HCR's Theorem for dihedral angles in a regular n-gonal right pyramid

Harish Chandra Rajpoot 10 July 2015

M.M.M. University of Technology, Gorakhpur-273010 (UP), India

Abstract

In this paper, a theorem is formulated and proved that yields generalized closed-form expressions for the dihedral angle between any two arbitrary lateral faces of a regular n-gonal right pyramid. The dihedral angles are expressed in terms of the apex angle, defined as the angle between two adjacent lateral edges meeting at the apex. The proposed formulation establishes a direct analytical relationship between the edge geometry at the apex and the corresponding dihedral angles of the pyramid. Due to its generality, the theorem applies to all regular and uniform polyhedra whose vertex configuration coincides with that of a right pyramid, as well as to regular n-gonal right prisms with an arbitrary number of sides. The resulting formulas are useful for geometric modeling, construction of physical models, and the development of computational algorithms for the analysis of polyhedral structures and equally inclined sets of concurrent vectors in three-dimensional space.

Keywords: Dihedral angle, apex angle, regular n-gonal right pyramid, right prism, polyhedral geometry

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1. Introduction

A regular polygonal right pyramid is a right pyramid that has a regular polygonal base and all its lateral faces are congruent isosceles triangles equally inclined, at an acute angle, with its geometric axis [1]. The dihedral angle is the angle between two intersecting planes measured perpendicular to their line of intersection [2]. In this paper, a theorem has been proposed and derived by the author for computing the dihedral angle between any two arbitrary lateral faces of a regular n-gonal right pyramid, expressed in terms of the apex angle, defined as the angle between two adjacent lateral edges meeting at the apex, and the number of lateral faces. The stated assumes a tetrahedron with two extended and intersecting faces as arbitrary planes and third one is formed by the unextended edges and applying inverse cosine formula [3]. The formula derived here is equally applicable to regular n-gonal right prism, and is also useful for determining the dihedral angles between any two arbitrary faces meeting at the vertex of various regular and uniform polyhedra [4-8]. A pyamidal shell can be easily constructed/framed by continuously fixing all its adjacent (flat) triangular faces each two as a pair at their common edge at an angle equal to the dihedral angle between them.

2. Dihedral angles in a regular n-gonal right pyramid and prism

2.1. HCR's Theorem: The dihedral angle i.e. angle of inclination δ_{1r} between *1*rst face (i.e. reference face) and r^{th} face (i.e. arbitrary face) meeting at the apex of a regular n-gonal right pyramid with apex angle α (i.e. angle between any two adjacent lateral edges), is given by the following generalized formula,

$$\boldsymbol{\delta_{1r} = 2\cos^{-1}\left(\sin\left(\frac{(r-1)\pi}{n}\right)\sqrt{1-\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}\right)^2}\right)} \quad \forall\ r, n \in N,\ r \le n,\ n \ge 3,\ \alpha < \frac{2\pi}{n},\ \delta_{1r} \in (0, \pi] \quad \ldots\ldots(1)$$

Proof: Consider a right pyramid with a regular n-gonal base $A_1A_2A_3 \ldots . A_rA_{r+1} \ldots\ A_{n-1}A_n$ with each side a and

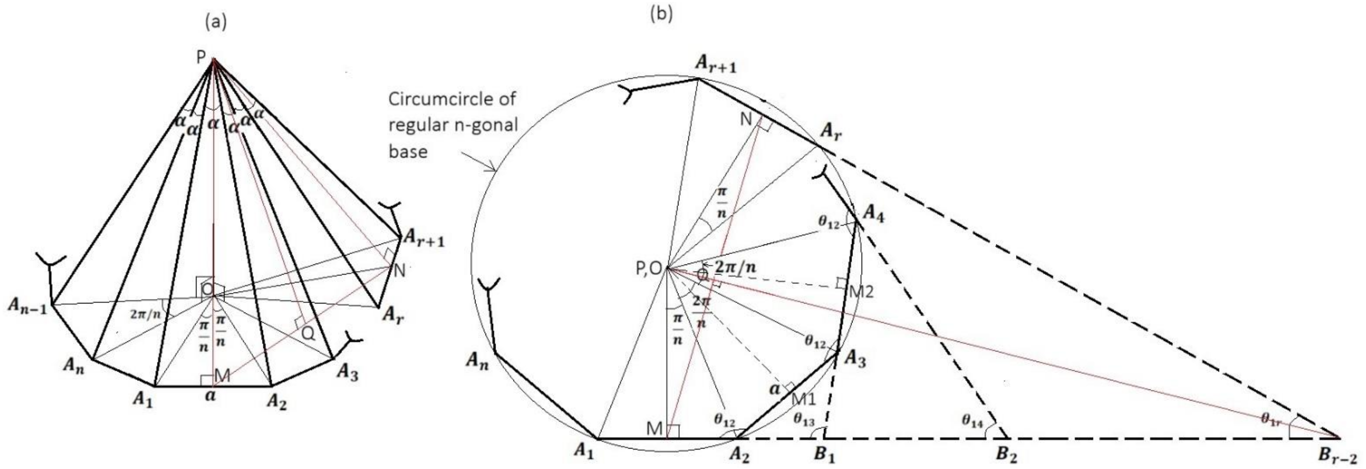

Figure 1: (a) The faces PA_1A_2 and PA_rA_{r+1} are 1rst and rth lateral faces, each with an apex angle α, in a regular n-gonal right pyramid, (b) regular polygonal base with n number sides each of length a (top view).

its lateral faces meeting at the apex P such that α is the apex angle i.e. the angle between any two consecutive lateral edges meeting at the apex P (as shown in Figure 1). Now, consider an arbitrary lateral face PA_rA_{r+1} and drop perpendiculars PO and PN from the apex P to the polygonal base and side A_rA_{r+1} respectively, and perpendicular ON from the centre O to the side A_rA_{r+1}(as shown in Figure 1(a)).

