Article of Research

Assuming $c < R^2$, The Explicit abc conjecture of Baker Is True, It Implies The abc Conjecture Is True

ABDELMAJID BEN HADJ SALEM*

ABSTRACT. In this paper, assuming that the conjecture $c < R^2$ is true, we give the proof that the explicit abc conjecture of Alan Baker is true and it implies that the abc conjecture is true. We propose the mathematical expression of the constant $K(\epsilon)$. Some numerical examples are provided.

RÉSUMÉ: Dans cette note, on assume que la conjecture $c < R^2$ est vraie, alors on donne une démonstration que la conjecture abc explicite d'Alan Barker est vraie. Cette dernière implique que la conjecture abc est elle aussi vraie, on propose l'expression mathématique de la constante $K(\epsilon)$. Des exemples numériques sont fournis.

Keywords: Elementary number theory, the $c < R^2$ conjecture, functions of one variable, Diophantine equations, explicit abc conjecture of A. Baker, the abc conjecture.

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To the memory of my Father who taught me arithmetic,
To my wife Wahida, my daughter Sinda, my son Mohamed Mazen and my granddaughter Rayhane

1. INTRODUCTION AND NOTATIONS

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ denoted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of the two integers along with their sum. The definition of the abc conjecture is given below:

Conjecture 1.1. *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{1.3}$$

where K is a constant depending only on ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [2]. It concerned the best example given by E. Reyssat [2]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{1.4}$$

A conjecture was proposed that $c < rad^2(abc)$ [3], it is one key to resolve the abc conjecture. In 2012, A. Nitaj [4] proposed the following conjecture:

Conjecture 1.2. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{1.5}$$

$$abc < rad^{4.42}(abc) \tag{1.6}$$

In 2004, Alan Baker [1], [5] proposed the explicit version of the abc conjecture namely:

Conjecture 1.3. *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \tag{1.7}$$

with $R = rad(abc)$ and ω denote the number of distinct prime factors of abc.

In the following, we assume the conjecture $c < R^2$ is true, we give the proof that the explicit abc conjecture of Alan Barker is true. Then we will give an elementary proof of the abc conjecture by verifying the below inequality:

$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} < K(\epsilon)R^{1+\epsilon}, \quad K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \tag{1.8}$$

Some numerical examples are provided.

2. The Proof of The Explicit abc Conjecture of Alan Baker

The case $c < R$ is trivial, it will use the parameters obtained in the following for the case $c > R$.

Proof. : We assume that $c < R^2$ is true. As $c > R$, we can write $c = m.R + m' < R^2 \Longrightarrow 0 < m < R$ and $0 < m' < R$ with $(m, m') \in \mathbb{N}^2$. We proceed by contradiction, let one triplet (a, b, c) of positive integers be relatively prime with $c = a + b$ and:

$$c > \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} \tag{2.9}$$

Let $A = \dfrac{(LogR)^\omega}{\omega!}$, we obtain:

$$R = e^{LogR} = 1 + LogR + \frac{(LogR)^2}{2!} + \cdots + A + \sum_{k=\omega+1}^{+\infty} \frac{(LogR)^k}{k!} \Longrightarrow$$

$$A = R - 1 - \sum_{k=1,\neq\omega}^{+\infty} \frac{(LogR)^k}{k!} \Longrightarrow$$

$$A = R\left(1 - \frac{1}{R}\left[1 + \sum_{k=1,\neq\omega}^{+\infty} \frac{(LogR)^k}{k!}\right]\right) = R(1-B) > 0,\ 0 < B < 1 \Rightarrow B = 1 - \frac{A}{R}$$

The equation (2.9) becomes :
(2.10)

$$c > \frac{6}{5}R.R(1-B) \Longrightarrow \frac{5}{6} \cong 0.83333 > \frac{R^2(1-B)}{c} \Rightarrow contradiction \frac{5}{6} \ll \frac{R^2(1-B)}{c}$$

If $c \longrightarrow +\infty$, then $R \longrightarrow +\infty$ and $B \longrightarrow 1^-$, then we obtain $\dfrac{c}{1-B} > \dfrac{6}{5}R^2$ that becomes $\infty \leq \infty$. □

We announce the theorem:

Theorem 2.4. *Assuming the conjecture $c < R^2$ true, the explicit abc conjecture of Alan Baker (2004) is true: if a, b, c positive integers relatively prime with $c = a + b$, then :*

(2.11)
$$c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!}$$

ω is the number of distinct prime factors of abc.

