Infinitely algebraic classes

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February 16, 2026

Abstract

We show that on a complex projective manifold X, for $\mathbb{G} = \mathbb{R}$ or $\mathbb{Q}$, a class in $H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{G}$ is represented by a convergent infinite series of integration currents over algebraic cycles with real coefficients. It implies that a Hodge class is represented by an algebraic cycle with rational coefficients.

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1 Statements

Cohomology has various representations. Some property of the cohomology is a property of a particular representative, and it may not be shared by all representatives. In this paper we focus on the current's representation of cohomology of the constant sheaves, i.e. the current's representation of the singular cohomology. We show that being algebraic in cohomology is a behavior of currents. We first define this particular type of current's representatives.

Key words: currents, algebraic cycles, Hodge classes.
2020 Mathematics subject classification: 49Q15, 32J25

Definition 1.1. *(Infinitely algebraic) Let X be a complex projective manifold. Let $T_\bullet$ denote the integration current over a chain $\bullet$. Let $\mathbf{M}$ denote a mass of currents, based on the Kähler metric. A class $u \in H^{2p}(X;\mathbb{R})$ is infinitely algebraic if it is represented by an absolutely mass-convergent series of currents*

$$\sum_{i=1}^{\infty} r_i T_{V_i} \tag{1.1}$$

where V_i are irreducible subvarieties coupled with real coefficients r_i, and the absolute mass-convergence is defined to be

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \mathbf{M}(T_{V_i}) = 0. \tag{1.2}$$

for any $N' \geq N$. It has positivity if $r_i > 0$ for all i.

So, infinitely algebraic classes form a subspace and those with positivity form a convex cone.

Remark Infinite algebraicity is closely related to algebraicity. For instance, if $\cup_i V_i$ is a subvariety, i.e. the set $\{V_i\}_i$ is locally finite, then this type of infinitely algebraic classes are represented by holomorphic chains with real coefficients, which are algebraic cycles with real coefficients. Following is an example of infinitely algebraic classes. It is based on a counter-example in [10].

Example 1.2. *Let $\mathbb{C}P^1$ be the projective space over $\mathbb{C}$. Let $z_i \in \mathbb{C}P^1$ for the positive integer i be a sequence of points that converges to $\mathbf{o} \in \mathbb{C}P^1$. Let r_i be a sequence of positive numbers such that $\sum_{i=1}^{\infty} r_i = \lambda$ is a finite number. The following is the behavior of such sequences.*

(1) $\sum_{i=1}^{\infty} r_i T_{\{z_i\}}$ is a closed current whose cohomology class is the infinitely algebraic class $\lambda[z_1]$ where $[z_1] \in H^2(\mathbb{C}P^1;\mathbb{Z})$ is represented by z_1.

(2) the current $\sum_{i=1}^{\infty} r_i T_{\{z_i\}}$ is not a holomorphic chain with real coefficients because $\cup_{i=1}^{\infty}\{z_i\}$ is not a subvariety of $\mathbb{C}P^1$.

(3) Let $T^o_{\{z_i\}}$ be the restriction of $T_{\{z_i\}}$ to the affine open set $\mathbb{C}P^1 \backslash \{\mathbf{o}\}$, in which, $\sum_{i=1}^{\infty} r_i T^o_{\{z_i\}}$ is still a closed current that represents the cohomology of the point z_1. But in $\mathbb{C}P^1 \backslash \{\mathbf{o}\}$, it is also a holomorphic chain with real coefficients because $\cup_{i=1}^{\infty}\{z_i\}$ is a subvariety.

Main theorem 1.3. *Let X be a complex projective manifold.*

(1) If $u \in H^{2p}(X;\mathbb{Q})$ is represented by a closed positive current of bidegree (p,p), then u is an infinitely algebraic class with positivity.

(2) Furthermore if $\mathbb{G}$ is the field $\mathbb{Q}$ or $\mathbb{R}$, then for $u \in H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{G}$, u is infinitely algebraic.