In right ΔPNA_r (Fig. 1(a)),

$$\tan\angle A_rPN = \frac{NA_r}{PN} \Rightarrow \tan\frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{PN} \qquad \left(\because \angle A_rPN = \frac{\angle A_rPA_{r+1}}{2} = \frac{\alpha}{2},\ NA_r = \frac{A_rA_{r+1}}{2} = \frac{a}{2}\right)$$

$$\Rightarrow PN = \frac{a}{2}\cot\frac{\alpha}{2} \qquad \ldots\ldots\ldots..(2)$$

In right ΔONA_r (Fig. 1(b)),

$$\tan\angle A_rON = \frac{NA_r}{ON} \Rightarrow \tan\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{ON} \qquad \left(\because \angle A_rON = \frac{\angle A_rOA_{r+1}}{2} = \frac{\pi}{n},\ NA_r = \frac{A_rA_{r+1}}{2} = \frac{a}{2}\right)$$

$$\Rightarrow ON = \frac{a}{2}\cot\frac{\pi}{n} \qquad \ldots\ldots\ldots..(3)$$

From the above Figure 1(b), the angles between the perpendiculars drawn from the centre O to the side A_1A_2 (i.e. reference side) and any arbitrary side say A_rA_{r+1} are given as follows

$$\text{Angle between perpendiculars drawn from O to sides } A_1A_2 \text{ and } A_2A_3 = \angle MOM1 = \frac{2\pi}{n} = (2-1)\frac{2\pi}{n}$$

$$\text{Angle between perpendiculars drawn from O to sides } A_1A_2 \text{ and } A_3A_4 = \angle MOM2 = \frac{4\pi}{n} = (3-1)\frac{2\pi}{n}$$

$$\text{Angle between perpendiculars drawn from O to sides } A_1A_2 \text{ and } A_4A_5 = \frac{6\pi}{n} = (4-1)\frac{2\pi}{n}$$

$$\text{Angle between perpendiculars drawn from O to sides } A_1A_2 \text{ and } A_5A_6 = \frac{8\pi}{n} = (5-1)\frac{2\pi}{n}$$

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Similarly, angle between perpendiculars drawn from O to sides A_1A_2 and $A_rA_{r+1} = \angle MON = (r-1)\frac{2\pi}{n}$

$$\therefore\ \angle MON = (r-1)\frac{2\pi}{n} \quad \dots\dots\dots..(4)$$

Now, extend the base sides $A_3A_4, A_4A_5, \dots\dots,$ and A_rA_{r+1} so that these intersect the first side A_1A_2 at the points $B_1, B_2, \dots\dots, B_{r-2}$ respectively (as shown in Figure 1(b). Draw the straight lines MN, and OB_{r-2} that intersect each other at the point Q at a right angle (Fig. 1(b)).

In right ΔOQM (Fig. 1(b)),

$$\sin\angle MOQ = \frac{QM}{OM} \Rightarrow \sin\frac{\angle MON}{2} = \frac{QM}{ON} \quad \left(\because \angle MOQ = \frac{\angle MON}{2},\ OM = ON\right)$$

$$\sin\left(\frac{(r-1)\frac{2\pi}{n}}{2}\right) = \frac{QM}{\frac{a}{2}\cot\frac{\pi}{n}} \quad (\text{setting values from Eq}(3)\text{ and }(4))$$

$$\Rightarrow\ QM = \frac{a}{2}\cot\frac{\pi}{n}\sin\left(\frac{(r-1)\pi}{n}\right) \quad \dots\dots\dots..(5)$$

In right ΔOMB_{r-2} (Fig. 1(b)),

$$\tan\angle MOB_{r-2} = \frac{MB_{r-2}}{OM} \Rightarrow \tan\frac{\angle MON}{2} = \frac{MB_{r-2}}{ON} \quad \left(\because \angle MOB_{r-2} = \angle MOQ = \frac{\angle MON}{2},\ OM = ON\right)$$

$$\tan\left(\frac{(r-1)\frac{2\pi}{n}}{2}\right) = \frac{MB_{r-2}}{\frac{a}{2}\cot\frac{\pi}{n}} \quad (\text{setting values from Eq}(3)\text{ and }(4))$$

$$\Rightarrow\ MB_{r-2} = \frac{a}{2}\cot\frac{\pi}{n}\tan\left(\frac{(r-1)\pi}{n}\right) \quad \dots\dots\dots..(6)$$

Now, extend the first isosceles triangular lateral face PA_1A_2 (i.e. reference face) and the r[th] isosceles triangular lateral face PA_rA_{r+1}(i.e. arbitrary face) that intersect each other at the line PB_{r-2} at an angle i.e. dihedral angle δ_{1r}. Now, drop the perpendicular PQ from apex P to the line MN and let $\angle MPN = \beta$ (as shown in Figure 2).

In right ΔPQM (Fig. 2),

$$\sin\angle MPQ = \frac{QM}{PM}$$

$$\sin\frac{\angle MPN}{2} = \frac{QM}{PN} \quad (\because PM = PN)$$

Substituting the values from (2) & (5),

$$\Rightarrow\ \sin\frac{\beta}{2} = \frac{\frac{a}{2}\cot\frac{\pi}{n}\sin\left(\frac{(r-1)\pi}{n}\right)}{\frac{a}{2}\cot\frac{\alpha}{2}}$$

$$\sin\frac{\beta}{2} = \frac{\tan\frac{\alpha}{2}\sin\left(\frac{(r-1)\pi}{n}\right)}{\tan\frac{\pi}{n}}$$

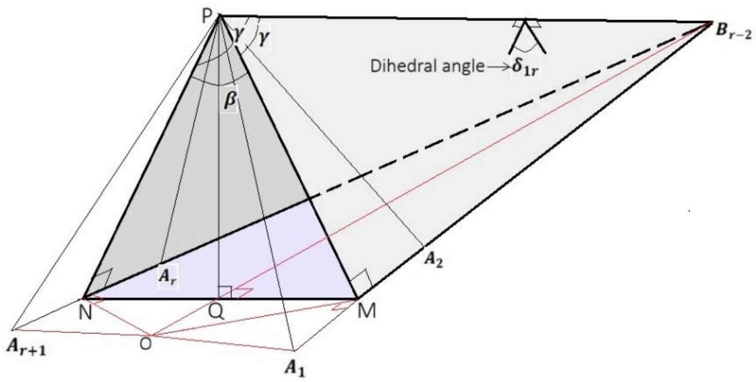

Figure 2: A tetrahedron $PMNB_{r-2}$ formed by joining apex P of right pyramid to the point M, N, and B_{r-2} that has faces as two congruent right triangles ΔPMB_{r-2} & ΔPNB_{r-2}, and two isosceles triangles ΔPMN & ΔMNB_{r-2}.

