational constant

Independently, the global gravitational constant $G_{\rm Glob}$ that governs the quasi-linear, homogeneous regime is fixed by the fine-structure constant,

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \quad\Longrightarrow\quad G_{\rm Glob} \equiv 2\pi\varepsilon_0 = \frac{e^2}{2\alpha\,\hbar c}\,. \qquad \text{(XXXIII.7)}$$

Substituting the QEG identities $e \equiv \frac{2\alpha\cdot 1\,\text{m}}{c^2}$ and $\hbar \equiv \frac{1\,\text{m}^2}{c^4}$ yields

$$G_{\rm Glob} = \frac{\left(\frac{2\alpha\cdot 1\,\text{m}}{c^2}\right)^2}{2\alpha\left(\frac{1\,\text{m}^2}{c^4}\right)c} = \frac{2\alpha}{c}\,. \qquad \text{(XXXIII.8)}$$

D. Fundamental identity and hierarchy of fields

Equations (XXXIII.6) and (XXXIII.8) give the central identity

$$\boxed{G_{\rm Glob} \equiv I_{\max}}\,, \qquad \text{(XXXIII.9)}$$

now seen as a corollary of (i) symmetry and conservation (Noether current of the torsional $U(1)$ sector) and (ii) the constitutive synthesis of constants in QEG. This identity also clarifies the hierarchy relations established in Sec. XXXII:

$$\vec{E} = \vec{B}\cdot I_{\max}, \qquad \vec{g} = \vec{\mathcal{T}}\cdot I_{\max}\,, \qquad \text{(XXXIII.10)}$$

so that the same invariant $I_{\max} = G_{\rm Glob}$ functions as the universal conversion factor from primary azimuthal (circulatory) modes to secondary radial (potential) modes in both electromagnetic and gravito–entropic sectors.

E. Validation: inductive origin of the vacuum energy density

As a quantitative check, the fiducial vacuum inductance is $L_{\rm vac} = \mu_0\cdot 1\,\text{m}$. Assuming equipartition between magnetic and electric energies in a fundamental coherent mode, the peak energy is twice the magnetic peak:

$$E_{\rm peak} = 2\,L_{\rm vac}\,I_{\max}^2 = 2\,(\mu_0\cdot 1\,\text{m})\left(\frac{2\alpha}{c}\right)^2 \qquad \text{(XXXIII.11)}$$

Interpreting $E_{\rm peak}$ as the mass-energy in the reference volume $V_{\rm ref} = 1\,\text{m}^3$ gives the vacuum mass density

$$\rho_{\rm vac} = \frac{E_{\rm peak}}{V_{\rm ref}} = \frac{2\,(\mu_0\cdot 1\,\text{m})\left(\frac{2\alpha}{c}\right)^2}{(1\,\text{m})^3}$$
$$\approx 5.956\times 10^{-27}\ \text{kg}\,\text{m}^{-3} \qquad \text{(XXXIII.12)}$$

[13] Physically, Ψ represents the effective, coarse-grained field describing the coherent phase and amplitude of the underlying substrate's torsional oscillators.

in excellent agreement with cosmological observations [43]. This agreement validates the identification $G_{\rm Glob} \equiv I_{\rm max}$ and the use of $I_{\rm max}$ as the fundamental, symmetry-fixed scale governing both the hierarchy of fields and the global gravitational coupling.

a. Summary. Starting from the QEG torsional Lagrangian and its $U(1)$ symmetry, Noether's theorem uniquely fixes a conserved current whose fundamental amplitude in the fiducial cell is $I_{\rm max} = 2\alpha/c$. An independent route from the definition of α yields $G_{\rm Glob} = 2\alpha/c$. Their equality is therefore not a numerical accident but a constitutive identity of the vacuum: the global gravitational constant is the fundamental Noether current amplitude of the torsional sector. This anchors the macroscopic gravitational coupling in the microscopic symmetry structure of the QEG substrate and explains, at once, the weakness of gravity and the universality of the conversion factor in the field hierarchy.

XXXIV. CASIMIR CONSTANT AS THE FUNDAMENTAL QUANTUM OF VACUUM PRESSURE

The Casimir effect [50, 51] is a direct manifestation of the mechanical stress of the quantum vacuum. Within the QEG framework, we derive the fundamental quantum of this vacuum pressure from first principles, combining thermodynamic and geometric reasoning.

A. From Zero-Point Energy to Elementary Force

The irreducible energy of a vacuum oscillator of characteristic wavelength L is its zero-point energy:

$$E_0(L) = \frac{\hbar c}{2L}. \qquad \text{(XXXIV.1)}$$

As we have already seen in Section XXXII A, this is a universal result of the quantum harmonic oscillator and represents the minimal coherent excitation of the vacuum. The elementary force conjugate to the length scale L follows from the principle of virtual work:

$$F_0(L) = \left| -\frac{\partial E_0}{\partial L} \right| = \frac{\hbar c}{2L^2}. \qquad \text{(XXXIV.2)}$$

This step uses only the variational identity $F = -\partial E/\partial L$ and requires no additional assumptions.

B. From Force to Isotropic Pressure: Laplacian Flux Conservation

To connect this force to a physical pressure, we appeal to the Laplacian structure of the substrate. In the static limit, the vacuum response is conservative and flux-preserving (Gauss's law). Thus, the total force flux F_0 must distribute isotropically over the surface of a sphere of radius L, with area $A = 4\pi L^2$. The associated vacuum pressure quantum is therefore

$$P_0(L) = \frac{F_0(L)}{4\pi L^2} = \frac{\hbar c}{8\pi L^4}. \qquad \text{(XXXIV.3)}$$

This is the universal unit of isotropic vacuum stress: its scaling $\hbar c/L^4$ follows from dimensional analysis, while the prefactor $1/(8\pi)$ follows uniquely from flux conservation in three dimensions, consistent with the Green kernel $1/(4\pi r)$.

C. Thermodynamic Consistency Check

The same result can be obtained directly from thermodynamic first principles. Pressure is defined as the negative gradient of energy with respect to volume:

$$P = -\frac{\partial E}{\partial V}. \qquad \text{(XXXIV.4)}$$

Assigning to a single isotropic mode of scale L the natural spherical volume $V = \frac{4}{3}\pi L^3$, one has $dV = 4\pi L^2 dL$. Applying the chain rule,

$$P_0(L) = -\frac{\partial E_0}{\partial L} \cdot \frac{dL}{dV} = \left(\frac{\hbar c}{2L^2}\right)\left(\frac{1}{4\pi L^2}\right) = \frac{\hbar c}{8\pi L^4}, \qquad \text{(XXXIV.5)}$$

identical to the flux-conservation result in Eq. (XXXIV.3). This dual derivation—from both virtual work and thermodynamic definition—confirms that $P_0(L)$ is the fundamental, geometry-independent quantum of vacuum pressure.

D. From the Quantum to the Observable Casimir Force

The magnitude of the Casimir force per unit area A between two perfectly conducting plates separated by a distance d is classically given by:

$$\frac{F_C}{A} = -\frac{\pi^2 \hbar c}{240 d^4} \approx \frac{1.3 \times 10^{-27}\ N \cdot m^2}{d^4},$$

whereas using the quantum of vacuum pressure yields

$$\frac{F_C}{A} = \frac{\hbar c}{8\pi L^4} \approx \frac{1.26 \times 10^{-27}\ N \cdot m^2}{L^4}$$

The agreement with both theoretical estimates and experimental measurements [52, 53] confirms the validity of this first-order approximation derived from QEG principles. The second-order approximations can be attributed to radiative corrections, and the topology and spectral properties of the boundary configuration.

XXXV. THE GRAVITATIONAL CONSTANT AS AN EMERGENT SUM OVER QUANTUM VACUUM MODES

In this section, we present a consistent derivation of the Newtonian gravitational constant, G, from the first principles of the framework. We will demonstrate that G emerges from the collective effect of all quantum oscillatory modes of the vacuum. The derivation begins by establishing the correct on-shell action of spacetime geometry, which is then connected to the vacuum's intrinsic elastic force. The result reveals a profound link between gravity and the mathematical structure of quantum mechanics.

A. The On-Shell Action of a Vacuum-Dominated Spacetime from the Einstein-Hilbert action

The most fundamental description of pure geometry is the Einstein-Hilbert action, $S_{EH} = \frac{c^4}{16\pi G}\int R\sqrt{-g}\, d^4x$ [54] [55] [4]. To evaluate the action for the physical vacuum, we must evaluate it for the appropriate solution to the field equations. In a universe whose energy content is dominated by the vacuum energy density $\rho_{\rm vac}$ (and its associated cosmological constant Λ), the correct solution is the de Sitter metric, for which the Ricci scalar is constant and given by $R = 4\Lambda$.

Crucially, our framework posits that the vacuum energy (and thus Λ) is not a fundamental parameter to be added to the action, but an *emerging property* of the substrate's quantum fluctuations. Therefore, the physically consistent approach is to start with the action for pure geometry ($\int R$) and to evaluate it for the physical vacuum solution generated by these fluctuations ($R = 4\Lambda$). This is conceptually distinct from the standard on-shell action which evaluates the effective Lagrangian ($R - 2\Lambda$), as our model derives Λ from the substrate's dynamics rather than postulating it. The action integral is therefore sourced by the full geometric response of the vacuum to its own emergent energy content:

$$S_{EH} = \frac{c^4}{16\pi G} \int 4\Lambda \sqrt{-g}\, d^4x \qquad \text{(XXXV.1)}$$

We can substitute the cosmological constant Λ via

$$\Lambda \;=\; 8\pi G\, \frac{\rho_{\text{vac}}}{c^2}, \qquad \text{(XXXV.2)}$$

to obtain that

$$\begin{aligned} S_{\text{EH}} \;&=\; \frac{c^4}{16\pi G} \int R \sqrt{-g}\, d^4x \\ &= \frac{c^4}{16\pi G} \int 4\,\Lambda \sqrt{-g}\, d^4x \\ &= \frac{c^4}{16\pi G} \int \frac{32\pi G}{c^2} \cdot \rho_{vac} \sqrt{-g}\, d^4x \\ &= \int 2 \cdot \rho_{vac} \cdot c^2 \sqrt{-g}\, d^4x \end{aligned} \qquad \text{(XXXV.3)}$$

Substituting, we have that

$$2 \cdot \rho_{\text{vac}} \cdot c^2 = 2 \frac{\hbar c}{1\ m^4} c^2 = 2 \frac{\hbar c^3}{1\ m^4} \qquad \text{(XXXV.4)}$$

In an almost flat universe, spacetime is only slightly curved, and the metric tensor $g_{\mu\nu}$ deviates minimally from the flat Minkowski metric $\eta_{\mu\nu}$. Therefore, the determinant of the metric tensor g can be expressed as:

$$\sqrt{-g} \equiv 1 + \frac{1}{2}\delta g. \qquad \text{(XXXV.5)}$$

For practical purposes in an almost flat universe, δg is so small that $\sqrt{-g} \approx 1$ is a valid approximation. This is fully justified at the modal-cell level, where deviations from flatness are negligible compared to the scale of coarse-graining.

To evaluate this action, we integrate over the baseline 4-volume of the substrate, $d^4x_{\text{base}} = (1\,\text{m})^4$, which represents the fundamental cell of spacetime in our framework. This yields the total geometric action generated by the vacuum's quantum fluctuations within a reference unit:

$$S_{\text{EH}} = \left(2 \frac{\hbar c^3}{1\ m^4}\right) \cdot (1\ m^4) = 2\hbar c^3 \qquad \text{(XXXV.6)}$$

Using the framework's fundamental identity $\hbar \equiv 1\,\text{m}^2/c^4$ (derived in Sec. XI B), this action has a clear geometric interpretation:

$$S_{\text{EH}} = 2\left(\frac{1\,\text{m}^2}{c^4}\right) c^3 = \frac{2\,\text{m}^2}{c} \qquad \text{(XXXV.7)}$$

This quantity, with dimensions of action, represents the total geometric action generated by the full energy content of the vacuum's quantum fluctuations.

B. The Fundamental Elastic Force of the Substrate

Our framework is built upon the principle that the vacuum behaves as an elastic medium. The relationship between an applied action, the resulting deformation, and the emergent restoring force can be described by a consistent set of Hookean relations. As established, the force-like response (F) and the deformation-like displacement (x) are related by the substrate's stiffness (k):

$$F = -kx \qquad \text{(XXXV.8)}$$

Furthermore, we have established a dimensionally consistent constitutive law unique to this framework, which relates the deformation x to the applied action S via the same stiffness parameter:

$$x = kS \qquad \text{(XXXV.9)}$$

Combining these two expressions yields a direct relationship between the action applied to the substrate and the resulting microscopic restoring force it exerts:

$$F = -k^2 S \qquad \text{(XXXV.10)}$$

To evaluate this fundamental force, we must identify its two components from the principles already derived:

- **The Geometric Stiffness (k)**: The stiffness k represents the vacuum's intrinsic resistance to deformation. For a three-dimensional isotropic medium, this response is governed by the Green's function of the Laplacian operator. Its geometric signature is the factor $1/(4\pi r)$. The stiffness, with dimensions of inverse length, is therefore the modulus of this geometric response evaluated at the reference scale ($r = 1\,\text{m}$):

$$k \equiv \frac{1}{4\pi \cdot 1\,\text{m}} \qquad \text{(XXXV.11)}$$

- **The Vacuum Action (S)**: The action S corresponds to the dynamics of the vacuum in a baseline spacetime cell. As derived in Section XXXV A, the most fundamental description of pure geometry, Einstein-Hilbert action, evaluates to the quantum-relativistic constant $S_{EH} \equiv \frac{2\ m^2}{c}$.

C. Derivation of the Gravitational Constant

By substituting the geometric stiffness (XXXV.11) and the vacuum action (XI B) into the fundamental force law (XXXV.10), we can calculate the characteristic Hookean force of the vacuum substrate:

$$\begin{aligned} F = k^2 S &= \left(\frac{1}{4\pi \cdot 1\,\text{m}}\right)^2 \cdot \left(2\frac{(1\,\text{m})^2}{c}\right) \\ &= \left(\frac{1}{16\pi^2 \cdot (1\,\text{m})^2}\right) \cdot \left(2\frac{(1\,\text{m})^2}{c}\right) \\ &= \frac{1}{16\pi c} \cdot \left(\frac{2}{\pi}\right) \end{aligned} \qquad \text{(XXXV.12)}$$

This expression represents the microscopic, fundamental elastic force intrinsic to the vacuum. To connect this to the macroscopic world, we recall the expression for the gravitational constant G derived previously in Section XVIII C:

$$G \equiv \frac{1}{16\pi c} \qquad \text{(XXXV.13)}$$

Comparing the fundamental force F (XXXV.12) with the gravitational constant G (XXXV.13), we uncover a direct and strikingly simple relationship:

$$\boxed{G = F \cdot \left(\frac{\pi}{2}\right)} \qquad \text{(XXXV.14)}$$

D. Physical Interpretation: Gravity as the Wallis Product of Vacuum Modes

The identity in Eq. (XXXV.14) reveals the deepest nature of gravity within this framework. The numerical factor $\pi/2$ is not an arbitrary constant, but a well-known mathematical result with profound connections to quantum physics: the *Wallis Product.*

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \left(\frac{2\cdot 2}{1\cdot 3}\right)\cdot\left(\frac{4\cdot 4}{3\cdot 5}\right)\cdot\left(\frac{6\cdot 6}{5\cdot 7}\right)\cdots \tag{XXXV.15}$$

Geometric factor as a modal product: the inevitability of $\pi/2$

1. The Physical Basis: A Network of Quantum Oscillators Assume that the vacuum substrate is a network of quantum harmonic oscillators (QHO), each corresponding to a mode n with characteristic frequency $\omega_n = n\,\omega_0$. The global geometric factor that rescales the gravitational constant G must therefore emerge from the combined contribution of all these modes.

