# A BRIEF STUDY ON SOLITAIRE MODULO 3

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**Abstract**

In this short article, we will discuss a card game, from now on namely *Solitaire modulo 3.* After having described how it works, through a probabilistic calculation, we will arrive at determining the probability of victory. In particular, we will use the *rook polynomials*, which will allow us to finally obtain a closed form for calculating the probability of winning at *Solitaire modulo 3.* Finally, we will study the case where the number of cards in play is much more greater than the number of constraints present in the game format. Under this assumption, the *Solitaire modulo 3* mechanism becomes asymptotically equivalent to a binomial distribution.

## Game formats

The game is played as follows. Choose any face-down deck of cards and begin turning over the cards one at a time, repeatedly counting "one", "two", "three", "one", "two", "three"..., for each card turned. The game ends, and you win, if an ace (with a value of one) never corresponds to the "one" position, or a two never corresponds on the "two" position, or a three never corresponds to the "three" position. Otherwise, the game stops, and you lose, as soon as one of the above conditions occurs.

We understand that this is just one possible version of a game with a similar functioning: the number of cards can vary, the number of counts can vary (thus becoming a *Solitaire modulo n*), the number of suits of the cards in the same deck can vary....

## Estimating win probability

In this dissertation, we will base the game on a classic *French* deck of 52 cards, divided into four different suits (hearts, diamonds, clubs, spades) with values as indicated below.

$$A \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad J \quad Q \quad K$$

The question at the basis of this article is about the probability of winning in a game like *Solitaire module 3.* The question, although it's trivial, doesn't have an equally obvious answer. In fact, the difficulty of the answer lies in calculating all the possible losing combinations in the game.

To do this, we will introduce the so-called *rook polynomials*, which are generating polynomials whose coefficients determine the number of ways to arrange non-attacking rooks on a chessboard [F B91]. At the moment, the connection between this concept and *Solitaire modulo 3* may not be immediately clear; It will become so as we continue with the solution, as it will come out naturally.

Let us start by analyzing the event space. Let $\Omega$ be the set of all possible permutations of the 52 cards, that is, $|\Omega| = 52!$. Let $\Omega_k$, with $k = 1, 2, 3, \ldots, 12$, be the set of all permutations of the possible forbidden pairs (e.g., the ace of hearts is in the "one" position). Consequently, we can state

$$\left|\Omega \setminus \bigcup_{k=1}^{12} \Omega_k\right| = |\Omega| - \left|\bigcup_{k=1}^{12} \Omega_k\right|$$

From the inclusion-exclusion principle [And02], we get

$$\left|\Omega \setminus \bigcup_{k=1}^{12} \Omega_k\right| = |\Omega| + \sum_{k=1}^{12} (-1)^k \left( \sum_{1 \le i_1 < \cdots < i_k \le 12} \left|\Omega_{i_1} \cap \cdots \cap \Omega_{i_k}\right| \right)$$

The previous writing can be compacted by defining the set $\Omega$ as an empty intersection, namely in the case $k = 0$, whence

$$\left|\Omega \setminus \bigcup_{k=1}^{12} \Omega_k\right| = \sum_{k=0}^{12} (-1)^k \left( \sum_{1 \le i_1 < \cdots < i_k \le 12} \left|\Omega_{i_1} \cap \cdots \cap \Omega_{i_k}\right| \right)$$

Let us imagine, now, a chessboard with some forbidden positions and a certain number of $n$ cards to be placed in that same number of positions. The problem now involves counting all permutations in which no card lands on a forbidden position. In terms of *Solitaire modulo 3*: a row of the aforementioned chessboard represents a card (e.g., A, 2, 3,. . . ,K); a column represents a position (e.g., "one", "two", "three"); a forbidden position is a pair of the form A-"one", 2-"two", 3-"three".

The board just defined has dimensions $52 \times 52$. This representation also takes into account the cards not in play, that is, the cards from 4 to K. To simplify the reasoning and maintain a cleaner writing, we will think in blocks. Let us consider three blocks defined as follows.