$$\Rightarrow \cos\beta = 1 - 2\sin^2\frac{\beta}{2} = 1 - \left(\frac{\tan\frac{\alpha}{2}\sin\left(\frac{(r-1)\pi}{n}\right)}{\tan\frac{\pi}{n}}\right)^2$$

$$\cos\beta = \frac{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}\sin^2\left(\frac{(r-1)\pi}{n}\right)}{\tan^2\frac{\pi}{n}} \quad \dots\dots\dots..(7)$$

In right ΔPMB_{r-2} (Fig. 2),

$$\tan\angle MPB_{r-2} = \frac{MB_{r-2}}{PM} \Rightarrow \tan\gamma = \frac{MB_{r-2}}{PN} \qquad (\text{Let } \angle MPB_{r-2} = \angle NPB_{r-2} = \gamma)$$

Substituting values of PN and MB_{r-2} from Eq(2) and (6) respectively into above equation,

$$\tan\gamma = \frac{\frac{a}{2}\cot\frac{\pi}{n}\tan\left(\frac{(r-1)\pi}{n}\right)}{\frac{a}{2}\cot\frac{\alpha}{2}} = \frac{\tan\frac{\alpha}{2}\tan\left(\frac{(r-1)\pi}{n}\right)}{\tan\frac{\pi}{n}}$$

$$\Rightarrow \cos\gamma = \frac{1}{\sec\gamma} = \frac{1}{\sqrt{1+\tan^2\gamma}} = \frac{1}{\sqrt{1+\left(\frac{\tan\frac{\alpha}{2}\tan\left(\frac{(r-1)\pi}{n}\right)}{\tan\frac{\pi}{n}}\right)^2}} \qquad \left(\because\ 0<\gamma<\frac{\pi}{2}\right)$$

$$\cos\gamma = \frac{\tan\frac{\pi}{n}}{\sqrt{\tan^2\frac{\pi}{n} + \tan^2\frac{\alpha}{2}\tan^2\left(\frac{(r-1)\pi}{n}\right)}} \quad \dots\dots\dots..(8)$$

$$\Rightarrow \sin\gamma = \frac{\tan\gamma}{\sec\gamma} = \frac{\tan\gamma}{\sqrt{1+\tan^2\gamma}} = \frac{\frac{\tan\frac{\alpha}{2}\tan\left(\frac{(r-1)\pi}{n}\right)}{\tan\frac{\pi}{n}}}{\sqrt{1+\left(\frac{\tan\frac{\alpha}{2}\tan\left(\frac{(r-1)\pi}{n}\right)}{\tan\frac{\pi}{n}}\right)^2}} \qquad \left(\because\ 0<\gamma<\frac{\pi}{2}\right)$$

$$\sin\gamma = \frac{\tan\frac{\alpha}{2}\tan\left(\frac{(r-1)\pi}{n}\right)}{\sqrt{\tan^2\frac{\pi}{n} + \tan^2\frac{\alpha}{2}\tan^2\left(\frac{(r-1)\pi}{n}\right)}} \quad \dots\dots\dots..(9)$$

If α_1, α_2 and α_3 are the apex angles i.e. angles between the consecutive lateral edges meeting at an apex A of a tetrahedron ABCD, the dihedral angle θ_{12} between the triangular faces ABC and ACD (as shown in Figure 3 below) is given from the inverse cosine formula [3],

$$\theta_{12} = \cos^{-1}\left(\frac{\cos\alpha_3 - \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}\right) \quad \dots\dots\dots..(10)$$

Now, the dihedral angle $\delta_{1r}\ (=\theta_{12})$ between triangular lateral faces $\mathrm{PMB_{r-2}}$ and $\mathrm{PNB_{r-2}}$ of tetrahedron $\mathrm{PMNB_{r-2}}$ (see Fig. 2) is obtained by substituting the corresponding values $\alpha_1 = \gamma, \alpha_2 = \gamma$ and $\alpha_3 = \beta$ from Eq.(9) and (7) respectively into Eq(10) as follows

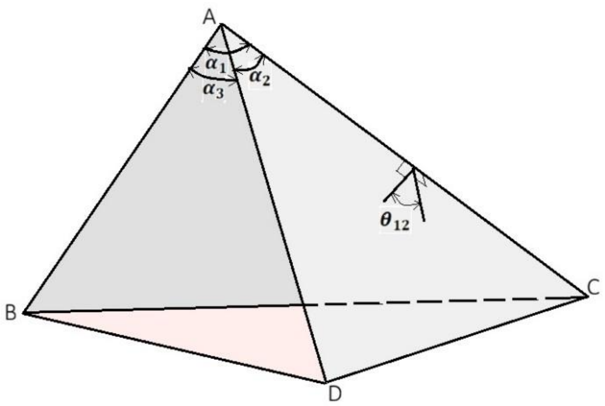

Figure 3: A tetrahedron ABCD with apex angles α_1, α_2 and α_3 of the triangular faces ABC, ACD, and ABD respectively meeting at the apex A. The triangular faces ABC and ACD are inclined at an angle i.e. dihedral angle θ_{12}.