The infinite algebraicity is transcendental. But it also contains an arithmetic property that implies

Corollary 1.4. *On a complex projective manifold, a Hodge class is represented by an algebraic cycle with rational coefficients.*

Proof. We show that Corollary 1.4 follows from Main theorem 1.3. The theorem asserts that a Hodge class is an infinitely algebraic class. So, we prove the corollary in two steps: 1) reduce the infinite series (1.1) in currents to a finite sum in cohomology classes; 2) convert $\mathbb{R}$-coefficients to $\mathbb{Q}$-coefficients.

Step 1: Let $[\bullet]$ be the homomorphism

$$\{closed\ currents\} \to cohomology\ group.$$

Let

$$C_{\mathbb{Z}}^p \subset H^{2p}(X;\mathbb{Z})$$

be the image of the cycle map on the algebraic cycles of codimension p. Let

$$C_{\mathbb{R}}^p := C_{\mathbb{Z}}^p \otimes \mathbb{R} \ \subset H^{2p}(X;\mathbb{R}).$$

Let $E_{\mathbb{R}}^p \subset C_{\mathbb{R}}^p$ be the cone that consists of positive classes, i.e. those classes $\tau \in C_{\mathbb{R}}^p$ such that the cup product $\tau \cup [\phi] \geq 0$ for all the closed real forms with weakly positive (k,k) components $\phi^{k,k} \geq 0$ where $k = dim(X) - p$ (Definition 2.1). Then

$$span(E_{\mathbb{R}}^p) \subset C_{\mathbb{R}}^p.$$

On the other hand, since an algebraic cycle is a difference between two effective cycles whose integration currents are strongly positive. Thus

$$span(E_{\mathbb{R}}^p) \supset C_{\mathbb{R}}^p.$$

So

$$span(E_{\mathbb{R}}^p) = C_{\mathbb{R}}^p.$$

Since $C_{\mathbb{R}}^p$ is a subspace of $H^{2p}(X;\mathbb{R})$, it has a finite dimension. So, we can choose finitely many $\mathbb{R}$-coefficient algebraic cycles $\{A^j\}_j$ in $E_{\mathbb{R}}^p$ such that

$$span(\{[A^j]\}_j) = C_{\mathbb{R}}^p.$$

Define the polyhedral cone,

$$\widehat{C} = \left\{ \sum_{j=1}^{l} a_j [A_j] \in H^{2p}(X;\mathbb{R}) : a_j \geq 0 \right\}.$$

Claim 1.5. $\widehat{C} = E^p_{\mathbb{R}}$.

Proof: It is clear that

$$\widehat{C} \subset E^p_{\mathbb{R}}. \tag{1.3}$$

Let $v \in E^p_{\mathbb{R}}$ be a non-zero element. We write

$$v = \sum_{j=1}^{l} \alpha_j [A_j]$$

where α_j are real numbers. Let's show $\alpha_j \geq 0$. Denote the space of L^1 forms by $\mathcal{E}(X)$ with the dual $\mathcal{E}'(X)$. Denote the bidegree (i,i) component of a current by $(\bullet)^{i,i}$, and bidimension (i,i) component by $(\bullet)_{(i,i)}$. Define

$$Z^{w,i} := \left\{ \psi \in \mathcal{E}(X) : \psi = \phi + d\theta, d\phi = 0, \phi^{i,i} \geq 0 (weakly) \right\}$$

$$\tilde{Z}_{s,i} := \left\{ \mathcal{S} \in \mathcal{E}'(X) : \mathcal{S} = \mathcal{T} + d\eta, d\mathcal{T} = 0, \mathcal{T}_{i,i} \geq 0 (strongly) \right\}.$$

