Integer Encoding Symmetries of the Riemann Zeta Function

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Abstract:

Here I present a derivation of an equation whose solution sets are the trivial and nontrivial zeros of the Riemann Zeta Function. I demonstrate how the trivial solutions are directly encoded by integer inputs and how these can be mapped by a symmetry to positive odd integers. I extend this insight to encode the even integers, and map these to the negative odd integers, which provides an explicit connection between particular values of the Riemann Zeta Function which have historical and ongoing research interest. I then extend this symmetry to the nontrivial zeroes, and demonstrate the dependence of the critical line in producing this symmetry. Finally, I note the distribution of the nontrivial zeroes have a correspondence with the distribution of trivial zeroes, and provide a first order approximation of this correspondence.

Introduction:

The zeroes of the Riemann Zeta Function rely on the unique property of the ratio (1):

$$\frac{a+b\,i}{(a+b\,i)-1} = \frac{2\,b-i}{2\,b+i} \text{ where } a = \frac{1}{2} \tag{1}$$

In that for any choice of b, this ratio produces a unit scaled complex number with decreasing exponent c for increasing b.

$$\frac{2\,b-i}{2\,b+i} = e^{-c\,i}$$

$$c = i\log\left(\frac{2\,b-i}{2\,b+i}\right) \tag{2}$$

This exponent c, can be associated with a unique value q:

$$\log(c + 2\pi) = \log\left(\frac{1}{q} + 1\right)$$

$$i\log\left(\frac{2b - i}{2b + i}\right) + 2\pi = \frac{1}{q} + 1$$

$$q = \frac{1}{i\log\left(\frac{2b-i}{2b+i}\right) + 2\pi - 1} \tag{3}$$

This value q, can be replicated with the equation (4):

$$q = \frac{1}{-1 + e^{i(2\pi+x)\, e^{i(2\pi+y)}}} \tag{4}$$

Where,

$$\mathrm{Im}\left(\frac{1}{-1 + e^{i(2\pi+x)\, e^{i(2\pi+y)}}}\right) = 0$$

Which gives an expression for x in terms of y:

$$x = -2\pi + 2\pi\sec(y) \tag{5}$$

Substituting (5) for x in (4).

$$q = \frac{1}{-1 + e^{2i\, e^{iy}\, \pi\sec(y)}} \tag{6}$$

We can then equate (3) and (6):

$$\frac{1}{i\log\left(\frac{2b-i}{2b+i}\right)+2\pi-1}=\frac{1}{-1+e^{2ie^{iy}\pi\sec(y)}} \tag{7}$$

Is it possible to solve for b in terms of y:

$$i\log\left(\frac{2b-i}{2b+i}\right)+2\pi-1=-1+e^{2ie^{iy}\pi\sec(y)}$$

$$i\log\left(\frac{2b-i}{2b+i}\right)=e^{-2\pi\tan(y)}-2\pi$$

$$b=\frac{i\left(1+e^{ie^{-2\pi\tan(y)}}\right)}{2\left(-1+e^{ie^{-2\pi\tan(y)}}\right)} \tag{8}$$

Likewise, you can solve for y in terms of b:

$$y=-\tan^{-1}\left(\frac{\log\left(2\pi-i\log\left(\frac{i+2b}{-i+2b}\right)\right)}{2\pi}\right) \tag{9}$$

A list of corresponding y values for the first 40 zeroes of the Zeta Function is included in Appendix A.

Integer mapping of the Trivial Zeroes of the Zeta Function

Finally, the point being that it is possible to construct the function (10) from (4) and (5):

$$f(y) = \frac{1}{1 - e^{i e^{-2\pi \tan(y)}}} \tag{10}$$

Such that it generates the solutions to the zeroes of the zeta function.

$$\zeta(f(y)) = 0$$

This function is particularly useful because it has 2 valid solution sets. The first solution set encodes the trivial zeroes of the Zeta Function through integer inputs:

$$y = \pi - i \tanh^{-1}\left(\frac{1}{2\pi}\left(2\pi - i \log\left(2\pi - i \log\left(1 + \frac{1}{2n}\right)\right)\right)\right) \tag{11}$$

$$f(y) = -2n$$

Most remarkable is that it is possible to map from the trivial zeroes to the odd integers with single sign change for y:

$$y = \pi - i \tanh^{-1}\left(\frac{1}{2\pi}\left(2\pi - i \log\left(2\pi + i \log\left(1 + \frac{1}{2n}\right)\right)\right)\right)$$