$$\delta_{1r} = \cos^{-1}\left(\frac{\cos\beta - \cos\gamma\cos\gamma}{\sin\gamma\sin\gamma}\right) \quad \Rightarrow \cos\delta_{1r} = \frac{\cos\beta - \cos^2\gamma}{\sin^2\gamma}$$

$$\Rightarrow \cos\delta_{1r} = \frac{\dfrac{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}\sin^2\left(\frac{(r-1)\pi}{n}\right)}{\tan^2\frac{\pi}{n}} - \left(\dfrac{\tan\frac{\pi}{n}}{\sqrt{\tan^2\frac{\pi}{n} + \tan^2\frac{\alpha}{2}\tan^2\left(\frac{(r-1)\pi}{n}\right)}}\right)^2}{\left(\dfrac{\tan\frac{\alpha}{2}\tan\left(\frac{(r-1)\pi}{n}\right)}{\sqrt{\tan^2\frac{\pi}{n} + \tan^2\frac{\alpha}{2}\tan^2\left(\frac{(r-1)\pi}{n}\right)}}\right)^2}$$

$$\cos\delta_{1r} = \frac{\left(\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}\sin^2\left(\frac{(r-1)\pi}{n}\right)\right)\left(\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}\sin^2\left(\frac{(r-1)\pi}{n}\right)\right) - \tan^4\frac{\pi}{n}}{\tan^2\frac{\alpha}{2}\tan^2\frac{\pi}{n}\tan^2\left(\frac{(r-1)\pi}{n}\right)}$$

$$\cos\delta_{1r} = \frac{\tan^4\frac{\pi}{n} - 2\tan^2\frac{\alpha}{2}\tan^2\frac{\pi}{n}\sin^2\left(\frac{(r-1)\pi}{n}\right) + \tan^2\frac{\alpha}{2}\tan^2\frac{\pi}{n}\tan^2\left(\frac{(r-1)\pi}{n}\right) - 2\tan^4\frac{\alpha}{2}\sin^2\left(\frac{(r-1)\pi}{n}\right)\tan^2\left(\frac{(r-1)\pi}{n}\right) - \tan^4\frac{\pi}{n}}{\tan^2\frac{\alpha}{2}\tan^2\frac{\pi}{n}\tan^2\left(\frac{(r-1)\pi}{n}\right)}$$

$$\cos\delta_{1r} = 1 - 2\left(\cos^2\left(\frac{(r-1)\pi}{n}\right) + \frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}\sin^2\left(\frac{(r-1)\pi}{n}\right)\right)$$

$$2\cos^2\frac{\delta_{1r}}{2} - 1 = 1 - 2\left(\cos^2\left(\frac{(r-1)\pi}{n}\right) + \frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}\sin^2\left(\frac{(r-1)\pi}{n}\right)\right)$$

$$\cos^2\frac{\delta_{1r}}{2} = 1 - \left(1 - \sin^2\left(\frac{(r-1)\pi}{n}\right) + \frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}\sin^2\left(\frac{(r-1)\pi}{n}\right)\right)$$

$$\cos^2\frac{\delta_{1r}}{2} = \left(1 - \frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}\right)\sin^2\left(\frac{(r-1)\pi}{n}\right)$$

$$\Rightarrow \cos\frac{\delta_{1r}}{2} = \sqrt{\left(1-\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}\right)\sin^2\left(\frac{(r-1)\pi}{n}\right)} \quad (\because 0<\delta_{1r}\le\pi)$$

$$\Rightarrow \delta_{1r} = 2\cos^{-1}\left(\sin\left(\frac{(r-1)\pi}{n}\right)\sqrt{1-\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}}\right)$$

The above expression of δ_{1r} is the dihedral angle between the 1rst lateral face PA_1A_2 and rth lateral face PA_rA_{r+1} in the given regular n-gonal right pyramid (Fig. 1(a)).

The dihedral angle δ_{ij} between ith and jth arbitrary lateral faces, counted/marked in the same direction (CW or CCW) with respect to a reference lateral face, in a regular n-gonal right pyramid with apex angle α, is obtained by substituting $1 = i$ and by $r = |i-j|+1$ in the above generalized formula as follows

$$\delta_{ij} = 2\cos^{-1}\left(\sin\left(\frac{|i-j|\pi}{n}\right)\sqrt{1-\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}}\right) \quad \ldots\ldots(11)$$

$$\forall\ i,j,n\in N,\ \ i,j\le n,\ \ n\ge 3,\ \ \alpha<\frac{2\pi}{n},\ \ \delta_{ij}\in(0,\pi]$$

The variations of dihedral angle δ_{1r} with respect to the apex angle α and face number r at different values of number of faces n in a regular n-gonal right pyramid have been shown in Figure 4.

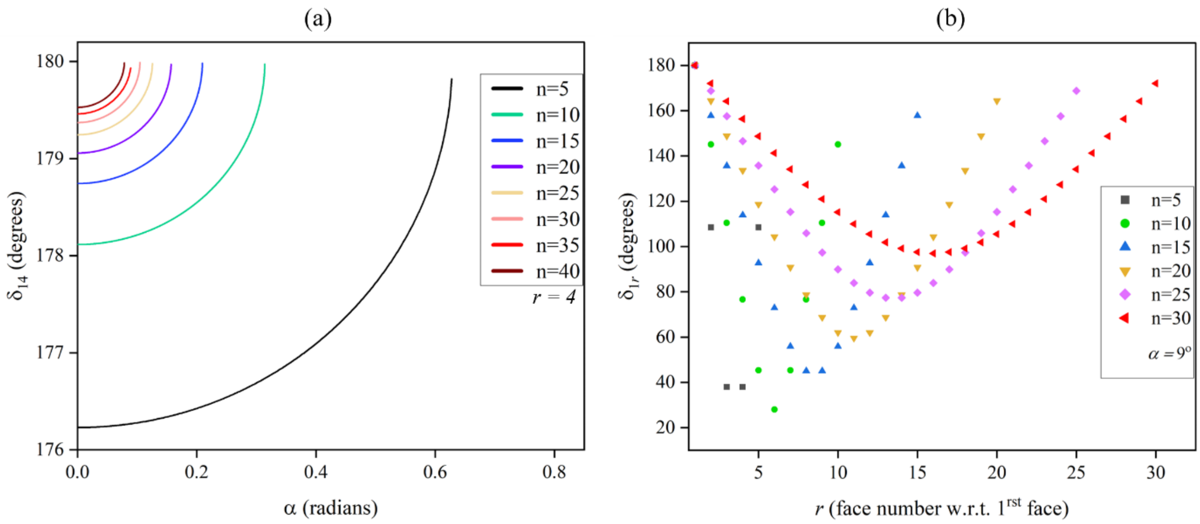

Figure 4: (a) The dihedral angle δ_{14} between 1rst and 4th faces varying with apex angle α for different number of faces n, (b) dihedral angle δ_{1r} between 1rst and rth faces varying with face number r for different number of faces n at given apex angle $\alpha = 9^o$.