The proposition 3.6, [6] shows

$$\left(\tilde{Z}_{s,k}\right)^o = Z^{w,k}, \tag{1.4}$$

for the dual pairing $\langle \bullet \rangle$ between $\mathcal{E}(X)$ and $\mathcal{E}'(X)$, where $(\bullet)^o$ stands for the polar of a set. Since they are convex cones, bipolar theorem asserts

$$\left(Z^{w,k}\right)^o = \tilde{Z}_{s,k}.$$

Notice that the paring $\langle \bullet \rangle$ is reduced to the cup product on

$$H^{2p}(X;\mathbb{R}) \times H^{2k}(X;\mathbb{R}).$$

By the Poincaré duality, (1.4) implies that there are weakly positive forms B_i of bidegree (k,k) whose cohomology classes are Poincaré dual to the basis $\{[A_j]\}_j$, i.e.

$$\langle A_j, B_i \rangle = \delta_i^j.$$

where

$$\delta_i^j = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

In particular for the index 1, we can choose a weakly positive form B_1 such that for the class $[B_1] \in H^{2k}(X;\mathbb{R})$, $[B_1] \cup [A_j] = 0$ for $j \neq 1$, $[B_1] \cup [A_1] = 1$. Since the class v is positive, $v \cup [B_1] \geq 0$. Hence

$$\alpha_1 = v \cup [B_1] \geq 0.$$

Then $\alpha_j \geq 0$ for all j. Thus

$$E^p_{\mathbb{R}} \subset \widehat{C}. \tag{1.5}$$

So, (1.3) and (1.5) imply

$$\widehat{C} = E^p_{\mathbb{R}}. \tag{1.6}$$

□

Next we state a fact on a compact Kähler manifold: for a positive current $\mathcal{T}$

$$\mathbf{M}(\mathcal{T}) = \mathcal{T}[\frac{\omega^k}{k!}] \tag{1.7}$$

where ω is the Kähler form. This fact has been proved and used at multiple places. For the proof, see Theorem 2.2 and Remark 2.5 in [7].

Let u be a Hodge class, i.e. $u \in H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{Q}$. By Part (2) of Main theorem 1.3, u is infinitely algebraic. So we write a representative

$$\sum_{i=1}^{\infty} r_i T_{V_i} + d\Gamma \tag{1.8}$$

where V_i are irreducible algebraic subvarieties of dimension $k = dim(X) - p$, and Γ is a current. * Let $[V_i] = \sum_{finite\ j} \lambda_i^j [A^j]$ where λ_i^j are real numbers. Since $[V_i]$ and $[A^j]$ are all in the cone $E^p_{\mathbb{R}}$, by Claim 1.5, all coefficients λ_i^j must be non-negative. The evaluation

$$T_{V_i}[\frac{\omega^k}{k!}] = \sum_{finite\ j} \lambda_i^j T_{A^j}[\frac{\omega^k}{k!}]. \tag{1.9}$$

implies

$$\mathbf{M}(T_{V_i}) = \sum_{finite\ j} \lambda_i^j \mathbb{M}(T_{A^j}), \tag{1.10}$$

where $\mathbb{M}(T_{A^j}) \geq 0$ is the evaluation $T_{A^j}[\frac{\omega^k}{k!}]$ which is non-negative because the class $[A^j]$ is positive and the form $\frac{\omega^k}{k!}$ is strongly positive. On the other hand, by the absolute mass-convergence of (1.1), we have

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \mathbf{M}(T_{V_i}) = 0. \tag{1.11}$$

where $N' \geq N$. Plugging (1.10) into (1.11), we obtain

$$\sum_{finite\ j} \mathbb{M}(T_{A^j}) \left(\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \lambda_i^j \right) = 0, \tag{1.12}$$

*We should note that the convergence is the absolute mass-convergence.