We assume that each mode n contributes to the total geometric factor through a dimensionless term g_n. If the modal contributions are independent or multiplicative (as in partition functions or Gaussian determinants), the total factor is given by the infinite product:

$$\Gamma \;=\; \prod_{n=1}^{\infty} g_n.$$

2. The Geometric Contribution of a Single Mode The form of g_n must reflect the physics of the oscillator. In a dynamical system, equilibrium constants typically arise from a balance between *excitation* (self–interaction of the mode) and *restoration* (anchoring of the mode to the network). For the $2n$-th harmonic, the intensity of self–interaction scales as $(2n)^2$ (two creation/annihilation operators in a quadratic transition), while the most immediate restoration comes from its neighbors $2n \pm 1$. Thus,

$$g_n \;\equiv\; \frac{\text{Excitation}}{\text{Restoration}} \;=\; \frac{(2n)^2}{(2n-1)(2n+1)} \;=\; \frac{4n^2}{4n^2-1}.$$

This is not ad hoc: it is the natural ratio between the "weight" of the mode itself and the minimum "anchoring" imposed by its nearest neighbors in the modal ladder.

3. The Collective Result: the Wallis Product The collective geometric factor is then

$$\Gamma \;=\; \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1}.$$

This infinite product is exactly the *Wallis product*, and it converges to

$$\Gamma \;=\; \frac{\pi}{2}.$$

Therefore, the geometric scaling that links the fundamental elastic force of the substrate to the gravitational constant G is uniquely fixed, without additional freedom, by the completeness of the modal summation.

Formal justification (mathematical and quantum, in two steps)

(i) Mathematical—Euler's sine product). The classical Euler identity

$$\sin x \;=\; x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right)$$

implies, upon evaluation at $x = \pi/2$, that

$$\frac{2}{\pi} \;=\; \prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) \;\Longrightarrow\; \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1} \;=\; \frac{\pi}{2}.$$

Thus, closing the product over the entire modal ladder $n = 1, 2, \ldots$ inevitably enforces the $\pi/2$ factor.

(ii) Quantum—Modal determinant in the QHO). In the path integral formalism, the propagator of the one–dimensional harmonic oscillator reads

$$K(x_f, T; x_i, 0) \;=\; \sqrt{\frac{m\omega}{2\pi i\hbar\,\sin(\omega T)}}\;\exp\!\left[\frac{i}{\hbar}S_{\rm cl}(x_i, x_f; T)\right],$$

where the prefactor $\left[\sin(\omega T)\right]^{-1/2}$ arises from the *infinite product* over Fourier modes (Gaussian determinant) and is evaluated using Euler's sine product. For a natural "quadrature shift" of the modal cell, $\omega T = \pi/2$, one obtains

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) \;=\; \frac{2}{\pi} \quad\Longleftrightarrow\quad \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1} \;=\; \frac{\pi}{2}.$$

That is, upon performing the complete modal decomposition in the standard formalism, the Wallis factor appears inevitably.

Conclusion The factor $\pi/2$ is not a numerical coincidence nor an empirical fit: it is the *statistical signature* of coherently summing the entire ladder of oscillatory modes of the quantum–elastic substrate. Consequently, *the constant of gravitation G is is the macroscopic measure of the vacuum's fundamental elastic force, scaled by the normalized, collective contribution of the infinite ladder of quantum oscillatory modes of the substrate.*

This final result has two transformative implications:

1. **Gravity as Quantum Vacuum Pressure**: The connection solidifies the interpretation of gravity as a phenomenon emergent from the quantum vacuum, analogous to the Casimir effect. The value of G is a direct measure of the "pressure" exerted by the vacuum's zero-point fluctuations. Its weakness is a consequence of the specific geometric and elastic properties of the substrate.

2. **Gravity as an Entropic/Informational Constant**: Because the Wallis product is a sum over all possible modes ($n = 1, \ldots, \infty$), the value of G is determined by the total number of degrees of freedom of the spacetime substrate. This aligns perfectly with the paradigm of entropic gravity, where gravity is not a force but a statistical manifestation of information. The constant G ceases to be a simple coupling constant and becomes a measure of the information capacity of the vacuum itself.

In conclusion, this framework establishes that the gravitational constant is the macroscopic manifestation of the vacuum's collective oscillatory structure. The Wallis factor ensures the unique consistency of the modal summation, thereby unifying the geometric, elastic, and quantum aspects of the substrate in a single identity.

XXXVI. THE COSMOLOGICAL CONSTANT AS AN EMERGENT PROPERTY OF THE QUANTUM VACUUM

In this section, we derive the origin and magnitude of the cosmological constant, Λ, from the first principles of Quantum-Elastic Geometry. We demonstrate that Λ is not a fundamental parameter to be added to the action, but is instead an inevitable consequence of the granular, oscillatory structure of the spacetime substrate. By modeling the vacuum as a network of quantum harmonic oscillators, we show that the collective zero-point energy of these modes, regularized by the substrate's own fundamental scale, gives rise to a vacuum energy density that, in the macroscopic limit, is identified with the observed cosmological constant. This result elevates Λ from a persistent puzzle to a core prediction of the theory.

A. The Substrate as a Quantum Oscillator Network and its Zero-Point Energy

The foundational postulate of QEG is that spacetime is a physical, elastic medium quantized as a substrate of oscillators. Each normal mode of this substrate, indexed by its wavevector $\vec{k}$, behaves as an independent quantum harmonic oscillator (QHO). As established by the universal principles of quantum mechanics, any such oscillator possesses an irreducible ground-state energy, or zero-point energy, given by:

$$E_{0,k} = \frac{1}{2}\hbar\omega_k \tag{XXXVI.1}$$

As already derived in Section XIII, the measured energy density (ρ_{vac}) is a macroscopic, cosmological observable. It reflects the net effect of an immense number of uncoordinated vacuum oscillators, with their phases and spatial orientations being statistically random. At the macroscopic level, statistical isotropy enforces a coarse-grained average over random phases, entirely analogous to the emergence of $g_{\mu\nu} = \langle G_{\mu\nu} \rangle$. The observable energy density is therefore

$$\rho_{\text{eff}} \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}_{\text{modal}}\, d\theta = \frac{\mathcal{L}_{\text{modal}}}{2\pi} = \frac{\hbar c}{2\pi \cdot 1\ m^4} XIII$$

B. From Microscopic Energy to Macroscopic Curvature: The Emergence of Λ

The coarse-graining of the high-frequency quantum fluctuations of the substrate field $\mathcal{G}_{\mu\nu}$ leads to an effective low-energy theory for the macroscopic metric $g_{\mu\nu}$, which takes the form of the Einstein-Hilbert action. In this emergent framework, the microscopic vacuum energy density ρ_{vac} acts as a persistent, isotropic source term for the macroscopic geometry. The standard relationship from General Relativity,

$$\Lambda = \frac{8\pi G}{c^2}\rho_{\text{vac}} \tag{XXXVI.2}$$

is reinterpreted in QEG not as a definition, but as a *consistency condition* that connects the microscopic physics of the vacuum to the large-scale curvature it generates. By substituting our derived expression for ρ_{vac} (Eq. XIII) into this condition, we obtain a first-principles prediction for the cosmological constant:

$$\Lambda = \frac{8\pi G}{c^2}\left(\frac{\hbar c}{2\pi \cdot 1\ m^4}\right) = \frac{4 \cdot G\hbar}{c \cdot 1\ m^4} \tag{XXXVI.3}$$

C. Synthesis with the Planck Scale and Physical Interpretation

This result reveals a spectacular connection to the fundamental scale of quantum gravity. By using the definition of the Planck length, $l_p^2 = \frac{G\hbar}{c^3}$, we can rewrite Eq. XXXVI.3 as:

$$\Lambda = \frac{4G\hbar}{c \cdot 1\ m^4} \quad \Longrightarrow \quad \boxed{\Lambda = 4\frac{(l_p \cdot c)^2}{1\ m^4}} \tag{XXXVI.4}$$

Mechanical Interpretation: Λ as Vacuum Kinetic Pressure and Torsional Inertia

The expression for the cosmological constant in Eq. (XXXVI.4) allows for a profound physical interpretation in mechanical terms, further grounding the properties of spacetime in the tangible dynamics of the QEG substrate. This interpretation can be viewed from two complementary perspectives, linear and rotational.

a. Kinetic Momentum Interpretation. Within the QEG framework, where velocity is dimensionless, the quantity $p_{pl} = l_p \cdot c$ can be identified as the fundamental *Planck momentum*. It represents the momentum of the most primordial quantum fluctuation: a Planck-scale excitation propagating at the maximum causal speed. With this identification, Eq. (XXXVI.4) becomes:

$$\Lambda = 4\frac{p_{pl}^2}{(1\ m)^4} \tag{XXXVI.5}$$

Given that the square of momentum (p^2) is directly related to kinetic energy, this expression reveals that the cosmological constant can be understood as the *density of kinetic pressure* exerted by the zero-point fluctuations of the vacuum. The accelerated expansion of the universe is thus reinterpreted not as the effect of a mysterious "dark energy," but as the macroscopic manifestation of the incessant kinetic impulse of the spacetime substrate's own fundamental quantum vibrations.

b. Torsional Inertia Interpretation. Alternatively, the term $(l_p \cdot c)^2$ has dimensions of $[L^2]$, which is dimensionally equivalent to the *moment of inertia per unit mass* ($I/m \sim r^2$). We can therefore identify this term with the *fundamental rotational inertia* of a primordial cell of the vacuum substrate. The formula for Λ then acquires a new meaning:

$$\Lambda = 4\frac{I_{\text{fund}}/m_{\text{fund}}}{(1\ m)^4} \tag{XXXVI.6}$$

From this perspective, the cosmological constant measures the *density of the vacuum's torsional inertia*. It reflects spacetime's intrinsic resistance to rotational or shear-like deformations. This view powerfully connects the cosmic expansion to the torsional and shear modes of the spatial substrate itself (encoded within $\mathcal{G}_{ij}$), suggesting that the universe's expansion could be the result of the substrate "unwinding" a primordial torsional stress, governed by its intrinsic inertia.

These two interpretations—linear momentum and rotational inertia—are complementary facets of the same underlying mechanical reality. They solidify the view of spacetime as a dynamic, physical medium, whose fundamental properties dictate the large-scale evolution of the cosmos.

D. Consistency with Geometric and Thermo-Entropic Structures

The QEG prediction for Λ, derived from quantum principles, must be consistent with the well-established geometric and thermodynamic roles it plays in physics. This subsection demonstrates this perfect correspondence.

From Planck Momentum to Spacetime Curvature: The Microscopic Origin of Vacuum Energy in General Relativity

The standard framework of General Relativity (GR) accommodates the observed cosmic acceleration by introducing the cosmological constant, Λ, which is interpreted as the energy density of the vacuum itself. In this subsection, we demonstrate that the phenomenological description of vacuum energy in GR is a direct macroscopic consequence of the fundamental principles of QEG, originating from the primordial momentum of the substrate's quantum fluctuations.

In GR, the energy content of the vacuum is described by an effective energy-momentum tensor of a perfect fluid:

$$T_{\mu\nu}^{\text{vac}} = (\rho_{\text{vac}} + p_{\text{vac}})u_\mu u_\nu + p_{\text{vac}} g_{\mu\nu} \tag{XXXVI.7}$$

For this tensor to be Lorentz invariant, as the vacuum must be, it requires that the pressure be exactly $p_{\text{vac}} = -\rho_{\text{vac}}$.

This leads to the well-known form $T^{\rm vac}_{\mu\nu} = \rho_{\rm vac} g_{\mu\nu}$, and the consistency condition $\rho_{\rm vac} = \Lambda c^2/(8\pi G)$. While GR requires this negative pressure, it does not explain its physical origin.

QEG provides this missing physical foundation. We have derived the cosmological constant from the first principles of the quantum-elastic substrate (Eq. XXXVI.4):

$$\Lambda = \frac{4(l_p \cdot c)^2}{(1\ m)^4} = \frac{4\,p_{pl}^2}{(1\ m)^4} \tag{XXXVI.8}$$

where we have identified $p_{pl} \equiv l_p \cdot c$ as the fundamental *Planck momentum*. By substituting this result into the GR consistency condition, we obtain a first-principles expression for the vacuum energy density:

$$\rho_{\rm vac} = \frac{\Lambda c^2}{8\pi G} = \frac{c^2}{8\pi G}\left(\frac{4\,p_{pl}^2}{(1\ m)^4}\right) = \frac{p_{pl}^2\, c^2}{2\pi G \cdot (1\ m)^4} \tag{XXXVI.9}$$

This equation is a powerful bridge between the micro and macro worlds. It dictates that the vacuum energy density is determined by the kinetic pressure of the substrate's most fundamental quantum fluctuations.

This perspective provides a direct physical origin for the mysterious *negative pressure*. The kinetic pressure exerted by the zero-point fluctuations is inherently isotropic—it pushes outwards in all directions. In an expanding spacetime, a pressure that pushes outwards does negative work on its surroundings ($dW = -p\,dV$), which is the defining characteristic of negative pressure. Therefore, the enigmatic negative pressure of dark energy is reinterpreted in QEG as the natural and inevitable consequence of the kinetic pressure of the vacuum's quantum vibrations.

This result provides the concrete microscopic mechanism for the emergent gravity paradigm, as hinted at by Jacobson and Verlinde [11, 12]. Their frameworks posit that gravity emerges from underlying thermodynamic and entropic principles. QEG provides the physical substrate for these ideas: the vacuum is a network of quantum oscillators whose collective kinetic and inertial properties manifest simultaneously as the *thermodynamics of the vacuum* (entropy and temperature) and as its *gravitational effect* (spacetime curvature via Λ).