2.2. Corollary 1: The dihedral angle i.e. angle of inclination δ_{1r} between *1*rst face (i.e. reference face) and rth face (i.e. arbitrary face) of a regular n-gonal right prism, is given by the generalized formula as follows

$$\delta_{1r} = \frac{|n-2r+2|\pi}{n} \quad \forall\ r,n\in N,\ \ r\le n,\ \ n\ge 3,\ \ \delta_{1r}\in(0,\pi] \quad \ldots\ldots(12)$$

Proof 1: A regular n-gonal right pyramid can be transformed into a regular n-gonal right prism by making all its lateral edges parallel to its geometric axis. In such a case, the angle between any two adjacent lateral edges becomes zero i.e. $\alpha = 0$. Therefore, the dihedral angle δ_{1r} between *1*rst face (i.e. reference face) and rth face (i.e. arbitrary face) of a regular n-gonal right prism, is obtained by substituting $\alpha = 0$ in the above generalized Eq.(1) of a regular n-gonal right pyramid as follows

$$\delta_{1r} = 2\cos^{-1}\left(\sin\left(\frac{(r-1)\pi}{n}\right)\sqrt{1-\frac{\tan^2 0}{\tan^2\frac{\pi}{n}}}\right)$$

$$= 2\cos^{-1}\left(\sin\left(\frac{(r-1)\pi}{n}\right)\right)$$

$$= 2\cos^{-1}\left(\cos\left(\frac{\pi}{2}-\frac{(r-1)\pi}{n}\right)\right)$$

$$= 2\left|\frac{\pi}{2}-\frac{(r-1)\pi}{n}\right|$$

$$= \frac{|n-2r+2|\pi}{n}$$

Proof 2: A regular n-gonal right prism has all its (rectangular) lateral faces parallel to its geometric axis or perpendicular to an arbitrary cross-section or its regular polygonal base. Now, consider a right prism with a regular n-gonal base $A_1A_2A_3 \dots . A_rA_{r+1} \dots A_{n-1}A_n$ (same as the regular polygonal base of a right pyramid as shown in the above Figure 1(b)). Each side of regular n-gonal base represent a rectangular lateral face of right prism perpendicular to the plane of paper. In such a case, the dihedral angle δ_{1r} between the *1*rst face (i.e. reference face) and r^{th} face (i.e. arbitrary face) of right prism will be equal to the angle θ_{1r} between *1*rst side A_1A_2, and r^{th} side A_rA_{r+1} that, when extended, intersect each other at the point B_{r-2} (refer to above Fig. 1(b)).

In right kite $MONB_{r-2}$ (Fig. 1(b)), the sum of all its interior angles is 2π as follows

$$\angle MON + \angle OMB_{r-2} + \angle ONB_{r-2} + \angle MB_{r-2}N = 2\pi$$

$$(r-1)\frac{2\pi}{n}+\frac{\pi}{2}+\frac{\pi}{2}+\theta_{1r} = 2\pi$$

$$\theta_{1r} = (n-2r+2)\frac{\pi}{n} \qquad \forall\, r < \frac{n+2}{2}, n \geq 3$$

$$\Rightarrow\ \delta_{1r} = \frac{|n-2r+2|\pi}{n} \qquad \forall\, r \leq n, n \geq 3$$

2.3. Corollary 2: The r^{th} and $(n-r+2)^{\text{th}}$ lateral faces are equally inclined with 1rst lateral face (i.e. reference face), in regular polygonal right pyramid and prism, at dihedral angles δ_{1r} given from Eq.(1) and (12) respectively.

Proof: The following two cases can be considered for regular polygonal right pyramid and prism.

Case-1: The dihedral angle between 1rst and $(n-r+2)^{\text{th}}$ lateral faces in a regular n-gonal right pyramid is obtained by substituting $r = n-r+2$ in Eq.(1) as follows

$$\delta_{1(n-r+2)} = 2\cos^{-1}\left(\sin\left(\frac{((n-r+2)-1)\pi}{n}\right)\sqrt{1-\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}}\right)$$

$$\delta_{1(n-r+2)} = 2\cos^{-1}\left(\sin\left(\frac{(n-r+1)\pi}{n}\right)\sqrt{1-\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}}\right)$$

$$\delta_{1(n-r+2)} = 2\cos^{-1}\left(\sin\left(\pi-\frac{(r-1)\pi}{n}\right)\sqrt{1-\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}}\right)$$

$$\delta_{1(n-r+2)} = 2\cos^{-1}\left(\sin\left(\frac{(r-1)\pi}{n}\right)\sqrt{1-\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}}\right)$$

$$\delta_{1(n-r+2)} = \delta_{1r}$$

Case-2: The dihedral angle between 1rst and $(n-r+2)^{\text{th}}$ lateral faces in a regular n-gonal right prism is obtained by substituting $r = n-r+2$ in Eq.(12) as follows

$$\delta_{1(n-r+2)} = \frac{|n-2r+2|\pi}{n}$$

$$\delta_{1(n-r+2)} = \frac{|n-2(n-r+2)+2|\pi}{n}$$

$$\delta_{1(n-r+2)} = \frac{|2r-n-2|\pi}{n}$$

$$\delta_{1(n-r+2)} = \frac{|n-2r+2|\pi}{n}$$

$$\delta_{1(n-r+2)} = \delta_{1r}$$

2.4. Corollary 3: The angles of inclination of the lateral face and lateral edge, δ_{fa} and δ_{ea} respectively with the geometric axis, and the angle of inclination of lateral face with polygonal base, δ_{fb} in a regular n-gonal right pyramid with apex angle α (i.e. angle between any two adjacent lateral edges), are given by the following generalized formulas

$$\delta_{fa} = \sin^{-1}\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}\right), \qquad \delta_{ea} = \sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\pi}{n}}\right), \qquad \delta_{fb} = \frac{\pi}{2}-\delta_{fa} = \cos^{-1}\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}\right)$$

$$\forall\ n \in N,\ n \geq 3,\ \alpha < \frac{2\pi}{n},\ \delta_{fa}, \delta_{ea}, \delta_{fb} \in \left(0, \frac{\pi}{2}\right)$$

Proof: The above three mathematical results can be obtained as given below.