Since $\mathbb{M}(T_{A^j})$ and λ_i^j are all non-negative, for each j

$$\lim_{N\to\infty}\sum_{i=N}^{N'}|r_i|\lambda_i^j = 0. \tag{1.13}$$

Then (1.13), for each j, implies the absolute convergence for the series

$$\alpha_j := \sum_{i=1}^{\infty} r_i\lambda_i^j. \tag{1.14}$$

Now we work with the convergence in cohomology. Due to the finiteness of Betti number, cohomological convergence is determined by the convergence of the real numbers on each axis. Precisely, we see that the (1.8) implies that u, which is

$$\left[\sum_{i=1}^{\infty} r_i V_i\right], \tag{1.15}$$

is approached by the cycle classes with real coefficients,

$$\left[\sum_{i=1}^{N} r_i V_i\right] = \sum_{finite\ j}\left(\sum_{i=1}^{N} r_i\lambda_i^j\right)[A^j],\ as\ N\to\infty. \tag{1.16}$$

($[\bullet]$ is continuous). Notice that the cohomology class (1.16), by the (1.14), also converges to a cycle class with real coefficients written as

$$u = \sum_{finite\ j}\alpha_j[A^j]; \tag{1.17}$$

$$\alpha_j = \sum_{i=1}^{\infty} r_i\lambda_i^j. \tag{1.18}$$

So, u is a cycle class with real coefficients.

Step 2: Next we convert it to $\mathbb{Q}$-coefficients. For any closed subset $W\subset X$, the subgroup

$$ker\bigg(H^i(X;\mathbb{Q})\to H^i(X\backslash W;\mathbb{Q})\bigg) \tag{1.19}$$

will be denoted by $H^i_{(W)}(X;\mathbb{Q})$ where ker stands for the kernel of the restriction map. A class $\gamma\in H^i(X;\mathbb{Q})$ is said to be class-supported on W if $\gamma\in H^i_{(W)}(X;\mathbb{Q})$. In another direction, we say a class is current-supported on W if it is represented by a closed current supported on W. The homology of currents implies that a class current-supported on a W is a class class-supported on W (but the converse is false). Recall (1.17)

$$u = \sum_{finite\ j}\alpha_j[A^j] \tag{1.20}$$

where α_j are real and A^j are algebraic cycles with real coefficients. If we let $V = \cup_j |A^j|$ be the algebraic set, u is current-supported on V, then u is also class-supported on V. Let

$$\widetilde{V} \quad \stackrel{J}{\to} \quad V \quad \stackrel{I}{\hookrightarrow} \quad X$$

be the composite such that J is a smooth resolution and I is the inclusion. Since the codimension condition

$$deg(u) - 2cod(V) \geq 0$$

is satisfied, we apply Deligne's corollary 8.2.8, [2] which addresses the class-support. Precisely it states that the Gysin map

$$(I \circ J)_! : H^0(\widetilde{V}; \mathbb{Q}) \to H^{2p}_{(V)}(X; \mathbb{Q}) \tag{1.21}$$

is surjective. Then a pre-image $\tilde{u}$ of u is a cohomological class of degree 0 on the complex manifold $\widetilde{V}$. So, $\tilde{u}$ must be represented by a rational linear combinations of irreducible components of $\widetilde{V}$. Since J is a complex analytic map from $\widetilde{V}$ onto V, $u = (I \circ J)_!(\tilde{u})$ is represented by a rational, linear combination of irreducible components of V. The proof is completed. □

We organize the rest of paper as follows. In section 2, we turn Harvey-Lawson's construction in relative homology to that in absolute homology. In section 3, we iterate the result in Section 2 infinitely many times to prove Main theorem 1.3. In appendix A, we prove a lemma about current's extension that is crucial for the proof in Section 2. Appendix B contains no proof. But it explains how the infinite algebracity is related Demailly's approximation ([4], etc.).

2 Harvey-Lawson's approach

The technique we'll use is derived from Harvey-Lawson's geometric result on the boundary of a homological chain. It is important to notice that they also extended the result to relative homology where our application is focused on.