$$f(y) = 2n + 1$$

Likewise, the positive even integers (although not zeroes) can be encoded by:

$$y = \pi - i \tanh^{-1}\left(\frac{1}{2\pi}\left(2\pi - i \log\left(2\pi + i \log\left(1 + \frac{1}{-1 + 2n}\right)\right)\right)\right)$$

$$f(y) = 2n$$

And mapped to the negative even integers with a similar single sign change:

$$y=\pi-i\tanh^{-1}\left(\frac{1}{2\pi}\left(2\pi-i\log\left(2\pi-i\log\left(1+\frac{1}{-1+2n}\right)\right)\right)\right)$$

$$f(y)=1-2n$$

Dependence on Critical line

The second solution set encodes the nontrivial zeroes through the ratio (1), where a p_n is the nth nontrivial zero of the Zeta Function.

$$y=\pi-i\tanh^{-1}\left(\frac{1}{2\pi}\left(2\pi-i\log\left(2\pi-i\log\left(1-\frac{1}{\rho_n}\right)\right)\right)\right) \tag{12}$$

However, we cannot rely on the cyclical knowledge of already knowing the nth zero. We can, however, explore what this solution set maps to following the same method as with the trivial zeroes.

We find that:

$$y=\pi-i\tanh^{-1}\left(\frac{1}{2\pi}\left(2\pi-i\log\left(2\pi-i\log\left(1-\frac{1}{a+ib}\right)\right)\right)\right)$$

$$f(y)=a+b\,i$$

and:

$$y=\pi-i\tanh^{-1}\left(\frac{1}{2\pi}\left(2\pi-i\log\left(2\pi+i\log\left(1-\frac{1}{a+ib}\right)\right)\right)\right)$$

$$f(y)=-a-b\,i+1$$

Which reveals the dependence on a=1/2 in mapping values to their horizontal reflection.

Distribution of Nontrivial Zeroes

The symmetry of the trivial and nontrivial solutions sets indicates a correspondence between the distribution of the trivial zeroes and the nontrivial zeroes. In particular, the trivial zeroes generated by integer inputs in (11):

$$i \log\left(1 + \frac{1}{2\,n}\right)$$

Seems to have some correspondence with the nontrivial zeroes in (12):

$$i \log\left(1 - \frac{1}{\rho_n}\right) \tag{13}$$

I have not yet found an exact pattern but have explored that (14) provides a decent first order approximation of (13), which drifts with increasing zero values.

$$\frac{1}{5\,\sqrt{2}\,(n+1)} \tag{14}$$

References

Meyer, D. (n.d.). Euler's product formula and the Riemann zeta function. https://davidmeyer.github.io/qc/Euler_product_formula_for_the_Riemann_zeta_function.pdf

Riemann, B. (1859). On the Number of Prime Numbers less than a Given Quantity. (D. R. Wilkins, Trans.).*Monatsberichte Der Berliner Akademie*.

Appendix A. Solutions for (8) and (10)

ZERO	B	Y
1	14.13473	-0.286208711
2	21.02204	-0.285673812
3	25.01086	-0.285498204
4	30.42488	-0.285333282
5	32.93506	-0.285275162
6	37.58618	-0.285187938
7	40.91872	-0.285137607
8	43.32707	-0.285106044
9	48.00515	-0.285053767
10	49.77383	-0.285036558
11	52.97032	-0.285008366
12	56.44625	-0.284981327
13	59.34704	-0.284961183
14	60.83178	-0.284951615
15	65.11254	-0.284926468
16	67.07981	-0.284915986
17	69.5464	-0.284903681
18	72.06716	-0.284891974
19	75.70469	-0.284876454
20	77.14484	-0.284870713
21	79.33738	-0.284862373
22	82.91038	-0.284849726
23	84.73549	-0.284843677
24	87.42527	-0.284835222
25	88.80911	-0.284831071
26	92.4919	-0.28482063
27	94.65134	-0.284814885
28	95.87063	-0.284811755
29	98.83119	-0.284804478
30	101.3179	-0.284798693
31	103.7255	-0.284793357
32	105.4466	-0.284789691
33	107.1686	-0.284786141
34	111.0295	-0.284778582
35	111.8747	-0.284776997
36	114.3202	-0.284772542
37	116.2267	-0.284769199
38	118.7908	-0.284764872
39	121.3701	-0.284760704
40	122.9468	-0.284758242