(i) Angle of inclination of lateral face with the geometric axis ($\delta_{fa} = \angle OPN$)

In right ΔPON (Fig. 1(a)),

$$\sin\angle OPN = \frac{ON}{PN}$$

$$\Rightarrow\ \sin\delta_{fa} = \frac{\frac{a}{2}\cot\frac{\pi}{n}}{\frac{a}{2}\cot\frac{\alpha}{2}} \qquad \text{(setting values fron (2) and (3))}$$

$$\sin\delta_{fa} = \frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}$$

$$\delta_{fa} = \sin^{-1}\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}\right) \quad \dots\dots\dots..(13)$$

(ii) Angle of inclination of lateral edge with the geometric axis ($\delta_{ea} = \angle OPA_r$)

In right ΔPNA_r (Fig. 1(a)),

$$\sin\angle A_rPN = \frac{NA_r}{PA_r} \Rightarrow \sin\frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{PA_r} \qquad \left(\because \angle A_rPN = \frac{\angle A_rPA_{r+1}}{2} = \frac{\alpha}{2},\ NA_r = \frac{A_rA_{r+1}}{2} = \frac{a}{2}\right)$$

$$\Rightarrow PA_r = \frac{a}{2}\operatorname{cosec}\frac{\alpha}{2}$$

In right ΔONA_r (Fig. 1(b)),

$$\sin\angle A_rON = \frac{NA_r}{OA_r} \Rightarrow \sin\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{OA_r} \qquad \left(\because \angle A_rON = \frac{\angle A_rOA_{r+1}}{2} = \frac{\pi}{n},\ NA_r = \frac{A_rA_{r+1}}{2} = \frac{a}{2}\right)$$

$$\Rightarrow OA_r = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$$

In right ΔPOA_r (Fig. 1(a)),

$$\sin\angle OPA_r = \frac{OA_r}{PA_r}$$

$$\Rightarrow \sin\delta_{ea} = \frac{\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}}{\frac{a}{2}\operatorname{cosec}\frac{\alpha}{2}}$$

$$\sin\delta_{ea} = \frac{\sin\frac{\alpha}{2}}{\sin\frac{\pi}{n}}$$

$$\delta_{ea} = \sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\pi}{n}}\right) \quad \dots\dots\dots..(14)$$

(iii) Angle of inclination of lateral face with the regular polygonal base ($\delta_{fb} = \angle PNO$)

In right ΔPON (Fig. 1(a)),

$$\cos\angle PNO = \frac{ON}{PN}$$

$$\Rightarrow \cos\delta_{fb} = \frac{\frac{a}{2}\cot\frac{\pi}{n}}{\frac{a}{2}\cot\frac{\alpha}{2}} \qquad \text{(setting values fron (2) and (3))}$$

$$\cos\delta_{fb} = \frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}$$

$$\delta_{fb} = \cos^{-1}\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}\right) = \frac{\pi}{2} - \delta_{fa} \qquad \ldots\ldots\ldots..(15)$$

The Eq.(13), (14) and (15) are generalized formulas for finding the required angles of inclination δ_{fa} , δ_{ea} , and δ_{fb} in a regular n-gonal right pyramid.

2.5. Corollary 4: In a regular n-gonal right prism, the r^{th} lateral face will be parallel to the 1rst lateral face (i.e. reference face) only if n is even.

Proof: Let the r^{th} lateral face be parallel to the 1rst lateral face (i.e. reference face) in a regular n-gonal right prism. In such a case, the dihedral angle $\delta_{1r} = 0$.

$$\therefore\ \delta_{1r} = \frac{|n - 2r + 2|\pi}{n} = 0$$

$$n - 2r + 2 = 0 \quad \Rightarrow \quad r = \frac{n}{2} + 1$$

The above value of r will be positive integer only if n is divisible by 2 which implies that n is even. Therefore, in a regular polygonal right prism with $n = 2k$ number of lateral faces, $(k+1)^{\text{th}}$ face is always parallel to the first lateral face (i.e. reference face).

3. Application of Theorem on dihedral angles in regular polyhedra or Platonic solids

All the regular polyhedra or Platonic solids have identical vertices at each of which n number of congruent regular polygonal faces, each with an apex angle (i.e. interior angle) α, meet one another. Therefore, each vertex of a regular polyhedron can be assumed to be an apex of a regular n-gonal right pyramid with an apex angle α and hence the above generalized formula i.e. Eq.(1) can be applied to find the dihedral angle between any two faces meeting at a vertex of regular polyhedron. The generalized formula of HCR's Theorem is used to find the dihedral angles of all five platonic solids.

3.1. Dihedral angles in regular tetrahedron

A regular tetrahedron has 4 identical vertices at each of which 3 congruent equilateral triangular faces meet one another. In this case, each vertex is analogous to the apex of a regular n-gonal right pyramid with apex angle $\alpha = \frac{\pi}{3}$ and $n = 3$ (as shown in Figure 5(a)). Therefore, the dihedral angle δ_{12} between first ($r = 1$) and second ($r = 2$) triangular faces meeting at any vertex of the regular tetrahedron is obtained by substituting $\alpha = \pi/3$, $n = 3$ and $r = 2$ in the generalized formula i.e. Eq(1) as follows