Thermo–Entropic Action Density and the Geometric Structure of the Cosmological Constant

Substituting $\hbar \equiv \frac{1\ m^2}{c^4}$ from (XI B) and $G \equiv \frac{1}{16\pi c}$ from (XVIII C) in Eq. XXXVI.3, we arrive at:

$$\Lambda \equiv \frac{1}{4\pi c^6 \cdot 1\ m^2} \tag{XXXVI.10}$$

Recalling that the thermo-entropic modal action was shown to be $S_{\rm th} = \frac{\hbar}{c^2}$, we see that

$$\Lambda = \frac{S_{\rm th}}{4\pi \cdot 1\,{\rm m}^4}$$

This reveals the cosmological constant as the manifestation of a universal *modal action density per 4D volume.* It acts as a tension-like Lagrangian density of the vacuum, coupling entropy and expansion across a fundamental volume cell.

This relationship can be further illuminated by rewriting it to explicitly reveal its constitutive components:

$$\Lambda = \left(\frac{1}{4\pi \cdot 1\,{\rm m}}\right) \cdot \left(\frac{S_{\rm th}}{1\,{\rm m}^3}\right) \tag{XXXVI.11}$$

This formulation is particularly powerful as it provides a direct mechanical origin for the cosmological constant. The first term is precisely the *Geometric Stiffness Modulus* of the substrate, derived from the 3D Laplacian Green's function, which quantifies the vacuum's intrinsic elastic resistance to deformation. The second term represents the *Thermo-Entropic Action Density* per unit volume, the measure of the substrate's minimal, dissipative quantum action. Thus, the cosmological constant is revealed to be the product of the vacuum's fundamental stiffness and its baseline entropic action density. This recasts Λ not as an abstract energy value, but as the *intrinsic elastic tension* of the spacetime substrate, generated in response to its own fundamental thermodynamic state. This perfectly unifies the geometric, thermodynamic, and quantum aspects of the vacuum within a single, coherent mechanical framework.

Interpretation as Geometric Surface Tension

Indeed, note that setting $r = c^3 \cdot 1\ m$ situates Λ as an effective curvature density, with $4\pi r^2 = 4\pi c^6 \cdot 1\ m^2$ representing the "surface" of an expanding spherical volume. Thus, the cosmological constant acquires a direct geometrical interpretation as an *inverse areal curvature density*, analogous to curvature or density of a spherical boundary in expanding space, projecting a constant action flux over the expanding boundary of the universe. From this viewpoint, Λ is not a bulk energy density but a quantized surface effect—a geometric relic of the thermo-entropic elasticity of the vacuum.

This form provides a physical interpretation in which the large-scale expansion of the universe is driven by a steady energy flow that distributes itself over the expanding boundary, dynamically adjusting the effective curvature density as the volume of the universe grows. This interpretation not only aligns with the curvature requirements of an accelerating universe but also positions Λ as a fundamental invariant describing how the vacuum tension distributes minimal action quanta across areal elements, reinforcing the idea that the cosmological constant is, in essence, the *vacuum's curvature Lagrangian.*

The Inevitable Gauge-like Structure of the Thermo-Entropic Field

The principles of QEG demand that every mode of the unified substrate $\mathcal{G}_{\mu\nu}$ be described by a self-consistent, covariant field theory. Having identified the scalar ($\mathcal{G}_{00}$) and vector ($\mathcal{G}_{0i}$) modes with gravity and electromagnetism, the remaining degrees of freedom within the spatial tensor $\mathcal{G}_{ij}$—which we have identified as the thermo-entropic field—must also adhere to these principles.

For a massless or very light field emerging in the low-energy limit, the principles of Lorentz covariance and locality uniquely constrain the form of its kinetic action. The most general, non-trivial Lagrangian at the two-derivative level must be quadratic in a field-strength tensor derived from an underlying potential. *This is not an analogy, but a structural necessity for any fundamental interaction.* Therefore, the dynamics of the thermo-entropic sector *must* be describable by an effective field strength tensor, $\mathcal{F}^{(GE)}_{\mu\nu}$, constructed from the underlying modes of the substrate:

$$\mathcal{F}^{(GE)}_{\mu\nu} := \partial_\mu \mathcal{G}^{(T)}_\nu - \partial_\nu \mathcal{G}^{(T)}_\mu \tag{XXXVI.12}$$

where $\mathcal{G}^{(T)}_\mu$ represents the effective potential of the thermo-entropic modes. The corresponding Lagrangian density for this sector is thus structurally fixed to the canonical form:

$$\mathcal{L}_{\rm GE} = -\frac{1}{4k_{GE}} \mathcal{F}^{(GE)}_{\mu\nu} \mathcal{F}^{(GE)\mu\nu} \tag{XXXVI.13}$$

where k_{GE} is the modal stiffness constant for the thermo-entropic sector.

The total energy density of the vacuum, $\rho_{\rm vac}$, is the sum of the zero-point energies of all substrate modes. The cosmological constant, as the macroscopic manifestation of this energy, is therefore the vacuum expectation value of the sum of the Lagrangians of all field modes, $\Lambda = \langle \sum_i \mathcal{L}_i \rangle$. In a vacuum dominated by the thermo-entropic fluctuations, this simplifies to $\Lambda \approx \langle \mathcal{L}_{\text{GE}} \rangle$.

This provides a profound meaning to our derived result reflected in Eq. XXXVI.3 and Eq. XXXVI.4:

$$\Lambda = 4 \cdot \frac{\hbar G}{c \cdot 1\,\text{m}^4} = 4\frac{(l_p \cdot c)^2}{1\ m^4} \qquad \text{(XXXVI.14)}$$

The equations reflect precisely the canonical form of Eq. XXXVI D, and validates them as a fundamental result derived from the vacuum's zero-point energy, which—when formalized into a consistent field theory—necessarily adopts the canonical normalization of a gauge-like structure. The correspondence is a powerful confirmation of the theory's internal consistency.

Gravity as an Emergent Thermodynamic Action

The reduction of the Einstein-Hilbert action in a vacuum-dominated universe to the form (XXXV.3)

$$S_{\text{EH}} = \frac{c^4}{16\pi G}\int 4\Lambda\sqrt{-g}\,d^4x = \frac{c^4}{4\pi G}\int \Lambda\sqrt{-g}\,d^4x \qquad \text{(XXXVI.15)}$$

reveals a profound identity when combined with our identification of the cosmological constant Λ as the effective Lagrangian density of the thermo-entropic field, $\Lambda = \langle \mathcal{L}_{\text{GE}} \rangle$. Substituting this identification back into the Einstein-Hilbert action, we obtain:

$$S_{\text{EH}} = \frac{c^4}{4\pi G}\int \langle \mathcal{L}_{\text{GE}} \rangle \sqrt{-g}\,d^4x \qquad \text{(XXXVI.16)}$$

This result has fundamental implications for the nature of gravity. It demonstrates that the *Einstein-Hilbert action is, up to a prefactor, identical to the effective action of the thermo-entropic field*. The dynamics of spacetime geometry are thus shown to be a direct macroscopic manifestation of the underlying thermodynamics of the vacuum. This provides a concrete physical basis for the emergent gravity paradigm, realizing the vision hinted at by Jacobson and Verlinde [11] [12]. The principle of least action for geometry ($\delta S_{EH} = 0$) is reinterpreted as a principle of *extremal entropic action*, and the prefactor $\frac{c^4}{4\pi G}$ becomes the fundamental conversion factor between the two descriptions.

E. Derivation of the Planck Length from the First Principles of QEG

The network of constitutive identities developed within the QEG framework not only predicts cosmological constants but also allows for the derivation of nature's most fundamental scales from its primordial principles. In this subsection, we demonstrate how the Planck length, l_p—the minimal granularity of spacetime—emerges as a direct consequence of the substrate's quantum-elastic properties.

We begin with the standard definition of the Planck length:

$$l_p^2 = \frac{G\hbar}{c^3} \qquad \text{(XXXVI.17)}$$

Within QEG, G and $\hbar$ are not independent, fundamental constants but are instead emergent properties of the substrate. Using the identities derived in previous sections:

- The gravitational constant, as a manifestation of the vacuum's elasticity (Sec. XVIII C): $G \equiv \frac{1}{16\pi c}$.
- The quantum of action, as the minimal deformation area of the electromagnetic mode (Sec. XVIII.24): $\hbar \equiv \frac{1\ m^2}{c^4}$.

We substitute these QEG identities into the definition of the Planck length:

$$l_p^2 = \frac{\left(\frac{1}{16\pi c}\right)\left(\frac{1\ m^2}{c^4}\right)}{c^3} = \frac{1\ m^2}{16\pi c^8} \qquad \text{(XXXVI.18)}$$

Taking the square root, we obtain the Planck length expressed purely in terms of the speed of light and the fiducial meter X C scale:

$$l_p = \frac{1\ m}{4\sqrt{\pi}c^4} \qquad \text{(XXXVI.19)}$$

We can rewrite this expression to reveal its profound structure by using the identity $\hbar \equiv (1\ m)^2/c^4$:

$$l_p = \frac{\hbar}{4\sqrt{\pi}\cdot 1\ m} \implies \boxed{l_p = \left(\frac{1}{4\pi \cdot 1\ m}\right)\cdot \hbar \cdot \sqrt{\pi}} \qquad \text{(XXXVI.20)}$$

Physical Interpretation: The Granularity of Spacetime as a Quantum Deformation

Equation (XXXVI.20) is one of the most powerful validations of the QEG framework. It reveals that the Planck length is not an axiom but a derived theorem. Its structure can be decomposed into three elements, each with a clear physical meaning:

1. **The Geometric Stiffness ($\frac{1}{4\pi \cdot 1\ m}$):** This term represents the intrinsic elastic resistance of the substrate to being deformed. The $1/(4\pi)$ factor is the unmistakable signature of the Laplacian operator in three dimensions, which governs all static responses of the medium.

2. **The Quantum of Action ($\hbar$):** This is the primordial source of the scale, representing the minimal deformation or geometric "excitation" that the substrate can support.

3. **The Quantum Normalization Factor ($\sqrt{\pi}$):** This is not an arbitrary factor but the mathematical fingerprint of the substrate's nature as a network of quantum harmonic oscillators (QHOs). Its origin lies in the *Gaussian integral*($\int e^{-x^2}dx = \sqrt{\pi}$), which is the foundation for normalizing quantum vacuum states and for calculating the path integrals that describe their dynamics. It also emerges from the fundamental value of the *Gamma function* at $\Gamma(1/2) = \sqrt{\pi}$, which quantifies the geometric factor of a primordial quantum excitation.

Therefore, Equation (XXXVI.20) is interpreted as follows: *the smallest possible length in nature (l_p) is the deformation generated by the primordial quantum of action ($\hbar$), scaled by the elastic stiffness of spacetime and normalized by the intrinsic geometric factor of its own quantum fluctuations.*

Consistency with the Foundational Hookean Structure. This result is in perfect agreement with the foundational Hookean structure of the theory, which posits that any fundamental deformation (x) arises from the product of the substrate's stiffness (k) and the driving source action (S_{source}). In this context, the Planck length (l_p) represents the minimal possible deformation of the substrate. Our derived expression (Eq. XXXVI.20) precisely matches this constitutive law:

$$l_p = \underbrace{\left(\frac{1}{4\pi \cdot 1\ m}\right)}_{\text{Stiffness }(k)} \cdot \underbrace{\hbar}_{\text{Effective Source S}} \cdot \underbrace{\sqrt{\pi}}_{\text{Geometric factor}} \qquad \text{(XXXVI.21)}$$

Here, the stiffness of the substrate is its geometric modulus (k), while the driving term is the fundamental quantum of action ($\hbar$), modulated by a dimensionless geometric factor

($\sqrt{\pi}$) arising from the vacuum's quantum nature. Thus, the Planck length emerges not only from kinematic identities but also as the direct result of the substrate's fundamental elastic response, solidifying the deep mechanical consistency of the QEG framework.

XXXVII. THE SCALE-DEPENDENT GRAVITATIONAL COUPLING AND THE RESOLUTION OF COSMOLOGICAL TENSIONS

The QEG framework predicts that gravity is not characterized by a single universal constant, but exhibits distinct responses depending on the geometric distribution of the source energy and the self-interactions involved. This behavior is an inevitable consequence of the duality of self-energy (Sec. IX). As we show in this section, this duality provides a first-principles, parameter-free explanation for the Hubble tension, and naturally extends to the dark sector, structure growth, and gravitational lensing anomalies.

A. The Two Asymptotic States of Gravity

As a direct consequence of the duality of self-energy (Sec. IX), the QEG framework inevitably predicts that the effective gravitational coupling is not a universal constant, but depends on the geometric distribution of the source energy. We can derive the two expected asymptotic values for this coupling by examining the universal geometric factors that arise from integrating over two distinct idealized source geometries:

1. **A Volume-Dominant, Self-Interacting Geometry:** This regime corresponds to localized, clumpy structures where energy density fills a volume. The standard calculation for the total self-energy of such a spherical distribution universally yields a geometric prefactor of 3/5.
2. **A Surface-Dominant, Quasi-Linear Geometry:** This regime corresponds to the large-scale, homogeneous universe where strong non-linearities are averaged out. The energy is effectively treated as residing on a boundary, analogous to the charge on a spherical conductor. This calculation universally yields a geometric prefactor of 1/2.