3. The Proof of the abc conjecture

Proof. We recall the definition of the abc conjecture:
For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

(3.12)
$$c < K(\epsilon).rad^{1+\epsilon}(abc)$$

where K is a constant depending only on ϵ.
We propose the constant $K(\epsilon)$ as follow:

- for $\epsilon \in]0, 1[,\ K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$,
- for $\epsilon \geq 1,\ K(\epsilon) = 1.2e^e$.

We write the equation (2.11) as:

$$c < \frac{6}{5(\epsilon^\omega)}R\frac{(LogR^\epsilon)^\omega}{\omega!}$$

As seen above, let $\epsilon \in]0,1[$ and $A' = \dfrac{(LogR^\epsilon)^\omega}{\omega!}$ and $B' = 1 - \dfrac{A'}{R^\epsilon}$, the above equation becomes:

$$c < \frac{6}{5(\epsilon^\omega)} R \frac{(LogR^\epsilon)^\omega}{\omega!} = \frac{6}{5(\epsilon^\omega)} R.R^\epsilon(1-B') \tag{3.13}$$

We recall the following proposition [4]:

Proposition 3.1. *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*

$$lim_{\epsilon\rightarrow 0} K(\epsilon) = +\infty \tag{3.14}$$

The chosen constant $K(\epsilon)$ verifies the proposition above. Now, is the following inequality true? :

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B') \overbrace{<}^{?} 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \tag{3.15}$$

We proceed by contradiction, we suppose that :

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B') > \frac{6}{5}e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \Longrightarrow 1 > (1-B') > \epsilon^\omega.e^{e^{\left(\frac{1}{\epsilon^4}\right)}}$$

As $\omega \geq 4 \Longrightarrow \omega = 4\omega' + r,\ 0 \leq r < 4, \omega' \geq 1$, we write $\epsilon^\omega.e^{e^{(1/\epsilon)^4}}$ as:

$$\epsilon^\omega.e^{e^{(1/\epsilon)^4}} = \frac{e^{e^{(1/\epsilon)^4}}}{(1/(\epsilon^4))^{\omega'}}.\epsilon^r = \frac{e^{e^X}}{X^{\omega'}}.\epsilon^r \tag{3.16}$$

where $X = \dfrac{1}{\epsilon^4}$ and $1 \ll X$. Or we know that $X^{\omega'} \ll e^X \Longrightarrow X^{\omega'} \ll e^{e^X}$. As $0 \leq r < 4$ and $0 < \epsilon < 1$, then $\epsilon^r > (\epsilon^4 = \dfrac{1}{X})$. The equation (3.16) becomes:

$$\epsilon^\omega.e^{e^{(1/\epsilon)^4}} = \frac{e^{e^{(1/\epsilon)^4}}}{(1/(\epsilon^4))^{\omega'}}.\epsilon^r = \frac{e^{e^X}}{X^{\omega'}}.\epsilon^r > \frac{e^{e^X}}{X^{\omega'+1}} > 1 \tag{3.17}$$

It follows the contradiction and we obtain:

$$\frac{6}{5}\frac{1}{\epsilon^\omega}(1-B') < 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} \Longrightarrow c < \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} < 1.2e^{e^{\left(\frac{1}{\epsilon^4}\right)}} R^{1+\epsilon} \tag{3.18}$$

Finally, the choice of the constant $K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$ is acceptable for $\epsilon \in]0,1[$. As the conjecture $c < R^2$ is true, we adopt $K(\epsilon) = 1.2e^e$ for $\epsilon \geq 1$, and the abc conjecture is true for all $\epsilon > 0$.

The proof of the abc conjecture is finished.

The End of the Mystery
of the abc Conjecture!

□

4. Examples

We give below some numerical examples.

4.1. **Example 1. of Eric Reyssat.**

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6\,436\,343 \tag{4.19}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19\,683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2, c = 23^5 = 6\,436\,343 \Rightarrow rad(c) = 23$.
Then $R = rad(abc) = 2 \times 3 \times 109 \times 23 = 15\,042$.

* $\omega = 4 \Longrightarrow A = (Log 15\,042)^4/24 = 356.6452\,953, 1 - B = A/R = 0.0237\,099$.
$\dfrac{5}{6} = 0.8333\,833 < \dfrac{R^2(1-B)}{c} = 0.8334\,925$.