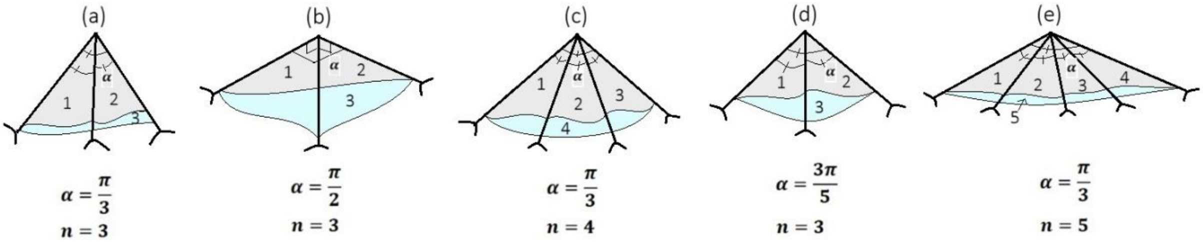

Figure 5: n number of congruent faces, each with apex angle α, meeting at the vertex of (a) regular tetrahedron, (b) regular hexahedron/cube (c) regular octahedron, (d) regular dodecahedron, (e) regular icosahedron.

$$\delta_{12} = 2\cos^{-1}\left(\sin\left(\frac{(2-1)\pi}{3}\right)\sqrt{1 - \frac{\tan^2\frac{\pi/3}{2}}{\tan^2\frac{\pi}{3}}}\right) = 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1 - \frac{\left(\frac{1}{\sqrt{3}}\right)^2}{\left(\sqrt{3}\right)^2}}\right) = 2\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

From the above Corollary 2, the dihedral angle between face-1 and face-3 is equal to dihedral angle between face-1 and face-2 meeting at a vertex of regular tetrahedron i.e. $\delta_{13} = \delta_{12}$ (refer to Fig. 5(a)).

$$\therefore\ \delta_{12} = \delta_{13} = 2\cos^{-1}\left(\sqrt{\frac{2}{3}}\right) \approx 70.52877937^{o}$$

3.2. Dihedral angles in regular hexahedron/cube

A regular hexahedron/cube has 8 identical vertices at each of which 3 congruent square faces meet one another. In this case, each vertex is analogous to the apex of a regular n-gonal right pyramid with apex angle $\alpha = \frac{\pi}{2}$ and $n = 3$ (as shown in Figure 5(b)). Therefore, the dihedral angle δ_{12} between first ($r = 1$) and second ($r = 2$) square faces meeting at any vertex of the cube is obtained by substituting $\alpha = \pi/2,\ n = 3$ and $r = 2$ in the generalized formula i.e. Eq(1) as follows

$$\delta_{12} = 2\cos^{-1}\left(\sin\left(\frac{(2-1)\pi}{3}\right)\sqrt{1-\frac{\tan^2\frac{\pi/2}{2}}{\tan^2\frac{\pi}{3}}}\right) = 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1-\frac{(1)^2}{\left(\sqrt{3}\right)^2}}\right) = 2\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}$$

From the above Corollary 2, the dihedral angle between face-1 and face-3 is equal to dihedral angle between face-1 and face-2 meeting at a vertex of regular hexahedron i.e. $\delta_{13} = \delta_{12}$ (refer to Fig. 5(b)).

$$\therefore\ \delta_{12} = \delta_{13} = \frac{\pi}{2}$$

3.3. Dihedral angles in regular octahedron

A regular octahedron has 6 identical vertices at each of which 4 congruent equilateral triangular faces meet one another. In this case, each vertex is analogous to the apex of a regular n-gonal right pyramid with apex angle $\alpha = \frac{\pi}{3}$ and $n = 4$ (as shown in Figure 5(c)). Therefore, the dihedral angle δ_{12} between first ($r = 1$) and second ($r = 2$) triangular faces meeting at any vertex of the regular octahedron is obtained by substituting $\alpha = \pi/3$, $n = 4$ and $r = 2$ in the generalized formula i.e. Eq(1) as follows

$$\delta_{12} = 2\cos^{-1}\left(\sin\left(\frac{(2-1)\pi}{4}\right)\sqrt{1-\frac{\tan^2\frac{\pi/3}{2}}{\tan^2\frac{\pi}{4}}}\right) = 2\cos^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1-\frac{\left(\frac{1}{\sqrt{3}}\right)^2}{(1)^2}}\right) = 2\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

From the above Corollary 2, the dihedral angle between face-1 and face-4 is equal to dihedral angle between face-1 and face-2 meeting at a vertex of regular octahedron i.e. $\delta_{14} = \delta_{12}$ (refer to Fig. 5(c)).

$$\therefore\ \delta_{12} = \delta_{14} = 2\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 109.4712206^{o}$$

Similarly, the dihedral angle δ_{13} between first ($r = 1$) and third ($r = 3$) triangular faces meeting at any vertex of the regular octahedron is obtained by substituting $\alpha = \pi/3,\ n = 4$ and $r = 3$ in the generalized formula i.e. Eq(1) as follows

$$\delta_{13}=2\cos^{-1}\left(\sin\left(\frac{(3-1)\pi}{4}\right)\sqrt{1-\frac{\tan^2\frac{\pi/3}{2}}{\tan^2\frac{\pi}{4}}}\right)=2\cos^{-1}\left((1)\sqrt{1-\frac{\left(\frac{1}{\sqrt{3}}\right)^2}{(1)^2}}\right)=2\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

From the above Corollary 2, there is no face symmetric to face-3 about face-1 meeting at a vertex of regular octahedron (refer to Fig. 5(c)).

$$\therefore\ \delta_{13}=2\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)\approx 70.52877937^o$$

3.4. Dihedral angles in regular dodecahedron

A regular dodecahedron has 20 identical vertices at each of which 3 congruent regular pentagonal faces meet one another. In this case, each vertex is analogous to the apex of a regular n-gonal right pyramid with apex angle $\alpha=\frac{3\pi}{5}$ and $n=3$ (as shown in Figure 5(d)). Therefore, the dihedral angle δ_{12} between first ($r=1$) and second ($r=2$) regular pentagonal faces meeting at any vertex of the regular dodecahedron is obtained by substituting $\alpha=3\pi/5,\ n=3$ and $r=2$ in the generalized formula i.e. Eq(1) as follows

$$\delta_{12}=2\cos^{-1}\left(\sin\left(\frac{(2-1)\pi}{3}\right)\sqrt{1-\frac{\tan^2\frac{3\pi/5}{2}}{\tan^2\frac{\pi}{3}}}\right)=2\cos^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1-\frac{\left(\sqrt{\frac{5+2\sqrt{5}}{5}}\right)^2}{\left(\sqrt{3}\right)^2}}\right)=2\cos^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)$$