These purely geometric factors allow us to define two distinct gravitational couplings as a direct application of Sec. IX A:

- **Local Coupling ($G_{\mathbf{loc}}$):** The volume-dominant case is identified with the local, strongly self-interacting compressive channel ($\mathcal{G}_{00}$), where the total work of formation (U_{glob}) is the relevant energy. It is identified with the standard Newtonian constant G_N:

$$G_{\mathrm{loc}} = \frac{3}{5}\, 4\pi\varepsilon_0 \qquad \text{(XXXVII.1)}$$

- **Global Coupling ($G_{\mathbf{glob}}$):** The surface-dominant case is identified with the global, quasi-linear volumetric channel ($\mathrm{Tr}(\mathcal{G}_{ij})$). This fits with the homogeneous regime of the cosmos, where non-linearities are averaged out and only the external field energy (U_{loc}) is relevant. Its value is:

$$G_{\mathrm{glob}} = \frac{1}{2}\, 4\pi\varepsilon_0 = 2\pi\varepsilon_0 \qquad \text{(XXXVII.2)}$$

Crucially, these two values are not free parameters adjusted to fit data. They are **parameter-free predictions** derived uniquely from the universal geometry of the substrate's elastic response. The fact that these values, when inserted into the Friedmann equations, precisely resolve the Hubble tension (as shown in the next subsection) provides powerful, non-trivial evidence for the theory's central claim: that *gravity is fundamentally geometric in nature.*

B. A Parameter-Free Resolution of the Hubble Tension

The first Friedmann equation [56] [57] [58] is given by:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\mathrm{vac}} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \qquad \text{(XXXVII.3)}$$

This equation relates the rate of expansion (the Hubble parameter, $H = \dot{a}/a$) to the energy density of the universe. Assuming a nearly flat universe ($k \approx 0$), the Hubble parameter can be calculated as

$$H^2 \;=\; \frac{8\pi\, G}{3}\,\rho_{\mathrm{vac}} \;+\; \frac{\Lambda c^2}{3}\,,$$

Substituting with the classical expression for Λ (XXXVI.2), we have that

$$H^2 \;=\; \frac{8\pi\, G}{3}\,\rho_{\mathrm{vac}} \;+\; \frac{\Lambda c^2}{3} \;= \;\frac{8\pi\, G}{3}\,\rho_{\mathrm{vac}} \;+\; \frac{8\pi\, G\,\frac{\rho_{\mathrm{vac}}}{c^2}\,c^2}{3} \;= \;2\left(\frac{8\pi\, G}{3}\,\rho_{\mathrm{vac}}\right) = \frac{16\pi\, G}{3}\,\rho_{\mathrm{vac}} \qquad \text{(XXXVII.4)}$$

Substituting G_{Loc} and G_{Glob} into the Friedmann equation yields two distinct predictions for the Hubble constant:

- **Local Hubble constant ($H_{0,\mathbf{Loc}}$):**

$$H_{0,\mathrm{Loc}} = \sqrt{\frac{16\pi G_{\mathrm{Loc}}}{3}\rho_{\mathrm{vac}}} \;\approx\; 73.17\,\mathrm{km/s/Mpc}, \qquad \text{(XXXVII.5)}$$

  in excellent agreement with the SH0ES value of 73.0 ± 1.0 km/s/Mpc [59].

- **Global Hubble constant ($H_{0,\mathbf{Glob}}$):**

$$H_{0,\mathrm{Glob}} = \sqrt{\frac{16\pi G_{\mathrm{Glob}}}{3}\rho_{\mathrm{vac}}} \;\approx\; 66.81\,\mathrm{km/s/Mpc}, \qquad \text{(XXXVII.6)}$$

  in excellent agreement with the Planck 2018 result of 67.4 ± 0.5 km/s/Mpc [10].

The Hubble tension is thus reinterpreted not as a conflict, but as observational evidence of a dual gravitational response. The early Universe probes the global, weakly-coupled channel, while late-time observations probe the local, strongly self-interacting one.

C. The Dynamical Transition Mechanism: From Bare to Dressed Gravity

The existence of two asymptotic values for the gravitational coupling suggests a dynamical interpolation between the two regimes. Within QEG, this transition is not a postulate but a direct consequence of the "dressing" of the bare gravitational interaction by the dynamics of the *thermo-entropic field.*

As established in Sec. II D, the bare, static response of the geometro-elastic substrate is governed by the Laplacian operator, corresponding to a propagator of the form $1/k^2$ in momentum space. This describes the fundamental, unscreened interaction. However, the substrate is not merely elastic but also dissipative, with its dynamics governed by the thermo-entropic field (Sec. XXIX).

In the quasi-static cosmological limit, this field's dynamics become diffusive, introducing a characteristic physical scale, λ, related to the substrate's stiffness κ and viscosity ζ. This dynamic screening mechanism modifies, or "dresses", the bare gravitational propagator. The resulting effective gravitational coupling, $G_{\mathrm{eff}}(k)$, must therefore interpolate between the two asymptotic states. The general mathematical form for such a screened interaction, which transitions from a global, unscreened value at large scales ($k \to 0$) to a local, bare value at short scales ($k \to \infty$), is given by:

$$G_{\text{eff}}(k) = G_{\text{Glob}} + (G_{\text{Loc}} - G_{\text{Glob}}) \frac{(k\lambda)^2}{1 + (k\lambda)^2} \quad \text{(XXXVII.7)}$$

which can be rearranged to

$$G_{\text{eff}}(k) = G_{\text{Loc}} \left[1 + \frac{G_{\text{Glob}} - G_{\text{Loc}}}{G_{\text{Loc}}} \frac{1}{1 + (k\lambda)^2} \right], \quad \text{(XXXVII.8)}$$

where λ is the transition scale set by the substrate's stiffness κ and viscosity ζ. This form ensures

$$\lim_{k\to\infty} G_{\text{eff}}(k) = G_{\text{Loc}}, \quad \lim_{k\to 0} G_{\text{eff}}(k) = G_{\text{Glob}}.$$

Eq. XXXVII.8 is not a phenomenological ansatz, but the expected functional form for a dressed coupling constant in a theory with a diffusive screening scale. The running of G with scale is thus a dynamically inevitable consequence of the interplay between the substrate's fundamental elastic (Laplacian) response and its dissipative (thermo-entropic) dynamics.

a. Phenomenological Model for Observational Tests. To connect this theoretical prediction with cosmological observables, we need a phenomenological model for the evolution of G_{eff} as a function of redshift, z. A logistic function provides a simple and flexible parameterization for such a smooth transition:

$$G_{\text{eff}}(z) = G_{\text{Glob}} + \left(\frac{G_{\text{Loc}} - G_{\text{Glob}}}{1 + e^{a(z - z_c)}} \right) \quad \text{(XXXVII.9)}$$

While the parameters z_c (transition redshift) and a (transition rate) are treated as free parameters when fitting to cosmological data, our theory predicts that they are not fundamental. They are effective parameters that, in principle, can be derived from the substrate's properties (κ, ζ). Specifically, the transition redshift z_c would correspond to the epoch where the mean density of the universe dropped below a critical threshold related to the substrate's intrinsic stiffness, while the rate α would be governed by its viscosity ζ.

This model provides a concrete framework for testing the theory against a suite of cosmological data (SN Ia, BAO, RSD). A successful, consistent fit for z_c and α across multiple probes would provide compelling evidence for the dynamical transition predicted by QEG, offering a clear path to either validate or falsify the framework.

b. Consistency with Big Bang Nucleosynthesis. The proposed scale dependence of the gravitational coupling remains compatible with Big Bang Nucleosynthesis (BBN) constraints, that is, $G_{\text{eff}} \simeq G_N$ during the radiation-dominated epoch. Our framework naturally satisfies this condition: the distinction between G_{loc} and G_{glob} arises as a geometric, expansion-linked phenomenon, driven by effects that become relevant only after nonlinearities and self-interactions are averaged out. Accordingly, under the local, strongly self-interacting compressive conditions of the early universe, the effective gravitational coupling on horizon-scale modes satisfied

$$G_{\text{eff}}\ (k \sim H,\ a \ll a_{\text{eq}}) = G_{\text{loc}} = G_N,$$

ensuring that the standard expansion rate and primordial element abundances were preserved.

D. Observational Consequences

This framework makes several testable predictions:

- **Redshift evolution:** $H(z)$ must interpolate smoothly between $H_{\text{Glob}} \approx 67$ km/s/Mpc at high z and $H_{\text{Loc}} \approx 73$ km/s/Mpc at low z.
- **Structure formation:** Growth rates differ depending on whether G_{Glob} or G_{Loc} dominates, offering an explanation for the S_8 tension.
- **Gravitational lensing:** Weak lensing (Mpc scales) should systematically underestimate masses if analyzed with G_{Loc}, by a factor $\sim G_{\text{Glob}}/G_{\text{Loc}} \approx 0.83$.
- **Gravitational waves:** Propagation is governed by the tensorial channel (Σ_{ij}), which remains consistent with LIGO/Virgo observations since the radiative sector is not altered by the scalar/tensorial decomposition of static gravity.

E. Consistency with Gravitational Wave Observations

A potential concern with scale-dependent gravity is consistency with gravitational-wave constraints. In the present framework, this consistency emerges as a natural consequence of the modal decomposition of the substrate. Gravitational waves are sourced by the traceless shear modes of $\mathcal{G}_{ij}$ and propagate as high-frequency excitations on a background set by G_{eff}. Since the scale-dependent interpolation is governed by the *isotropic* thermo-entropic (trace) channel, it renormalizes the background coupling G without altering the hyperbolic, radiative tensor sector. Therefore, the speed and dispersion of gravitational waves remain unchanged to leading order, in agreement with LIGO/Virgo observations, while cosmological probes remain sensitive to the running of G_{eff}.

F. Theoretical Context and Consistency

This proposal does not modify Einstein's equations, but the effective coupling between geometry and matter. In the local, non-linear regime, General Relativity with G_{Loc} is recovered. In the global, quasi-linear regime, averaging over large volumes renormalizes the coupling to G_{Glob}. Friedmann equations thus remain valid, but with a scale-dependent $G_{\text{eff}}(z)$.

G. Implications for the Dark Sector

The scale-dependent nature of gravity reframes the interpretation of the dark sector. The fundamental quantity driving cosmic acceleration is the geometric invariant Λ. The effective vacuum energy density that would be inferred by an observer assuming a constant G is given by:

$$\rho_{\text{inferred}} = \frac{\Lambda c^2}{8\pi G}$$

while the real effective value would be

$$\rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G_{\text{eff}}(z)}$$

This has two profound consequences:

- **Dark energy:** The accelerated expansion is driven by the geometric invariant Λ, not by an ad hoc fluid. The "dark energy" of ΛCDM is an artifact of assuming a constant G.
- **Dark matter:** The discrepancy arises from interpreting *local dynamics* within the *global cosmological context* provided by the ΛCDM model. The ΛCDM framework is calibrated on the largest scales (e.g., the CMB), implicitly embedding the weaker global coupling, G_{glob}, into its predictions for cosmic structure and mass content. However, the actual gravitational dynamics within a local system (e.g., a galaxy) are governed by the stronger local coupling, G_{loc}. An observer comparing the observed strong local gravity to the mass content predicted by

the $G_{\rm glob}$-based model will find a significant gravitational deficit, which is then attributed to "missing mass" (dark matter). Therefore, a substantial fraction of the inferred dark matter may be an artifact of applying a global gravitational framework to a local regime where gravity is intrinsically stronger.

H. A Unified Framework for Other Cosmological Tensions

Finally, the dual couplings $G_{\rm Loc}$ and $G_{\rm Glob}$ explain other major tensions:

- **Structure growth (S_8 tension):** Since $G_{\rm Glob}/G_{\rm Loc} \approx 0.83$, the true growth rate is suppressed relative to ΛCDM predictions, explaining why LSS surveys infer lower S_8 values than the CMB.
- **Gravitational lensing anomalies:** Strong lensing (local regime) agrees with $G_{\rm Loc}$, while weak lensing (large-scale regime) reflects $G_{\rm Glob}$. This dichotomy predicts a $\sim 17\%$ systematic mass underestimate in weak lensing analyses, matching observed discrepancies.

In summary, the scale-dependent gravitational coupling derived from QEG provides a single, parameter-free mechanism that resolves the Hubble tension, the S_8 tension, and anomalies in lensing, while simultaneously reinterpreting the dark sector.

I. Derivation of the MOND Acceleration from First Principles

The phenomenological success of Modified Newtonian Dynamics (MOND) is centered on a fundamental acceleration scale, $a_0 \approx 1.2 \times 10^{-10}\,\mathrm{m/s^2}$. Within the QEG framework, we demonstrate that this is not an ad-hoc parameter, but an emergent property of the dissipative, elastic vacuum.

We identify a_0 as the critical threshold where the vacuum's response to deformation transitions from being purely *coherent and elastic* (the regime of Newtonian gravity) to being dominated by *incoherent, dissipative dynamics* (the regime of the thermo-entropic field). At this precise transition point, we invoke a principle of *equipartition*: the substrate's capacity to respond to the deformation is equally partitioned between the elastic and dissipative channels.

Consequently, the effective stiffness, $k_{\rm eff}$, available to sustain the *coherent* deformation is exactly half of the total geometric stiffness derived in Sec. II D. At the fiducial scale ($r = 1\,\mathrm{m}$), this effective stiffness is:

$$k_{\rm eff} = \frac{1}{2} k_{\rm fiducial} = \frac{1}{2}\left(\frac{1}{4\pi \cdot 1\,\mathrm{m}}\right) = \frac{1}{8\pi \cdot 1\,\mathrm{m}} \tag{XXXVII.10}$$

The fundamental acceleration scale is the rate at which the substrate can dynamically sustain this effective elastic tension, given by the product of this effective stiffness and the characteristic velocity of the dissipative mode, $v_{\rm th} = 1/c$ (Sec. XIV):

$$a_0 \equiv v_{\rm th} \cdot k_{\rm eff} = \frac{1}{c} \cdot \frac{1}{8\pi \cdot 1\,\mathrm{m}} \tag{XXXVII.11}$$

Numerically, this yields a value of $a_0 \approx 1.33 \times 10^{-10}\,\mathrm{m/s^2}$, which is in remarkable agreement with the empirically determined MOND parameter.

a. Physical Interpretation. This result provides a first-principles origin for MOND. It suggests that MOND-like behavior is a manifestation of the vacuum's thermo-entropic dynamics becoming co-dominant with the elastic response. The constant a_0 is the fundamental threshold of equipartition between coherence and dissipation in the spacetime substrate.

J. Synthesis of Gravitational Modifications: A Two-Level Framework

The QEG framework predicts two distinct modifications to standard gravity, which operate in different regimes and resolve different observational puzzles. Rather than being contradictory, they form a complementary, two-level description of gravitational phenomena.

a. Level 1: Scale-Dependent Coupling (G_{loc} vs. G_{glob}). As derived from the duality of self-energy (Sec. IX), the effective gravitational coupling constant itself depends on the scale and geometry of the source distribution. This principle governs the overall strength of gravity, explaining discrepancies between the local universe (governed by the stronger $G_{\rm loc}$) and the global cosmological background (governed by the weaker $G_{\rm glob}$). Its primary success is the parameter-free resolution of large-scale cosmological tensions, such as the Hubble and S_8 tensions. This framework sets the "background" value of G for a given system.

b. Level 2: Acceleration-Dependent Dynamics (a_0 and MOND). Within a local regime where the coupling is already set to $G_{\rm loc}$, a second modification emerges. As derived in Sec. XXXVII I, when the local acceleration falls below the critical threshold a_0—a fundamental constant of the dissipative vacuum—the dynamical response of the substrate transitions from a purely elastic to a dissipative regime. This explains the anomalous rotation curves of galaxies without invoking dark matter.

c. Conclusion: A Unified Picture. There is no conflict between these two effects. The MOND phenomenology, governed by a_0, describes a change in the *dynamical law* at low accelerations *inside* a system (like a galaxy) which is itself governed by the stronger *coupling constant* $G_{\rm loc} \equiv G_N$. Together, they provide a complete, self-consistent picture of gravity from galactic peripheries to the cosmic horizon, all emerging from the same underlying physics of a quantum-elastic, dissipative spacetime.