* $\omega = 4, \omega' = 1, r = 0$.

case $\epsilon = 0.5 \Longrightarrow X = \dfrac{1}{0.5^4} = 16, X^2 = 256 \ll \epsilon^X = 8\,886\,110.52 \Longrightarrow (1 - B') < 1$.
$\omega = 4 \Longrightarrow A' = \dfrac{(LogR^{0.63})^4}{24} = 0.252 \Longrightarrow \dfrac{6}{5}R\dfrac{(LogR)^\omega}{\omega!} = 6\,437\,590.238 > (c = 6\,436\,343)$. $R^{0.63} = 428.255 \Longrightarrow B' = \dfrac{428.255 - 0.252}{428.255} = 0.9994 \Longrightarrow 1 - B' = 0.000\,6 < \epsilon^4.e^{e^{16}}$.

4.2. **Example 2. of Nitaj.**

See [4]:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953$$
$$R = rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \Longrightarrow$$
$$*\,\omega = 10,\ \frac{6}{5}.R.\frac{(LogR)^{10}}{10!} = 7\,794\,478\,289\,809\,729\,132\,015,590 >$$
$$c = 613\,474\,845\,886\,230\,468\,750,\ A = 225\,312\,992.5562\,633 \Longrightarrow$$
$$1 - B = A/R = 7.815 \times 10^{-6} \Longrightarrow \frac{5}{6} = 0.8333 < \frac{R^2(1-B)}{c} = 10.587$$

* $\omega = 10 \Longrightarrow \omega' = 2, r = 2 \Longrightarrow$:
Case $\epsilon = 0.5 \Longrightarrow X = 16, X^3 = 4096 \ll e^X = 8\,886\,110.52 \Longrightarrow (1 - B') < 1$.
Case $\epsilon = 0.001 \Longrightarrow X = 10^{12} \Longrightarrow X^3 = 10^{36} \ll e^X \Longrightarrow (1 - B') < 1$.

4.3. **Example 3. of Ralf Bonse.** See [2] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$R = rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$R = 1.5683959920004546031461002610848 \times 10^{+33}$$
$$*\omega = 10 \Longrightarrow \alpha = \frac{6}{5}R\frac{(LogR)^\omega}{\omega!} = 3.5303452259448631166310839830891 \times 10^{+45} >$$
$$3.4136998783296235160378273576498 \times 10^{+44} = c,\ 1 - B = 1.1959\,816 \times 10^{-21} \Longrightarrow$$
$$\frac{5}{6} = 0.8333 \ll 8.618 = \frac{R^2(1-B)}{c}$$

* $\omega = 10, \omega' = 2, r = 2$, case $\epsilon = 0.5 \Longrightarrow X = 16, X^3 = 4096 \ll e^X = 8\,886\,110.521 \Longrightarrow (1 - B') < 1$.

Case $\epsilon = 0.001 \Longrightarrow X = 10^{12} \Longrightarrow X^3 = 10^{36} \ll e^X \Longrightarrow (1 - B') < 1$.

5. Conclusion

Assuming the explicit abc conjecture of Alan Baker true, we can announce the important theorem:

Theorem 5.5. *Assuming the explicit abc conjecture of Alan Baker true, then the abc conjecture is true:*
For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{5.20}$$

where K is a constant depending only on ϵ. For $\epsilon \in]0, 1[$, $K(\epsilon) = 1.2e^{e^{\left(\frac{1}{\epsilon}\right)^4}}$ and $K(\epsilon) = 1.2e^e$ if $\epsilon \geq 1$.

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- ORCID - ID:0000-0002-9633-3330.

References

[1] M. Waldschmidt: On the abc Conjecture and some of its consequences, presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013. (2013). https://zbmath.org/1395.11056.
[2] B. De Smit: https://www.math.leidenuniv.nl/ desmit/abc/. Accessed December 2020.
[3] P. Mihăilescu: Around ABC, European Mathematical Society Newsletter, N° 93, 29-34. (2014). https://zbmath.org/1384.11061.
[4] A. Nitaj: Aspects expérimentaux de la conjecture abc. Séminaire de Théorie des Nombres de Paris (1993-1994), London Math. Soc. Lecture Note Ser., Vol °235, 145-156, Cambridge Univ. Press. (1996). https://zbmath.org/0920.11090.
[5] A. Baker: Experiments on the abc-conjecture, Publ. Math. Debrecen 65:3-4, pp. 253-260. (2004). https://zbmath.org/1064.11050.

A. Ben Hadj Salem
Résidence Bousten 8, Av. Mosquée Raoudha,
Bloc B, Appart B8
1181 Soukra Raoudha, Soukra, Tunisia
Email address: abenhadjsalem@gmail.com