From the above Corollary 2, the dihedral angle between face-1 and face-3 is equal to dihedral angle between face-1 and face-2 meeting at a vertex of regular dodecahedron i.e. $\delta_{13}=\delta_{12}$ (refer to Fig. 5(d)).

$$\therefore\ \delta_{12}=\delta_{13}=2\cos^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)\approx 116.5650512^o$$

3.5. Dihedral angles in regular icosahedron

A regular icosahedron has 12 identical vertices at each of which 5 congruent equilateral triangular faces meet one another. In this case, each vertex is analogous to the apex of a regular n-gonal right pyramid with apex angle $\alpha=\frac{\pi}{3}$ and $n=5$ (as shown in Figure 5(e)). Therefore, the dihedral angle δ_{12} between first ($r=1$) and second ($r=2$) triangular faces meeting at any vertex of the regular icosahedron is obtained by substituting $\alpha=\pi/3$, $n=5$ and $r=2$ in the generalized formula i.e. Eq(1) as follows

$$\delta_{12}=2\cos^{-1}\left(\sin\left(\frac{(2-1)\pi}{5}\right)\sqrt{1-\frac{\tan^2\frac{\pi/3}{2}}{\tan^2\frac{\pi}{5}}}\right)=2\cos^{-1}\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\sqrt{1-\frac{\left(\frac{1}{\sqrt{3}}\right)^2}{\left(\sqrt{5-2\sqrt{5}}\right)^2}}\right)=2\cos^{-1}\left(\frac{\sqrt{5}-1}{2\sqrt{3}}\right)$$

From the above Corollary 2, the dihedral angle between face-1 and face-5 is equal to dihedral angle between face-1 and face-2 meeting at a vertex of regular icosahedron i.e. $\delta_{15}=\delta_{12}$ (refer to Fig. 5(e)).

$$\therefore\ \delta_{12}=\delta_{15}=2\cos^{-1}\left(\frac{\sqrt{5}-1}{2\sqrt{3}}\right)=2\tan^{-1}\left(\frac{3+\sqrt{5}}{2}\right)\approx 138.1896851^o$$

Similarly, the dihedral angle δ_{13} between first ($r=1$) and third ($r=3$) triangular faces meeting at any vertex of the regular icosahedron is obtained by substituting $\alpha=\pi/3,\ n=5$ and $r=3$ in the generalized formula i.e. Eq(1) as follows

$$\delta_{13}=2\cos^{-1}\left(\sin\left(\frac{(3-1)\pi}{5}\right)\sqrt{1-\frac{\tan^2\frac{\pi/3}{2}}{\tan^2\frac{\pi}{5}}}\right)=2\cos^{-1}\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\sqrt{1-\frac{\left(\frac{1}{\sqrt{3}}\right)^2}{\left(\sqrt{5-2\sqrt{5}}\right)^2}}\right)=2\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

From the above Corollary 2, the dihedral angle between face-1 and face-4 is equal to dihedral angle between face-1 and face-3 meeting at a vertex of regular icosahedron i.e. $\delta_{14}=\delta_{13}$ (refer to Fig. 5(e)).

$$\therefore\ \delta_{13}=\delta_{14}=2\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 109.4712206^o$$

Conclusion: The generalized formula for the dihedral angle derived in the stated theorem is broadly applicable beyond regular polygonal right pyramids. In particular, it extends naturally to all Platonic solids, right prisms, and other uniform polyhedra (both convex and concave) whose local vertex configurations are identical to that of a regular polygonal right pyramid. The generalized expression given in Eq. (1) enables the direct and systematic determination of dihedral angles, as well as related angles, for regular n-gonal right pyramids and right prisms. The corresponding values obtained using this formulation are summarized in the table below.

Table. Angles in a regular n-gonal right pyramid and prism

Angle	Generalized formula	Conditions
Dihedral angle between 1rst and rth triangular faces meeting at the apex of a regular n-gonal right pyramid with apex angle α	$\delta_{1r}=2\cos^{-1}\left(\sin\left(\frac{(r-1)\pi}{n}\right)\sqrt{1-\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\pi}{n}}}\right)$	$\forall\ r,n\in N,\ r\le n,\ n\ge 3,$ $\alpha<\frac{2\pi}{n},\ \delta_{1r}\in(0,\pi]$ where $\delta_{1r}=\pi=\delta_{11}$ is dihedral angle of a face with itself.
Dihedral angle between each triangular face and polygonal base in a regular n-gonal right pyramid with apex angle α	$\delta_{fb}=\frac{\pi}{2}-\delta_{fa}=\cos^{-1}\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}\right)$	$\forall\ n\in N,\ n\ge 3,\ \alpha<\frac{2\pi}{n},$ $0<\delta_{fb}<\frac{\pi}{2}$
Angle of inclination of each triangular face with the geometric axis in a regular n-gonal right pyramid with apex angle α	$\delta_{fa}=\sin^{-1}\left(\frac{\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}}\right)$	$\forall\ n\in N,\ n\ge 3,\ \alpha<\frac{2\pi}{n},$ $0<\delta_{fa}<\frac{\pi}{2}$
Angle of inclination of each lateral edge with the geometric axis in a regular n-gonal right pyramid with apex angle α	$\delta_{ea}=\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\pi}{n}}\right),$	$\forall\ n\in N,\ n\ge 3,\ \alpha<\frac{2\pi}{n},$ $0<\delta_{ea}<\frac{\pi}{2}$
Dihedral angle between 1rst and rth (rectangular) faces of a regular n-gonal right prism	$\delta_{1r}=\frac{\lvert n-2r+2\rvert\pi}{n}$	$\forall\ r,n\in N,\ r\le n,$ $n\ge 3,\ \delta_{1r}\in(0,\pi]$

Note: Above articles had been derived *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

M.M.M. University of Technology, Gorakhpur-273010 (UP) India *10 July 2015*

Email: *rajpootharishchandra@gmail.com*

Author's Home Page: *https://notionpress.com/author/HarishChandraRajpoot*

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