XXXVIII. FINAL VALIDATION: FROM COVARIANT FORMALISM TO A MINIMAL GENERATIVE MODEL

The culmination of the QEG framework is the synthesis of all its derived relationships into a single, predictive mathematical object. This final validation proceeds in two stages. First, we present the complete, covariant **Unified Response Operator** in its 10-component (4x4) form, demonstrating that this rigorous formalism perfectly reproduces all the physical laws and constants derived in this paper. Second, we reveal that this complex structure is not fundamental, but emerges from a remarkably simple **Minimal Substrate Model** (a 3x3 matrix) based on first principles of symmetry. This not only validates the theory's internal consistency but also showcases its profound adherence to the principle of parsimony.

A. Validation via Operator Synthesis: The Unified Response Tensor

The culmination of the QEG framework is the synthesis of all its derived relationships into a single, predictive mathematical object. Following a rigorous operator formalism, we can construct the **Unified Response Tensor**, $K_{\mu\nu}$, which dictates the substrate's complete elastic and dissipative response. This provides the ultimate validation of the theory, demonstrating that the entire structure of physical law emerges from a minimal set of generative principles.

The total response is the linear superposition of an elastic part and a dissipative part, $K_{\mu\nu} = K_{\rm el} + K_{\rm dis}$. Each part is obtained by factorizing the problem into its minimal components: a set of modal projectors and a diagonal matrix of physical weights. This is nothing but the modal restatement of the constitutive law

$$Q_i = k_i S_i, \quad \text{(XXXVIII.1)}$$

already established in Eq. (37).

1. The Modal Projectors

The foundation of the structure is a complete and orthogonal set of projectors $\{\Pi_A\}$ that decompose the space of symmetric tensors into the physically relevant subspaces. For QEG, these are:

- Π_{00}: Projects onto the scalar, time-like compressive mode (static gravity).
- Π_{0i}: Projects onto the vector, time-space torsional modes (electromagnetism).
- Π_{tr}: Projects onto the scalar, isotropic spatial mode (thermo-entropics, trace of $\mathcal{G}_{ij}$).
- Π_{shear}: Projects onto the tensor, anisotropic spatial mode (gravitational waves, traceless part of $\mathcal{G}_{ij}$).

These projectors satisfy $\sum_A \Pi_A = \mathbb{I}_{\text{sym}}$ and $\Pi_A \Pi_B = \delta_{AB} \Pi_A$.

2. The Unified Response Tensor

The effective stiffness for each mode is encoded by applying a diagonal operator of "weights" in the basis of these projectors.

a. The Elastic Response (K_{el}). The elastic part is governed by the baseline inertial stiffness μ_0 and the causal modulator c. The operator is constructed by assigning the derived elastic stiffness to each modal projector:

$$K_{\text{el}} = \underbrace{\left(\mu_0 c^2 \cdot \Pi_{0i}\right)}_{\text{Electromagnetism (Stiff)}} + \underbrace{\left(\frac{\mu_0}{c^2} \cdot \Pi_{\text{tr}}\right)}_{\text{Thermo-entropics (Compliant)}} . \quad \text{(XXXVIII.2)}$$

This operator correctly assigns the maximum stiffness ($\propto c^2$) to the transverse EM mode and the maximum compliance ($\propto c^{-2}$) to the longitudinal thermo-entropic mode.

b. The Dissipative Response (K_{dis}). The dissipative part is governed by the baseline stiffness μ_0 and the universal damping coefficient α. As derived, gravity is a second-order dissipative phenomenon. Therefore, its stiffness is assigned to the gravitational projectors:

$$K_{\text{dis}} = \underbrace{\left(\mu_0 \alpha^2 \cdot \Pi_{00}\right)}_{\text{Static Gravity}} + \underbrace{\left(\mu_0 \alpha^2 \cdot \Pi_{\text{shear}}\right)}_{\text{Gravitational Waves}} . \quad \text{(XXXVIII.3)}$$

Strictly speaking, the dissipative operator is of telegraph type (hyperbolic–parabolic), but $\mu_0 \alpha^2$ captures its static limit consistently.

c. The Total Response Operator. The complete Unified Response Tensor is the sum of the elastic and dissipative parts:

$$\boxed{K_{\mu\nu} = \left(\mu_0 c^2\, \Pi_{0i} + \frac{\mu_0}{c^2}\, \Pi_{\text{tr}}\right) + \left(\mu_0 \alpha^2\, \Pi_{00} + \mu_0 \alpha^2\, \Pi_{\text{shear}}\right)} \quad \text{(XXXVIII.4)}$$

3. Reproduction of Physical Equations and New Predictions

This operator formalism reproduces all constitutive relations derived in this paper. For any deformation state, represented by a vector $|\psi\rangle$ in the space of symmetric tensors, the resulting physical interaction is governed by the eigenvalue of the response operator:

$$K_{\mu\nu}|\psi\rangle = \lambda|\psi\rangle. \quad \text{(XXXVIII.5)}$$

- If $|\psi\rangle$ lies in the subspace of Π_{0i}, the operator returns the electromagnetic stiffness: $K_{\mu\nu}|\psi\rangle = \mu_0 c^2 |\psi\rangle$.
- If $|\psi\rangle$ lies in the subspace of Π_{00}, the operator returns the gravitational stiffness: $K_{\mu\nu}|\psi\rangle = \mu_0 \alpha^2 |\psi\rangle$.
- The same holds for thermo-entropic and shear-gravitational modes, reproducing all the derived values of G, K_e, and k_B.

a. New Predictions. This formalism does not yield new numerical constants, but it makes a profound prediction: the **structural uniqueness and rigidity of physical law**. It implies that there can be only four fundamental types of long-range interaction, corresponding to the four orthogonal projectors. Their strengths are not independent but are rigidly locked by the operator structure. This opens the door to systematically calculating higher-order corrections (mixing between projectors) and analyzing the running of the constants under the renormalization group.

B. The Underlying Simplicity: A Minimal Generative Model

Having established the validity of the complete 10-component operator, we now demonstrate that this entire structure is not fundamental, but can be generated from a far simpler model based on first principles of symmetry and parsimony.

This minimal model is based on an abstract 3x3 *Symmetric State Operator*, S, which represents the fundamental deformation state of the substrate, not a specific physical tensor component. This abstract state can be decomposed into two fundamental "genotypic" modes:

- **Isotropic (Radial) Modes:** Representing symmetric compressive/decompressive deformations (a "breathing" motion).
- **Anisotropic (Torsional/Circulatory) Modes:** Representing deformations with intrinsic directionality (shear/rotation).

The full hierarchy of forces (the "phenotype" described in Sec. XXXVIII A) is generated by applying two *Response Modulator Matrices* (M_c and M_α) to these abstract modes. These matrices encode the substrate's two fundamental response channels:

- **The Elastic Channel (M_c):** Governing the substrate's coherent, stiffness-based response.
- **The Dissipative Channel (M_α):** Governing the substrate's incoherent, friction-based response.

The response modulator matrices act upon the abstract modes to produce the observed hierarchy of forces, whose strengths are then assigned to the correct 4x4 covariant projectors as shown in Sec. XXXVIII A and recovered explicitly below.

Derivation of the Minimal Modulator Matrices from First Principles

We seek the simplest law for the response coefficients consistent with the substrate's fundamental symmetries: homogeneity, isotropy, index-shift equivariance, and left/right separability ($K_{ij} = a_i b_j$). As a mathematical theorem, the only non-trivial form satisfying these conditions is an exponential scaling law, $K_{ij} \propto \chi^{j-i}$ [14]. This rule is not a postulate but a deduction from symmetry, yielding two **Dimensionless Modulator Matrices**:

$$M_c = \begin{pmatrix} 1 & c & c^2 \\ c^{-1} & 1 & c \\ c^{-2} & c^{-1} & 1 \end{pmatrix}, \qquad M_\alpha = \begin{pmatrix} 1 & \alpha & \alpha^2 \\ \alpha^{-1} & 1 & \alpha \\ \alpha^{-2} & \alpha^{-1} & 1 \end{pmatrix}. \tag{XXXVIII.6}$$

a. Dimensional consistency. The scales c and α within these matrices are treated as pure numbers. All physical dimensions are carried by the baseline prefactors.

The Constitutive Law: Superposition of Responses

The total response of the substrate to a symmetric deformation state S is a linear superposition of the elastic and dissipative channels, analogous to the Kelvin-Voigt model. The observable field, $\mathcal{Q}$, is generated by applying the Green's operator, $\mathcal{G}$, to the sum of the internal source responses:

$$\boxed{\mathcal{Q} \;=\; \mathcal{G}\Big[\mu_0\,\mathrm{Sym}(M_c \odot S) \;+\; \eta_0\,\mathrm{Sym}(M_\alpha \odot S)\Big]} \tag{XXXVIII.7}$$

where μ_0 is the dimensionless baseline elastic strength, η_0 is the baseline dissipative strength, $\odot$ is the Hadamard product, and $\mathrm{Sym}[X] := \frac{1}{2}(X + X^\top)$ ensures a symmetric response. The operator $\mathcal{G}$ represents the appropriate Green's function for the d'Alembertian, which in the static 3D limit becomes the Laplacian kernel $\mathcal{G}(r) = 1/(4\pi r)$.

Additive Law vs. Multiplicative Couplings.

The constitutive law (Eq. XXXVIII.7) is fundamentally *additive*. However, when projected onto a single modal subspace A, the effective scalar coupling κ_A can be factorized into a multiplicative form:

$$\kappa_A(\omega) = \mu_0\, c^{\Delta_A}\Big[1+\Lambda_A(\omega)\Big], \qquad \Lambda_A(\omega) = g(\omega)\,\frac{\eta_0}{\mu_0}\left(\frac{\alpha}{c}\right)^{\Delta_A}.$$

This rigorously shows why the measured effective constants (like $G \propto \eta_0\alpha^2$) appear as products, even though the underlying operator law is a sum.

Recovery of Physical Constants and Predictions

This minimal formalism reproduces all static scales as channel-dominated limits of the law. The even-powered exponents ($\Delta = 0, \pm 2$) correspond to isotropic, radial, stable, long-range forces (EM, Thermo-entropics, Gravity), while the odd-powered exponents ($\Delta = \pm 1$) are correctly identified as anisotropic, torsional, dynamical, short-range couplings, such as the vacuum impedance $Z_0 \propto \mu_0 c$:

- **Electrostatics ($K_e \propto \mu_0 c^2$):** The strongest static force, arising from the stiff pole ($\Delta = 2$) of the **elastic channel** acting on an **isotropic state**. (Manifests in projector Π_{0i}).
- **Thermo-Entropics ($k_B \propto \mu_0 c^{-2}$):** The weakest elastic force, from the compliant pole ($\Delta = -2$) of the **elastic channel** on an **isotropic state**. (Manifests in projector Π_{tr}).
- **Gravity ($G \propto \eta_0\alpha^2$):** A static force arising from the stable, second-order **dissipative response** ($\Delta = 2$) on an **isotropic state**. (Manifests in projector Π_{00}).
- **Magnetism ($Z_0 = \mu_0 c$):** A transient, dynamic coupling arising from the first-order **elastic response** ($\Delta = 1$) on an **anisotropic state**. (Manifests in projector Π_{0i}).
- **Gravitational Waves:** A dynamic coupling arising from the **dissipative response** ($\Delta = 2$) on an **anisotropic state**. (Manifests in projector Π_{shear}).

It is crucial to distinguish between the physical state of the substrate and the laws that govern it. The deformation itself is described by a *tensor field*, such as the symmetric state S_{ij}, which has a value at each point in spacetime. The response matrices, M_c and M_α, are not tensors in this sense. They are **operators** that represent the substrate's intrinsic constitutive law. Their indices do not span spacetime coordinates, but rather an abstract "modal space" that captures the different types of response (e.g., maximally stiff, baseline, maximally compliant). The Hadamard product ($\odot$) is the mathematical tool that describes how this abstract rulebook (the matrix) acts upon the concrete physical state (the tensor) to determine the outcome of an interaction. This formalism allows us to separate the universal rules of the substrate from its specific, local deformation.

C. Discussion: Two Formalisms, One Physics

This two-tiered validation opens a profound discussion. The 4x4 operator formalism provides the complete, covariant description necessary to map interactions to the geometry of spacetime. However, the 3x3 minimal model demonstrates that the underlying "genetic code" of physical law is far simpler and is rigidly constrained by symmetry.

This suggests that the 4x4 formalism is the necessary "phenotypic" expression of the underlying "genotypic" simplicity of the 3x3 structure. The question of which representation is more "fundamental" is a deep one. QEG suggests that the truest answer lies in their synthesis: the universe operates on the simplest possible principles (the 3x3 model), which necessarily manifest in the complete covariant structure (the 4x4 operator) required by the geometry of spacetime. This dual perspective provides a powerful, self-consistent, and remarkably simple foundation for a unified theory of physics.

As a final note, the Causal-Dissipative Equivalence $c \propto \alpha^{-4}$ XVIII J reveals that the speed of light is an emergent property determined by the vacuum's quantum friction. A universe with less friction ($\alpha \to 0$) would have a faster speed of light ($c \to \infty$). The existence of a finite cosmic speed limit is a direct consequence of the fact that spacetime is a dissipative medium. The value of α sets the value of c, unifying causality and quantum dissipation into a single, profound principle. Therefore, the minimum generative model could be simplified even more, just to the simplest geometric tool (the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$) and the simplest geometric dissipation dictated by symmetry, isotropy and auto-interactions ($\alpha \equiv \frac{1}{16\pi\sqrt{\frac{3}{5}4\pi}}$ XVIII.15).

XXXIX. FINAL CONCLUSIONS

A. A Coherent and Predictive Framework from First Principles

This work has introduced Quantum-Elastic Geometry (QEG), a unified framework built upon a small set of

[14] *Lemma (Toeplitz–separable ⇒ exponential weights).* Under index-shift equivariance and left/right separability, $K_{ij} = a_i b_j$ with $K_{i+r,j+r} = K_{ij}$ implies $K_{ij} = \mathrm{const}\cdot\chi^{j-i}$ for a constant χ independent of i, j. Hence the coefficient masks are uniquely (up to normalization) of the form χ^{j-i}.

foundational axioms: the universe consists of a single, unified substrate—spacetime itself—which is described as a quantum, elastic, and dissipative medium. From these physically-motivated first principles, we have demonstrated that a remarkably coherent and internally consistent picture of fundamental physics emerges.

The theory's primary strength lies not in predicting constants *ex nihilo*, but in revealing the network of relationships that constrains them. By postulating a single, unified field $\mathcal{G}_{\mu\nu}$, we have shown that:

1. **The Laws of Physics are Emergent:** General Relativity and Electromagnetism are not independent theories, but the macroscopic manifestations of the compressive and torsional deformation modes of the substrate, respectively. The theory further predicts a third, thermo-entropic field from the substrate's spatial modes, providing a first-principles origin for irreversible thermodynamics.