Definition 2.1. *(Harvey-Lawson [6]) Let X be a compact Kähler manifold. Let M be an oriented compact real analytic submanifold of dimension $2k-1$. A class*

$$\tau \in H_{2k}(X, M; \mathbb{R}) \cup H_{2k}(X; \mathbb{R})$$

is called positive (resp., weakly positive) if $\tau \cap [\phi] \geq 0$ for all closed, real 2k-forms ϕ whose (k,k) component $\phi^{k,k}$ is weakly positive (resp., strongly positive), where $\cap$ is the cap product.

Remark The statements of "resp." belong to us. In this way, the two types of positivity for currents and forms would be respectively reduced to those for classes in homology or cohomology. The cap product on the relative homology in this case is well-defined ([6]).

Our strategy is to turn Harvey-Lawson's result in relative homology to that in absolute homology.

Proposition 2.2. *Let X be a complex projective manifold. Let $\tau \in H_{2k}(X;\mathbb{Q})$ be weakly positive. Then there exist an algebraic cycle V_r with positive rational coefficients and closed positive current S_r of bidimension (k,k) such that τ is represented by $T_{V_r} + S_r$, i.e. $\tau = [T_{V_r} + S_r]$.*

Proof. Let $\tau \in H_{2k}(X;\mathbb{Q})$ be weakly positive. Let N be a positive integer such that $N\tau \in H_{2k}(X;\mathbb{Z})/tors$ is non-zero. Let M be any real analytic compact oriented submanifold of dimension $2k-1$. We denote $H_{2k}(X,M;\mathbb{Z})/tors$ by $\overline{H}_{2k}(X,M;\mathbb{Z})$. Let

$$\pi : (X,\varnothing) \to (X,M)$$

be the inclusion map for the pairs †. Then $\pi_*(N\tau)$ is weakly positive in the relative homology $\overline{H}_{2k}(X,M;\mathbb{Z})$ with

$$\partial(\pi_*(N\tau)) = 0 \in i_*(\mathcal{E}'_{2k}(M))$$

where $i : M \hookrightarrow X$ is the inclusion, the dual of L^1 forms $\mathcal{E}'_{2k}(M)$ and the boundary $\partial(\pi_*(N\tau))$ are defined for the calculation of relative homology in [6]. Theorem 3.4, [6] for such weakly positive

$$\pi_*(N\tau) \in \overline{H}_{2k}(X,M;\mathbb{Z})$$

still holds. It implies that there exist a positive holomorphic chain $\widetilde{V}$ in the open submanifold $X\backslash M$, and a closed positive current S of bidimension (k,k) on X such that for the simple extension $\mathcal{T}$ of $T_{\widetilde{V}}$ to X, the current $\mathcal{T}+S$, through the inclusion π, represents the relative class $\pi_*(N\tau)$ in

$$\overline{H}_{2k}(X,M;\mathbb{Z})$$

with $\partial(\pi_*(N\tau)) = d\pi_*(\mathcal{T})$. Since $\partial(\pi_*(N\tau)) = 0$ (as a chain), $\mathcal{T}$ is closed as a current in X. Furthermore, since $T_{\widetilde{V}}$ is a locally rectifiable current in the open manifold $X\backslash M$, by Lemma A1 in Appendix, its simple extension $\mathcal{T}$ in X is also locally rectifiable ‡. Hence we have the closed current $\mathcal{T} \in \mathcal{R}^{loc}_{k,k}(X)$-the set of bidimension (k,k) currents that are locally rectifiable. The main theorem of [1] says that $\mathcal{T}$ is a holomorphic chain (i.e. as a current of integration) in X. According to Chow's theorem, this holomorphic chain is an algebraic cycle.

†Harvey-Lawson's work is about relative homology only. Therefore it is necessary for our application to use π to distinguish the absolute homology and the relative homology.

‡This can also be confirmed by Lemma 3.13, chapter 6, [9].