- $\alpha$ block: represents the forbidden A-"one" pairs, and it has size $4 \times 18$.
- $\beta$ block: represents the forbidden 2-"two" pairs, and it has size $4 \times 17$.
- $\gamma$ block: represents the forbidden 3-"three" pairs, and it has size $4 \times 17$.

At this point, we will use the newly defined block division to determine the last numerical series of the previous equality. To do this, for each block, we introduce the *rook polynomials*, whose coefficients determine the ways to select $k$ forbidden pairs, with $k = 0, 1, 2, 3, 4$. For example, let $k = 0, 1, 2, 3, 4$, in the case of the $\alpha$ block, the choice of cards to use can be made in $\binom{4}{k}$ distinct ways; the choice of positions can be made in $\binom{18}{k}$ distinct ways. Furthermore, once these choices are defined, which reduce the initial block into a new block of size $k^2$, the number of ways to arrange $k$ cards in this scheme such that there are exactly $k$ distinct violations is equal to $k!$. In other words, we obtain

$$r_k^\alpha = \binom{4}{k}\binom{18}{k}k!, \qquad k = 0, 1, 2, 3, 4.$$

In the case of the $\beta$ and $\gamma$ blocks, which are distinct for the purposes of the game, but identical for practical purposes, we have

$$r_k^\beta = r_k^\gamma = \binom{4}{k}\binom{17}{k}k!, \qquad k = 0, 1, 2, 3, 4.$$

Hence, for each block, we can define the rook polynomials as

$$R^\alpha(x) = \sum_{k=0}^{4} r_k^\alpha x^k, \qquad R^\beta(x) = R^\gamma(x) = \sum_{k=0}^{4} r_k^\alpha x^k.$$

We note that the three blocks are independent of each other, that is, they do not share rows or columns, as both the cards and the positions are different. Furthermore, the product of the rook polynomials defined above corresponds precisely to the independent combination of the correspondences in the three blocks. In other words, the coefficients of this new polynomial, obtained by *convolution* of the three starting polynomials, count all global configurations (of the entire board) and distinct ones relating to the forbidden pairs [Zen96]. From what has just been said, we can write

$$R(x) = R^\alpha(x) \cdot R^\beta(x)^2 = \sum_{k=0}^{12} r_k x^k.$$

With this last definition, the term $r_k$ is defined as

$$r_k = \sum_{\substack{i+j+l=k \\ 0 \le i,j,l \le 4}} r_i^\alpha r_j^\beta r_l^\gamma, \qquad k = 0, 1, 2, \dots, 12.$$

Thanks to this last part, we can say

$$\sum_{k=0}^{12} (-1)^k \left( \sum_{1 \le i_1 < \cdots < i_k \le 12} \left| \Omega_{i_1} \cap \cdots \cap \Omega_{i_k} \right| \right) = \sum_{k=0}^{12} (-1)^k r_k (52-k)!.$$

In fact, let $k = 1, 2, 3, \dots, 12$, $\Omega_k$ is the set of all permutations of the possible forbidden pairs. Hence $\sum_{1 \le i_1 < \cdots < i_k \le 12} \left| \Omega_{i_1} \cap \cdots \cap \Omega_{i_k} \right|$ is the number of all permutations in which each of the chosen cards falls into a position relative to a forbidden pair. Furthermore, if the chosen cards fall into positions that are not compatible with the dynamics of the game (e.g., the same card falls into two different positions), then

$$\bigcap_{1 \le i_1 < \cdots < i_k \le 12} \left( \Omega_{i_1} \cap \cdots \cap \Omega_{i_k} \right) = \emptyset,$$

whence

$$\sum_{1 \le i_1 < \cdots < i_k \le 12} \left| \Omega_{i_1} \cap \cdots \cap \Omega_{i_k} \right| = 0.$$

Conversely, if the chosen cards fall into positions compatible with the dynamics of the game, then exactly $k$ forbidden pairs are fixed. After placing these $k$ cards on the board, $52-k$ free cards and the same number of free positions in which to place these cards remain, and this can be done in $(52-k)!$ ways. Consequently, we can state

$$\left|\Omega \setminus \bigcup_{k=1}^{12} \Omega_k\right| = \sum_{k=0}^{12} (-1)^k r_k (52-k)!.$$