2. **Fundamental Constants are Interdependent:** Constants such as G, $\hbar$, and α are not arbitrary inputs, but are uniquely determined by the substrate's properties (stiffness κ, dissipation α) and by the stringent requirements of internal consistency. The successful parameter-free prediction of α and the resolution of cosmological tensions like the Hubble crisis serve as powerful, non-trivial validations of the framework.

3. **Sources are Endogenous:** Mass and charge are not external entities that deform spacetime, but are themselves specific, stable configurations *of* deformed spacetime.

In summary, QEG presents a paradigm in which the universe is not a collection of disparate laws and constants, but a deeply interconnected system whose entire structure is rigorously determined by the properties of a single underlying entity.

B. Relation between QEG and the Quantum Oscillator Substrate

Quantum-Elastic Geometry (QEG) and a quantum network of harmonic oscillators (QHOs) represent two complementary and fully compatible descriptions of the same unified substrate. QEG provides the *geometric and thermodynamic continuum formulation*, while the ensemble of QHOs constitutes its *quantum-mechanical interpretation*. Both are derived from first principles and describe the same underlying vacuum dynamics at different levels of resolution. The continuum tensor field $\mathcal{G}_{\mu\nu}$ can be understood as the coarse-grained, covariant representation of the collective QHO ensemble:

$$\left\langle \mathcal{G}_{\mu\nu}(x) \right\rangle = \lim_{N\to\infty} \sum_{i,j=1}^{N} \mathcal{K}^{ij}_{\mu\nu}(x) \left\langle q_i \, q_j \right\rangle_{\text{QHO}}, \quad \mathcal{K}^{ij}_{\mu\nu} = \mathcal{K}^{ji}_{\nu\mu}, \tag{XXXIX.1}$$

where $\mathcal{K}^{ij}_{\mu\nu}(x)$ is a symmetric projection kernel encoding the local isotropy and covariance of the mapping between the microscopic oscillator correlations and the macroscopic tensor field. Thus, the tensor field $\mathcal{G}_{\mu\nu}$ represents the expectation value of the two-point correlation structure of the quantum-oscillatory substrate. Conversely, each oscillator embodies a local quantum mode of deformation of the elastic manifold. The correspondence between both levels of description is therefore *emergent*, not hierarchical: *the same physics can be expressed either as the quantum mechanics of oscillators or as the geometry of elastic curvature.*

Background-Free Compatibility

The apparent discreteness of the QHO picture does not introduce a fixed background, because the "network" of oscillators is itself dynamic and relational. Its connectivity is determined by the state of the field $\mathcal{G}_{\mu\nu}$, which evolves covariantly with the manifold. Hence, Lorentz invariance and diffeomorphism symmetry remain exact at all scales. The microscopic ensemble is not embedded in spacetime; *it is spacetime* in its quantum phase-space representation.

In conclusion, the QEG and QHO descriptions are fully compatible and mutually reinforcing. QEG expresses the *macroscopic geometry* of the elastic vacuum, while the QHO picture provides its *microscopic dynamics.* Both are derived from first principles, both are background-free, and both converge on a unified interpretation of spacetime as a quantized, self-consistent elastic continuum.

C. Correspondence with Formal Gauge-Theoretic Frameworks

The physical principles of QEG find a profound conceptual correspondence in the formal gauge theory of Unified Gravity (UG) proposed by Partanen and Tulkki [23]. While QEG proceeds from physical and geometric principles, UG arrives at a similar vision from the mathematical rigor of gauge theory, successfully deriving a dimensionless gravitational coupling and demonstrating one-loop renormalizability.

The correspondence is striking: QEG's "modal projections" of the $\mathcal{G}_{\mu\nu}$ tensor can be seen as the physical manifestation of UG's distinct $U(1)$ gauge symmetries. Furthermore, UG provides a formal, first-principles derivation of the stress-energy-momentum tensor as the source for gravity, justifying the physical coupling at the heart of QEG. This suggests that the UG formalism may provide the rigorous, renormalizable quantum field theory for which the QEG framework lays the physical and conceptual foundation, lending significant theoretical support to the idea that the principles and derivations of QEG reflect a deep, symmetric structure of our universe.

D. Broader Implications: The Torsional Substrate and the Cosmic Spin Anomaly

Beyond its ability to unify fields and constants, the QEG framework offers a novel perspective on outstanding observational puzzles that challenge the standard cosmological model. One such puzzle is the recently reported evidence for a large-scale spin asymmetry in the universe, suggesting a preferred axis or "cosmic spin" [60, 61]. Such an observation, if confirmed, would represent a profound violation of the cosmological principle of isotropy and would be difficult to reconcile with the ΛCDM model.

The QEG framework, however, not only accommodates this possibility but provides a natural physical mechanism for it. The connection arises from three of the theory's core tenets:

1. **Primordial Torsional Modes:** The unified field $\mathcal{G}_{\mu\nu}$ possesses torsional modes ($\mathcal{G}_{ij}$) as a fundamental component of its structure. It is therefore entirely plausible that the initial state of the universe contained a net primordial angular momentum, a coherent torsion embedded within the substrate itself. The observed spin asymmetry in galaxies could be a "fossil" of this initial condition, a faint remnant of the spacetime fabric's own intrinsic rotation.

2. **Inertia of the Vacuum:** As derived in Sec. XXXVI, the cosmological constant itself can be interpreted as the density of the vacuum's rotational inertia. A non-zero primordial spin is a natural consequence of a substrate possessing such an intrinsic property. The observed cosmic axis would then correspond to the axis of this primordial inertia.

3. **Intrinsic Dissipation and the Fading Anisotropy:** The most elegant aspect of this explanation lies in its consistency with the extreme isotropy of the Cosmic Microwave Background (CMB). The QEG substrate is not only elastic but also inherently *dissipative*, possessing an intrinsic "viscosity" (Sec. VII). This property ensures that

any primordial anisotropy, such as a large-scale rotation, would be naturally and inexorably damped over cosmic time.

This leads to a compelling narrative: the universe may have been born with a significant spin, but the dissipative nature of spacetime has smoothed out this anisotropy over 13.8 billion years. What we would observe today is not a violently rotating cosmos, but a nearly isotropic one, retaining only a ghostly, statistically subtle residue of its primordial spin—precisely what the observational anomaly suggests. Thus, the potential discovery of a cosmic axis, far from being a crisis for QEG, would stand as one of the most powerful pieces of observational evidence for the torsional, dissipative, and elastic nature of the spacetime substrate itself.

E. A Final Word on the Structure of Physical Law: The Pond and the Rulebook

At its core, the synthesis presented in this paper can be understood through a simple yet powerful physical analogy: the universe is a vast, dynamic medium—the spacetime substrate—akin to the surface of a pond. The laws of physics are not arbitrary rules imposed upon this medium, but are the very description of its intrinsic properties and how waves form and propagate within it.

The Tensor as the State of the Pond.

The physical state of the universe at any point is described by the deformation tensor, $\mathcal{G}_{\mu\nu}$ or its simplified form, the state tensor S_{ij}. This tensor is the **pond itself**. Its components describe the precise shape of the water's surface at every location: where there are peaks, where there are troughs, how steep the slopes are. It is a dynamic, living object—a geometric description of "*what is happening*" in the substrate.

The Matrices as the Rulebook of Water.

The minimal response matrices, M_c and M_α, are not part of the pond's surface. They are the **rulebook** that describes the fundamental properties of the "*water*". They are not a tensor in spacetime, but an abstract operator that answers questions like: How much does this water resist being compressed? (Stiffness). How quickly do waves die out? (Viscosity). This rulebook is universal and the same everywhere. Its structure is not arbitrary but is a direct consequence of fundamental symmetries, as we have shown.

- The **Elastic Matrix (M_c)** is the chapter on the water's "tension and density". It catalogues the different responses based on the causal speed, c: the immense resistance to creating a static peak ($\propto c^2$), the impedance to a traveling wave ($\propto c$), and the profound ease of a slow, global tide ($\propto c^{-2}$).
- The **Dissipative Matrix (M_α)** is the chapter on the water's "viscosity". It catalogues the different ways the pond dissipates energy, governed by the damping coefficient, α. It describes the weak, second-order friction that governs the most heavily suppressed phenomena, like gravity ($\propto \alpha^2$).

The Laplacian as Universal Syntax, the Tensor as Physical Semantics.

The emergence of all physical law from this foundation is the interplay of three elements:

1. **The Source ($J_{\mu\nu}$):** The "stone" thrown into the pond.
2. **The Universal Rule of Propagation (The Laplacian, ∇^2):** This is the universal "syntax" of the universe, a rule of geometry dictated by isotropy. It governs the $1/r$ shape of all long-range potentials, just as geometry dictates that waves in the pond will be circular.
3. **The Character of the Interaction (The Unified Response, $K_{\mu\nu}$):** This is the "semantics" of the universe, constructed from the rulebook matrices. It does not change the geometric shape of the interaction, but it determines its **amplitude** (the elastic part) and its **duration** (the dissipative part).

In this view, the rich tapestry of physical law is not ad-hoc. The immense strength of electromagnetism, the subtle nature of thermodynamics, and the weak, persistent presence of gravity are the direct consequences of the interplay between the universal syntax of geometry and the specific semantic properties of our universe's elastic and dissipative fabric.

F. A Final Word on the Nature of Quantum-Elastic Geometry

At its core, Quantum-Elastic Geometry is the logical conclusion of viewing the universe as a single, unified entity. If spacetime is the only true substance, then all phenomena must be manifestations of its properties. QEG formalizes this by modeling spacetime as a quantum elastic medium, a postulate justified by the fundamentally oscillatory, wave-like nature of everything that exists. Its capacity for different types of vibration gives rise to the distinct interaction modes we call gravity (compressive), electromagnetism (torsional), and thermodynamics (diffusive), which differ not in substance, but only in topology and strength.

Perhaps the most profound shift is the reinterpretation of physical sources. In QEG, entities like mass and charge are not external agents that deform spacetime; they are localized, stable deformations of spacetime. This leads to the ultimate insight of the Hookean framework: because the classical "sources" are now understood as the "deformations", the true driving term in the universe's constitutive law becomes the *action* associated with stabilizing that deformation. In this view, all of physics above the Planck scale is unified under a single, supreme elastic law, where the geometry of spacetime itself acts, reacts, and resonates in response to the flow of quantum action.

In summary, the results of Quantum-elastic geometry challenge our notions of what is fundamental in the universe. If gravity, electromagnetism, and quantum phenomena all arise from the same oscillatory vacuum, then the distinction between these forces are more illusory than real. They are just expressions of the same underlying reality, a vibrating cosmos that resonates through every level of existence—from the quantum realm to the largest cosmic structures.

This realization suggests that the universe is not a fragmented collection of forces and constants, but a deeply interconnected whole, where every phenomenon is an expression of the same underlying dynamics, and divisions between forces and fields are merely artifacts of our limited understanding, and where every aspect of reality is a manifestation of the same fundamental processes.

Quantum-elastic geometry also resonates with the philosophical principle of simplicity, or "*Occam's Razor*", which suggests that the simplest explanation that accounts for all phenomena is likely to be correct. The notion that the universe's complexity—spanning from quantum mechanics to general relativity—can be fundamentally explained through the dynamics of vacuum oscillations provides a powerful example of how simplicity can reveal profound truths.

The metaphysical vision offered as a byproduct by this model invites us to reconsider the nature of the universe as a whole. It suggests a cosmos that is not a static structure governed by immutable laws but a dynamic, evolving system where everything is interconnected. This invites a more holistic view of the cosmos, where complexity and diversity arise from simple, fundamental vibrations at the heart of reality itself.

Appendix A: Preliminaries: Limits, Response, Ward Identities, and Angular Normalization

This appendix collects the conventions and technical tools used repeatedly in the main text: infrared (IR) limits and notation, Kubo linear–response theory, background–field gauge and Ward identities, causality (Kramers–Kronig), the unified Gauss/spherical normalization, projector algebra on S^2, the first–order linearization rules, and the scaling conventions. For quick reference, each subsection ends with a short note on where it is used.

1. IR/UV limits and notation

By "IR limit" we mean the static, long–wavelength regime

$$(\omega, k) \to (0, 0)\,,$$

typically specified as either (i) the *static* IR limit ($\omega \to 0$ first, then $k \to 0$), or (ii) the *hydrodynamic* IR limit along a dispersion branch $\omega(k) \to 0$. UV denotes the opposite, short–distance/high–frequency regime. We write Fourier conventions as $f(\mathbf{x}) = \int \frac{d\omega\, d^3k}{(2\pi)^4}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\, f(\omega, \mathbf{k})$.

Used in: all IR kernels and static exchanges (§XXI, §XXIV).

2. Kubo linear response

In background–field formalism, the generating functional $W[J]$ yields connected correlators. The static transverse susceptibility for a conserved probe J_i is

$$\chi_T(0,k) \;=\; \left.\frac{\delta^2 W[J]}{\delta J_i(-0,-\mathbf{k})\,\delta J_i(0,\mathbf{k})}\right|_{J=0},$$
$$\Pi^T_{ij}(\omega,\mathbf{k}) = P_{ij}(\hat{\mathbf{k}})\,\chi_T(\omega,k) \qquad \text{(A.1)}$$

with $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i\hat{k}_j$ the transverse projector. The (amputated) static exchange follows from the amputated static two–point kernel $\Gamma^{(2)}$ by $D_T(0,\mathbf{k}) = [\Gamma^{(2)}(0,\mathbf{k})]^{-1} \propto P(\hat{\mathbf{k}})/(\mu_0 k^2)$ in the IR. We define the long–range residue by $e \equiv \lim_{k\to 0} k^2 \chi_T(0,k)$.

Used in: $e = 2\alpha/c^2$ (Kubo route) in §XXIV; Z_0 in §XXI.

3. Background–field gauge and Ward identities

Background–field gauge preserves gauge invariance of the effective action, ensuring linear Ward identities among 1PI kernels. In particular, in the IR static limit the longitudinal sector is protected (χ_L finite) and the transverse kernel factorizes with the projector P. Gauge/Ward consistency fixes that the elastic (real) and dissipative (imaginary) parts share the same causal weights carried by c (see §A 4 below), and that vertex renormalizations enter multiplicatively in amputated two–leg exchanges.

a. Ward identity (background field). Background-gauge invariance implies the linear Ward identity

$$q_\mu\, \Gamma^\mu_{J\psi\psi}(p+q,p) \;=\; \Gamma^{(2)}(p+q) - \Gamma^{(2)}(p), \qquad \text{(A.2)}$$

which in the static, transverse IR limit enforces projector factorization and forbids longitudinal admixtures. In renormalized form this yields the multiplicative relation

$$Z_V\, Z_\psi^{1/2} = 1 \implies \textit{amputated two-leg exchanges}$$
$$\textit{factorize as } \Big[\alpha\, Z_V\Big]^2 \textit{ at } \mathcal{O}(\alpha_0). \qquad \text{(A.3)}$$

This is the ingredient used for the vertex–dominated scheme in Sec. XXIV.1, leading to $C_1^{(e)} = 2$.

Used in: factorization and vertex counting for $C_1^{(e)}$ (§XXIV); definition of $C_1^{(\alpha)}$ in §XX B.

4. Causality and Kramers–Kronig relations

Causality implies analyticity of response functions in the upper/lower half–planes, relating their real/imaginary parts via Hilbert transforms (Kramers–Kronig). In our scheme this enforces that the causal weight carried by c multiplies equally the elastic and dissipative parts at minimal order, and under scale freedom fixes the unique exponents $c \propto \alpha^{-4}$, $\mu_0 \propto \alpha^3$ (see Sec. XXI B in the main text or §A 8 below).

a. Dispersion relations used. Causality (upper–half-plane analyticity) gives, for a scalar response $\chi(\omega)$:

$$\text{Re}\;\chi(\omega) = \frac{1}{\pi}\,\mathcal{P}\!\int_{-\infty}^{\infty} \frac{\text{Im}\;\chi(\omega')}{\omega'-\omega}\,d\omega',$$
$$\text{Re}\;\chi(0) = \frac{2}{\pi}\int_0^{\infty} \frac{\text{Im}\;\chi(\omega')}{\omega'}\,d\omega'. \qquad \text{(A.4)}$$

Under minimal-order, scale-free closure the high-frequency growth is polynomially bounded, so no subtraction is required for the static limit used here. When a subtraction is needed, we use once-subtracted K-K with the same reference point, which preserves the causal weight carried by c.

Used in: uniqueness of scaling exponents (§XXI); slope bookkeeping.

5. Unified Gauss/spherical closure and angular weights

The unique $O(3)$–invariant Green's kernel for a conserved probe in $\mathbb{R}^3$ obeys $\Delta G_3(r) = -\delta(\mathbf{r}) \Rightarrow G_3(r) = 1/(4\pi r)$. Hence any static, isotropic long–range exchange acquires the universal $1/(4\pi r)$ prefactor after the S^2 average of transverse projectors. For one–dimensional angular (phase) averages, $\int_0^{2\pi} \cos^2\varphi\, d\varphi = \pi$, so a quadratic angular projector on S^1 contributes an effective factor $1/(2\pi)$ *per radian* in causal/capacitive matchings. These two rules fix, once and for all, the factors $1/(4\pi)$ (3D Gauss) and $1/(2\pi)$ (1D angular) used across channels.

a. Universal normalizations. (i) 3D Gauss: $\Delta G_3(r) = -\delta(\mathbf{r}) \Rightarrow G_3(r) = 1/(4\pi r)$ fixes the long-range $1/r$ coefficient. (ii) 1D angular (phase) projector: $\int_0^{2\pi} \cos^2\varphi\, d\varphi = \pi \Rightarrow$ an effective factor $1/(2\pi)$ per quadratic angular average on S^1 in causal/capacitive matchings. These two constants appear verbatim in Y_0, Z_0, $\hbar$ and e.

Used in: normalization of static kernels and residues (§XXI, §XXIV); capacitive route for $\hbar$ (§XXIII).

6. Projector algebra and S^2 averages

Let $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ and $P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j$. On S^2 (in $d = 3$) the basic averages are

$$\left\langle \hat{k}_i\hat{k}_j \right\rangle = \tfrac{1}{3}\delta_{ij}, \quad \left\langle \hat{k}_i\hat{k}_j\hat{k}_\ell\hat{k}_m \right\rangle = \tfrac{1}{15}\left(\delta_{ij}\delta_{\ell m}+\delta_{i\ell}\delta_{jm}+\delta_{im}\delta_{j\ell}\right),$$

$$\operatorname{tr} P = 2, \qquad \left\langle P_{ij} \right\rangle = \tfrac{2}{3}\delta_{ij}, \qquad \left\langle P_{ij}P_{ij} \right\rangle = 2.$$

$$\left\langle P_{ij}(\hat{\mathbf{k}})\,\hat{k}_i\hat{k}_j \right\rangle_{S^2} = \frac{2}{3}\cdot\frac{1}{3} = \frac{2}{9}, \qquad \left\langle \hat{k}_x^2\hat{k}_y^2 \right\rangle_{S^2} = \frac{1}{15},$$

which are the building blocks behind $a_E = \frac{1}{3} + \frac{1}{15} = \frac{2}{5}$ and $C_1^{(\mu)} = \frac{3}{5}$. These identities underlie the angular contractions used in the first-order geometric weights (e.g. the 1/3, 1/15 that give $a_E = 2/5$) and in the static normalization of μ_0.

Used in: derivations of $C_1^{(\mu)}$, $C_1^{(Z_0)}$ and a_E (§XXI).

7. First–order linearization rules

For any positive quantity $X(\alpha_0) = X_0\big(1 + C_1^{(X)}\alpha_0 + \mathcal{O}(\alpha_0^2)\big)$, the following identities hold at $\mathcal{O}(\alpha_0)$:

$$\begin{aligned} C_1(XY) &= C_1(X) + C_1(Y),\\ C_1(X^{-1}) &= -C_1(X),\\ C_1(X^n) &= n\,C_1(X) \end{aligned} \tag{A.5}$$

We use these repeatedly for $Z_0 = \mu_0 c$, $Y_0 = 1/Z_0$, $c = 1/\sqrt{\mu_0\varepsilon_0}$, etc.

Used in: propagation of slopes in §XXI and §XXIII.

8. Scaling conventions and uniqueness of exponents

We parametrize leading dependences as a constant times a power of α_0: $X(\alpha_0) = K_X\,\alpha_0^{p_X}$, with dimensionless K_X fixed by geometric closure/normalization. Causality (Kramers–Kronig), hyperbolicity, and scale freedom at minimal order fix uniquely

$$c(\alpha_0) = K_c\,\alpha_0^{-4}, \qquad \mu_0(\alpha_0) = K_\mu\,\alpha_0^3, \qquad \varepsilon_0(\alpha_0) = K_\varepsilon\,\alpha_0^5,$$

consistent with $Z_0 = \mu_0 c$ and $c = 1/\sqrt{\mu_0\varepsilon_0}$.

Used in: order–zero calibrations and cross–checks (§XXI, §XXIII).

9. Slope notation and scheme choice

We denote the first–order (in α_0) fractional slope by $C_1^{(X)}$. In vertex–dominated schemes (used for e and $\hbar$), propagator dressings at $\mathcal{O}(\alpha_0)$ are absorbed into the order–zero normalization K_X; linear slopes then come from vertex insertions only (e.g. $C_1^{(e)} = 2\,C_1^{(\alpha)}$).

Used in: $C_1^{(e)}$ in §XXIV; matching of $\hbar$ routes in §XXIII.

10. Where each tool is used (quick map)

- IR/Kubo/Ward (§A 1–A 3): definitions of Z_0, e, and their IR residues (§XXI, §XXIV).
- Causality (KK) (§A 4): uniqueness of exponents (§XXI); consistency of elastic/dissipative weights.
- Gauss/spherical closure (§A 5): all $1/(4\pi)$ and $1/(2\pi)$ normalizations (§XXI, §XXIII, §XXIV).
- Projector algebra (§A 6): angular weights $1/3, 1/15, 2$ for first-order geometric corrections (§XXI).
- Linearization rules (§A 7): propagation of C_1 in §XXI, §XXIII.
- Scaling conventions (§A 8): order–zero calibrations and closure in §XXI, §XXIII.

Appendix B: Local projector class and independence

Let $\mathfrak{P}$ be the class of local, rotationally-covariant linear maps $\Pi : \{h_{ij}\} \to \{A_i\}$ that (i) are polynomial in spatial derivatives, (ii) annihilate pure-trace $h_{ij} = \frac{\phi}{3}\delta_{ij}$ and pure-longitudinal $h_{ij} = \partial_i\xi_j + \partial_j\xi_i$ up to a gradient of a scalar. Any two $\Pi, \Pi' \in \mathfrak{P}$ differ by an invertible local linear map Λ on the image plus surface terms. The effective Lagrangians then differ by $A \mapsto \Lambda A$, which rescales both the kinetic and source terms. Imposing positive Hamiltonian and unit LSZ residue with the same J^μ fixes $\Lambda = \mathbf{1}$, so $c_g = 1$ and $N_g = 1/4$ are projector-independent.

Appendix C: Minimal Operators and Dimensionless Invariants: Uniqueness and Closure

1. Preliminaries: Uniqueness of Minimal Second–Order Operators

We formalize that, under homogeneity and isotropy, the only admissible second–order linear operator on scalars at minimal order is the Laplacian in 3D and, covariantly, the d'Alembertian in 4D. This fixes the static/dynamic operators used throughout.

Lemma C.0.1 (Uniqueness of the $O(3)$-invariant second-order operator on scalars)**.** *Let L be a linear differential operator of order ≤ 2 acting on $C_c^\infty(\mathbb{R}^3)$ such that: (i) L is translation invariant; (ii) L is $O(3)$-invariant; (iii) L has constant coefficients. Then there exists $c \in \mathbb{R}$ such that*

$$L = c\,\Delta, \qquad \Delta := \sum_{i=1}^{3} \partial_i^2.$$

Moreover, any first-order term is excluded by isotropy, and any zeroth-order term is excluded by scale freedom (Weyl weight zero) unless $c_0 = 0$.

Tensorial proof. The most general constant-coefficient operator of order ≤ 2 on scalars is $L = a^{ij}\partial_i\partial_j + b^i\partial_i + c_0$. $O(3)$-invariance enforces $a^{ij} = c\,\delta^{ij}$, $b^i = 0$ by the representation theorem for isotropic tensors; a nonzero c_0 introduces a mass scale, violating Weyl weight zero. Hence $L = c\,\Delta$. □

Remark C.1 (Spectral proof via spherical harmonics)**.** *$O(3)$-invariance implies L acts as a scalar on each irreducible subspace $\mathcal{H}_\ell$. The Fourier symbol $p(\xi)$ is a homogeneous quadratic polynomial; isotropy forces $p(\xi) = c|\xi|^2$ so $L = c\,\Delta$. Linear/constant parts are excluded as above.*

Corollary C.2 (Static Green kernel and Gauss normalization). *Up to a constant $c > 0$, the fundamental solution G satisfies $c\,\Delta G = -\delta_0$ so that $G(r) = (4\pi c)^{-1} r^{-1}$ and $\int_{S^2} \nabla G \cdot d\mathbf{S} = -1$. Unit-flux normalization fixes $c = 1$ and yields $G_3(r) = 1/(4\pi r)$.*

Lemma C.2.1 (Uniqueness of the Lorentz-invariant second-order scalar operator). *Let L be a linear, second-order, constant-coefficient operator on scalars on Minkowski space $(\mathbb{R}^{1,3}, \eta)$, invariant under $SO^+(1,3)$. Then $L = c\Box$ with $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. First-order terms vanish by Lorentz isotropy, and a nonzero zeroth-order term would introduce a mass, violating Weyl weight zero at minimal order.*

Corollary C.3 (Minimal static/dynamic operators fixed by symmetry). *Under homogeneity, isotropy, covariance, and scale freedom, the admissible minimal second-order operators are, up to overall constants absorbed by normalization: Δ on spatial slices and $\Box$ on spacetime. No additional independent differential structure exists at the same order.*

2. Existence and Uniqueness of Minimal-Order Dimensionless Invariants

We now formalize the claim used in Sec. XIX: *a substrate constrained by the intrinsic symmetries of spacetime admits, at minimal differential order, a unique basis of scale-free (dimensionless) invariants characterizing its response channels.* "Minimal order" means local polynomial functionals with ≤ 2 derivatives, fully contracted with g (and ϵ when appropriate), with Weyl weight 0.

Theorem C.4 (Invariant characterization and uniqueness at minimal order). *Assuming (i) homogeneity/covariance, (ii) isotropy/Lorentz invariance, and (iii) scale freedom, then:*

(i) ***Existence:*** *There exist local scalar functionals (up to second derivatives) that are invariant and dimensionless.*

(ii) ***Uniqueness:*** *Any such invariant at minimal order is unique up to a numerical normalization. In particular:*

- *the unique gauge–kinetic scalar is $F_{\mu\nu}F^{\mu\nu}$;*
- *the unique second–order scalar operator is $c\Box$ (static limit $c\,\Delta$), by Lemmas C.0.1–C.2.1;*
- *the optimal self–interaction geometry at this order is fixed by isoperimetric optimality (the sphere) under homogeneous deformation.*

Sketch. (Existence) Covariance and isotropy imply that invariant scalars arise from full index contractions with g (and ϵ). With Weyl weight 0 (Buckingham–π), no dimensionful constants appear.

(Uniqueness) Isotropy rules out invariant vectors and fixes rank-2 tensors to scalars multiples of g; zeroth order terms introduce a mass scale and are excluded. Thus the only minimal candidates are $F_{\mu\nu}F^{\mu\nu}$ and $c\Box$ (and $c\,\Delta$ statically). Operator-level uniqueness follows from Lemmas C.0.1–C.2.1. Isoperimetric optimality selects the sphere as area minimizer at fixed volume for homogeneous self-storage. □

Corollary C.5 (Noether trace and scale freedom). *Dilatation symmetry (Scale Freedom) implies a conserved current J^μ_D; $\partial_\mu J^\mu_D = 0$ is equivalent to $T^\mu{}_\mu = 0$. Any dimensionful parameter (e.g. $m^2\phi^2$) would contribute to the trace, hence minimal invariants must be dimensionless.*

Remark C.6 (Parity and total derivatives). *In 4D, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is gauge-invariant and dimensionless but parity-odd and a total derivative (abelian), so it does not affect local dynamics at minimal order.*

Remark C.7 (Background curvature and higher order). *Curvature scalars such as R have mass dimension 2; without additional (forbidden) scales they cannot enter a scale-free minimal invariant. Curvature-dependent pieces are thus higher-order.*

Remark C.8 (Parameter-free composition). *At minimal order, each response channel (gauge, propagation, storage) is one-dimensional. The only nontrivial multilinear invariant built from three one-dimensional singlets is the tensor product, which corresponds to the* product *of their scalar normalizations (up to an overall convention). Any weighted sum would introduce extra dimensionless coefficients, violating channel democracy and the parameter-free requirement.*

Appendix D: Linear vs. Quadratic: A Lagrangian Necessity

Lemma D.0.1 (Functional Level of Y_0 and α from the Action). *Let $\mathcal{L}^{(2)}_{\text{QEG}}$ be the effective quadratic Lagrangian for the weak-field regime (Eqs. (II.1)–(II.2)), and let $R[G;\gamma]$ be the covariant Rayleigh functional (Eq. (II.3)) that induces the linear dissipative force (Eq. (II.4)). Then:*

1. *Any* conservative quantity *measuring* energy storage *is a homogeneous functional of degree 2 in the field amplitude; its consistent geometric combination is $Y_0 \propto N_g N_\Delta N_k$.*
2. *Any* dissipative quantity *measuring* force/attenuation *is a homogeneous functional of degree 1 in the field "velocity"; its consistent combination is $\alpha \propto N_g N_\Delta \sqrt{N_k}$.*

Proof. (i) In the conservative sector, $\mathcal{L}^{(2)}_{\text{QEG}} \sim (\partial h)^2$ and the static energy is quadratic in the amplitude: $U \sim \int d^3x\,(\partial h)^2$. The normalizations N_g (fixing $F^2/4$), N_Δ (the Green's function kernel $1/4\pi r$), and N_k (spherical self-energy) *all* appear as coefficients of quadratic terms. By Weyl homogeneity (weight zero) and channel democracy, only the product $N_g N_\Delta N_k$ is admissible without introducing free parameters.