To conclude, after tedious calculations, we obtain that the probability of winning at *Solitaire modulo 3* is equal to

$$\frac{\left|\Omega \setminus \bigcup_{k=1}^{12} \Omega_k\right|}{|\Omega|} = \frac{1}{52!} \sum_{k=0}^{12} (-1)^k r_k (52-k)! \simeq 0,0082.$$

# Asymptotic approximations

In this final section, we will give an asymptotic estimate of the probability of winning at *Solitaire modulo 3* [I A81]. In particular, we will analyze the case where the number of cards is far greater than the number of constraints $k$, with $k = 0, 1, 2, 3, \ldots, 12$. More formally, calling $N$ the number of cards in play, we will now study the limit case $N \to +\infty$.

Without loss of generality, by reasoning as previously done in the case of *Solitaire modulo 3*, we can assume that the three blocks have the same size $4 \times (N/3)$. Consequently, for each block, we introduce the coefficients of the rook polynomials as

$$r_k^\alpha = r_k^\beta = r_k^\gamma = \binom{4}{k}\binom{N/3}{k} k!, \qquad k = 0, 1, 2, 3, 4.$$

From the starting hypothesis, for $N \to +\infty$, we obtain the following asymptotic estimate

$$\binom{N/3}{k} = \frac{(N/3)^k}{k!} + \mathcal{O}(N^{k-1}), \qquad k = 0, 1, 2, 3, 4.$$

Consequently, for $N \to +\infty$, we have

$$r_k^\alpha = r_k^\beta = r_k^\gamma = \binom{4}{k}\left(\frac{N}{3}\right)^k + \mathcal{O}(N^{k-1}), \qquad k = 0, 1, 2, 3, 4.$$

At this point, for $N \to +\infty$, we can define the rook polynomials as an asymptotic estimate

$$R^\alpha(x) = R^\beta(x) = R^\gamma(x) = \sum_{k=0}^{4} \binom{4}{k}\left(\frac{N}{3}\right)^k x^k + \mathcal{O}(N^4) = \left(1 + \frac{N}{3}x\right)^4 + \mathcal{O}(N^4).$$

For $N \to +\infty$, we can asymptotically estimate the product of the three rook polynomials above as

$$R(x) = R^\alpha(x) R^\beta(x) R^\gamma(x) = \left(1 + \frac{N}{3}x\right)^{12} + \mathcal{O}(N^{12}) = \sum_{k=0}^{12} \binom{12}{k}\left(\frac{N}{3}\right)^k x^k + \mathcal{O}(N^{12}).$$

In particular, for $N \to +\infty$, the coefficients of the polynomial just defined are asymptotically equivalent to

$$r_k = \binom{12}{k}\left(\frac{N}{3}\right)^k + \mathcal{O}(N^k), \qquad k = 0, 1, 2, 3, \ldots, 12.$$

Always following the reasoning made in the previous discussion regarding *Solitaire modulo 3*, we have that, for $N \to +\infty$, the probability of winning is asymptotically equivalent to

$$\begin{aligned}
\frac{1}{N!}\sum_{k=0}^{12}(-1)^k r_k (N-k)! &= \sum_{k=0}^{12}(-1)^k\left(\binom{12}{k}\left(\frac{N}{3}\right)^k + \mathcal{O}(N^k)\right)\left(\frac{1}{N^k} + \mathcal{O}\left(\frac{1}{N^{k+1}}\right)\right) \\
&= \sum_{k=0}^{12}(-1)^k\left(\binom{12}{k}\left(\frac{1}{3}\right)^k + \mathcal{O}\left(\frac{1}{N}\right)\right) \\
&= \left(\frac{2}{3}\right)^{12}.
\end{aligned}$$

In other words, this means that the *Solitaire modulo 3* is asymptotically equivalent to 12 independent experiments, each with a 2/3 probability of not causing any failure [Asm03].

# References


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