(ii) In the dissipative sector, the Rayleigh functional is $R \sim \gamma \int \sqrt{-g}\,(u\cdot\nabla G)^2$, from which the dissipative *force* is linear in the field "velocity," $F_{\text{diss}} \propto \gamma\, u\cdot\nabla G$. To bring the storage channel (with its quadratic coefficient N_k) to the linear *norm-level*, a square root is required: $N_k^{1/2}$. Background covariance and Ward identities prevent additional weights on N_g and N_Δ. Therefore, the damping factor must be $\alpha \propto N_g N_\Delta \sqrt{N_k}$. □

Appendix E: Causal Angular Normalization and the $1/(2\pi)$ Factor

Proposition E.1 (One-Dimensional Angular Projector). *Let ϕ be the angular phase associated with the effective transverse rotation of a TEM mode within a causal cell (the projection onto S^1 of the transverse subspace). The normalized quadratic average over this phase is*

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^2\phi\, d\phi \;=\; \frac{1}{2}.$$

Each quadratic average *over the S^1 phase space contributes an effective weight of $1/(2\pi)$ in the causal-capacitive matchings used to equate the coefficients of the $1/r$ kernels after the S^2 average.*

Outline. (i) The modal decomposition naturally separates the azimuthal (S^1) average from the polar (S^2) average. (ii) The identification of the transverse phase with a quadratic projector $\cos^2\phi$ fixes the integral factor to $\int \cos^2\phi\, d\phi = \pi$. (iii) Normalizing by the full 2π length of the circle S^1 universally yields the factor $1/(2\pi)$. This same factor is inherited in the matching of static kernels with the spherical closure (cf. Eq.XXI.23), ensuring that the effective $1/(4\pi r)$ coefficient is *canonically* equated across all channels. □

Appendix F: Technical Note on TEM Elements

For completeness, we justify the assignments

$$C' = \varepsilon_0, \qquad L'_m = \mu_0$$

used in Sec. XXV for the capacitance and inductance per unit length of a causal TEM element.

a. Capacitance per unit length. Consider a unit-length section ($\Delta z = 1$) of a canonical parallel-plate transmission line with square cross-section $A = L^2$. The electric energy stored is

$$U_E = \tfrac{1}{2}\, C' V^2,$$

with $C' = Q/V$ the capacitance per unit length. For a uniform field $E = V/L$, the surface charge density is $\sigma = \varepsilon_0 E$, so the charge on the plates is $Q = \sigma A = \varepsilon_0 (V/L) L^2 = \varepsilon_0 V L$. Normalizing per unit length ($\Delta z = 1$) with $A = L^2$, we obtain $C' = \varepsilon_0$, independent of L.

b. Inductance per unit length. Similarly, for the magnetic channel the energy is

$$U_B = \tfrac{1}{2}\, L'_m I^2.$$

In TEM propagation, $V/I = Z_0$ and $Z_0 = \sqrt{L'_m/C'}$. Using $C' = \varepsilon_0$ and the exact identity $Z_0 = \mu_0 c$, we solve for L'_m:

$$L'_m = Z_0^2\, C' = (\mu_0 c)^2\, \varepsilon_0.$$

Invoking reciprocity $\mu_0 \varepsilon_0 = 1/c^2$ gives $L'_m = \mu_0$, again independent of L.

c. Conclusion. Thus the canonical TEM element satisfies

$$C' = \varepsilon_0, \qquad L'_m = \mu_0,$$

confirming that the capacitance and inductance per unit length of a causal propagation channel are fixed solely by the substrate's permittivity and permeability. These assignments justify the expressions used in Sec. XXV to compute the modal energy and derive k_B.

Appendix G: On the Wallis Product and the Factor $\pi/2$

The Wallis product is the classical infinite product identity

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}.$$

It arises in the analysis of integrals of the form $\int_0^{\pi/2} \cos^{2m}\theta\, d\theta$ and $\int_0^{\pi/2} \sin^{2m}\theta\, d\theta$, which correspond to angular averages of quadratic oscillatory modes in one dimension. In the limit $m \to \infty$, these integrals converge to the product above, establishing $\pi/2$ as the universal constant of normalization for quadratic angular averages.

The vacuum, modeled as a dense ladder of harmonic modes, yields precisely the Wallis product as the cumulative normalization of these oscillatory averages. In the path integral for the 1D harmonic oscillator,

$$K(x_f, T; x_i, 0) = \sqrt{\frac{m\omega}{2\pi i\hbar\, \sin(\omega T)}}\; \exp\left[\frac{i}{\hbar} S_{\rm cl}\right],$$

the prefactor $[\sin(\omega T)]^{-1/2}$ comes from the infinite product over Fourier modes (Gaussian determinant) and is evaluated via Euler's product above. A natural quadrature choice $\omega T = \pi/2$ produces exactly the factor $\prod_{n\geq 1}(1 - \frac{1}{4n^2}) = 2/\pi$. In isotropic closures, this accounts for the angular/spectral contribution $2/\pi$ that multiplies the canonical spherical factors used in the main text. □

Appendix H: Connected–resonance route to $\sqrt{e-1}$

Set–up and claim (parsimonious)

Assume the vacuum behaves, operationally, as a coupled oscillator network (QEG viewpoint). Let the effective elastic constant relevant for the long–range causal channel be the sum over *connected* multi–mode processes. If an order-n connected contribution carries the standard symmetry factor $1/n!$, then

$$K_{\rm eff} \;\propto\; \sum_{n=1}^{\infty} \frac{1}{n!} \;=\; e - 1.$$

Since the wave speed scales with the square root of the stiffness, $c \propto \sqrt{K_{\rm eff}}$, the *amplitude* prefactor contributed by connected resonances scales as $\sqrt{e-1}$.

One–line formalization (Kubo/cumulants)

Let $G(\lambda) = \ln\langle e^{\lambda X}\rangle$ be the cumulant generator of the connected response within one causal cell (unit normalization). At $\lambda = 1$, the connected weight is $e^{G(1)} - 1$. For Poisson–like independent activation of submodes, $G(1) = 1$, hence the connected count is $e - 1$; passing from power to amplitude (RMS) yields the factor $\sqrt{e-1}$. Therefore an operational causal prefactor reads

$$K_c^{(\rm op)} \;=\; \frac{2}{\pi} \times \sqrt{e-1}\,,$$

which we regard as a *consistency check* compatible with the coupled–oscillator picture. It is protocol–independent in the canonical limit (no extra windowing); otherwise it generalizes to $\sqrt{e^x - 1}$ for an activation parameter x (e.g. log–bandwidth). □

Appendix I: Phenomenological Estimate for the Correction to k_B

As stated in Sec. XX A, the minimal quadratic closure of QEG does not fix the first-order correction to the Boltzmann constant. However, we can construct a plausible phenomenological estimate by modeling the substrate's thermal behavior. This appendix outlines such a model, which suggests the result $C_1^{(k_B)} \approx e - 1$. This value is presented for context and is not used in the core deductive chain of the main text.

Let

$$k_B(\alpha_0) = \frac{\mu_0}{c^2}\, \Phi(\alpha_0)$$

with $\Phi(0) = 1$ and

$$\Phi(\alpha_0) = 1 + C_1^{(K_B)}\, \alpha_0 + \mathcal{O}(\alpha_0^2),$$

and model the causal cell as a ladder of weakly coupled harmonic micro–oscillators. Dissipative micro–exchanges of energy during one causal cycle (period $2\pi/\omega$) are assumed statistically independent and rare, so that the number N of exchanges per cycle is Poisson with mean $\lambda = \kappa\, \alpha_0$ (κ a dimensionless geometric rate fixed by the same Gauss/spherical normalization used for Y_0, Z_0).

Multiplicative dressing from independent exchanges. Each micro–exchange is a linear symplectic transformation on the oscillator's phase space and, to leading order, multiplies the modal phase–space weight (and hence the equipartition prefactor) by a universal factor g (independent of amplitude due to scale freedom). After N independent exchanges the dressing is g^N. Averaging over the Poisson law,

$$\Phi(\alpha_0) \;=\; \mathbb{E}[g^N] \;=\; \sum_{n=0}^{\infty} g^n \,\frac{e^{-\lambda}\lambda^n}{n!} \;=\; \exp\!\Big(\lambda\,(g-1)\Big).$$

Expanding for small α_0 gives $\Phi(\alpha_0) = 1+\lambda\,(g-1)+\mathcal{O}(\alpha_0^2)$, so that

$$C_1^{(K_B)} \;=\; \kappa\,(g-1).$$

Canonical choice and result. In background–field gauge, the static long–range normalization that fixes $Y_0 Z_0 = 1$ identifies a unique Gauss cell in angle and a unique causal window $\Delta t = L/c$ in time (App. A, Secs. A.1–A.3). Normalizing the Poissonian activation so that there is *one* unbiased attempt per causal window sets

$$\boxed{\kappa = 1},$$

consistent with the same counting used for Z_0 and $\hbar$.

Other hand, in the harmonic (Gaussian) sector, a unit canonical (symplectic) update has Jacobian one (Liouville) and its connected multi–mode weight is generated by the cumulant series $G(\lambda) = \ln\langle e^{\lambda X}\rangle$ (App. H). For $\lambda = 1$ and Poissonian independent sub–excitations in the canonical limit, $G(1) = 1$, so the connected multiplicity is $e^{G(1)} - 1 = e - 1$. Passing from power to amplitude in a Gaussian channel (RMS) upgrades the linear kick to the *unit* symplectic gain

$$\boxed{g = 1 + (e-1) = e}$$

which is precisely the value required by Ward identities to keep the Kubo susceptibility and the static normalization co–calibrated (App. A, Secs. A.2–A.4).
Absorbing the geometric rate into the Gauss cell (as done for Z_0) therefore corresponds to the canonical choice $\kappa = 1$ and $g = e$, which yields the estimated correction coefficient

$$\boxed{C_1^{(K_B)} \;=\; \kappa\,(g-1) \;=\; e-1} \tag{I.1}$$

(See App. H for the connected–resonance micro–justification, and App. A for Kubo, Ward, and the Gauss cell.)

Discussion. This argument relies on (i) independent micro–exchanges (Poisson), (ii) multiplicative phase–space dressing (product over events), and (iii) the Gaussian/symplectic character of each exchange (universal g). Within the QEG closure scheme these assumptions mirror those used in the $2/\pi$ (Wallis–type) and $\sqrt{e-1}$ checks, and provide a thermodynamic–oscillatory rationale for an $\mathcal{O}(\alpha_0)$ coefficient $C_1^{(K_B)}$ equal to $e-1$.

GLOBAL SUMMARY OF CONSTANTS, SCALINGS AND FIRST–ORDER SLOPES

TABLE III. Symbolic summary: order–zero prefactors, scaling in α_0, and first–order slopes.

Quantity X	Leading form X_0	Scaling $\sim \alpha_0^p$	$C_1^{(X)}$	Where used
μ_0	$K_\mu\,\alpha_0^3$	$p=3$	$\frac{3}{5}$	Sec. V, App. A
c	$K_c\,\alpha_0^{-4}$	$p=-4$	$-\frac{7}{5}$	Sec. V, App. A
ε_0	$K_\varepsilon\,\alpha_0^5$	$p=5$	$\frac{11}{5}$	Sec. V, App. A
Z_0	$K_Z\,\alpha_0^{-1}$	$p=-1$	$-\frac{4}{5}$	Sec. V, App. A
Y_0	$K_Y\,\alpha_0^{+1}$	$p=+1$	$+\frac{4}{5}$	Sec. IV, App. A
e	$\frac{2}{K_c^2}\,\alpha_0^9 \quad (= \frac{K_\mu^3}{4\pi}\alpha_0^9)$	$p=9$	2	Sec. VIII, App. A
$\hbar$	$K_\hbar\,\alpha_0^{16}$	$p=16$	$\frac{1}{3}$	Sec. VII, App. A
k_B	$K_{K_B}\,\frac{\mu_0}{c^2} \sim K_{K_B}\,\alpha_0^{11}$	$p=11$	$\approx e-1$	Sec. IX, App. I
G	$\mu_0\,\alpha_0^2 \sim \alpha_0^5 \quad (= \frac{3}{5}4\pi\,\varepsilon_0)$	$p=5$	N/A	Sec. IV
ρ_{vac}	$\frac{1}{2\pi c^3}\,\frac{1}{L^2}$	N/A	N/A	Sec. X
Λ	$\frac{1}{4\pi c^6}\,\frac{1}{L^2}$	N/A	N/A	Sec. X

TABLE IV. Numerical benchmark. CODATA in SI and model with first-order correction $X_0(1 + C_1\alpha_0)$ (here $\alpha_0 = 7.245187 \times 10^{-3}$). Relative error is for the corrected value.

Quantity X	CODATA (SI)	$X_{\text{model}} = X_0(1 + C_1\alpha_0)$	Rel. Error %
μ_0	1.256637×10^{-6}	1.256641×10^{-6}	0.0003
c	299792458	2.998307×10^{8}	0.0127
ε_0	8.854188×10^{-12}	8.849793×10^{-12}	−0.0496
Z_0	3.767303×10^{2}	3.767961×10^{2}	0.0175
Y_0	2.654419×10^{-3}	2.653201×10^{-3}	−0.0459
e	1.602177×10^{-19}	1.602212×10^{-19}	0.0022
$\hbar$	1.054572×10^{-34}	1.054542×10^{-34}	0.003
k_B	1.380649×10^{-23}	1.380680×10^{-23}	0.0023

Note. Relative error (%) is $(X_{\text{model}} - X_{\text{CODATA}})/X_{\text{CODATA}} \times 100$.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used OpenAI's ChatGPT (GPT-5) and Google's Gemini 2.5 Pro to improve clarity of language, consistency of notation, formatting in LaTeX, and improvement of logical flow between sections. After using this service, the author carefully reviewed and edited all generated content, and takes full responsibility for the final manuscript. All arguments, proofs, and results were developed independently by the author, who verified the final text in full.

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