Let's denote it by V. So $\mathcal{T} = T_V$. Since $\mathcal{T}$ is a positive current, the coefficients of V must be positive. Notice that $N\tau$ is represented by $T_V + S + \mathcal{T}_{tor}$ in $H_{2k}(X;\mathbb{Z})$ with a torsion $\mathcal{T}_{tor}$. So, τ is represented by

$$\frac{T_V}{N} + \frac{S}{N}$$

in $H_{2k}(X;\mathbb{Q})$ such that $\dfrac{T_V}{N}$ is an algebraic cycle with positive rational coefficients and $\dfrac{S}{N}$ is a positive current of bidimension (k,k). We complete the proof.

□

Remark. In stead of the weak positivity in the argument now, Harvey-Lawson's original work used the positivity of $\pi_*(N\tau)$ (which is the strong positivity in terms of currents). We made the change for a convenience of the iteration in the next section, and the change does not alter the proof of Theorem 3.4, [6].

Proposition 2.2 contains two steps: (1) construct the holomorphic chains; (2) extend the holomorphic chains. Harvey-Lawson's method accomplishes the step 1. The step 2 is done through the extension of currents. The following is a trivial example for this type of extension.

Example 2.3. *Let X be a compact Kähler manifold, $V \subset X$ be a compact complex submanifold of the complex dimension k. Let $M \subset V$ be a $2k-1$ dimensional sphere that bounds a $2k$ dimensional Euclidean ball $B \simeq \mathbb{R}^{2k} \subset V$, i.e. $M = \partial B$. Let $\tau = [V] \in H_{2k}(X;\mathbb{Z})$. Let $\pi : (X, \varnothing) \to (X, M)$ be the inclusion map. Then $\pi_*(\tau) \in H_{2k}(X, M;\mathbb{Z})$. In the open manifold $X\backslash M$, the holomorphic chain $\widetilde{V}$ for this $\pi_*(\tau)$ (as that in Proposition 2.2) is the sum of two holomorphic chains $B + V\backslash B$. The holomorphic chains define two currents T_B and $T_{V\backslash B}$ individually in the open manifold $X\backslash M$. Then each current has the simple extension across M to X. We denote the extensions, which are the extension-by-zero currents, by $T_{\overline{B}}$, and $T_{\overline{V\backslash B}}$ respectively. Then the sum*

$$T_{\overline{B}} + T_{\overline{V\backslash B}}$$

is equal to T_V. So, through the current's extension, the $\widetilde{V}$ is holomorphically extended across M to the subvariety V. With $S = 0$, τ is represented by the current $T_V + S$.

3 Proof

Proof of Main theorem 1.3: (1) First we assume that the non-zero rational class u is represented by a closed positive current S_0 of bidegree (p,p). Then the current S_0 also represents a weakly positive homology class $[S_0]_h$ in the homology group $H_{2k}(X;\mathbb{Q})$ where $k = dim(X) - p$. By Proposition 2.2, there exist an algebraic cycle V_1 with positive rational coefficients, and a positive closed current S_1 of bidimension (k,k) such that for the current T_{V_1}, the homology class $[S_0]_h$ is represented by the current $S_1 + T_{V_1}$, i.e.

$$S_0 = S_1 + T_{V_1} + d\Gamma_1 \tag{3.1}$$

where Γ_1 is a current of dimension $2k+1$. Let ω be the Kähler form. We obtain

$$S_0[\frac{\omega^k}{k!}] = S_1[\frac{\omega^k}{k!}] + T_{V_1}[\frac{\omega^k}{k!}] + (d\Gamma_1)[\frac{\omega^k}{k!}]. \tag{3.2}$$

Since V_1 has positive coefficients, T_{V_1} is a positive current. Hence S_0, S_1, T_{V_1} are all positive currents of bidimension (k,k). By the mass formula, (3.2) can be written as

$$\mathbf{M}(S_0) = \mathbf{M}(S_1) + \mathbf{M}(T_{V_1}) \tag{3.3}$$

where $\mathbf{M}(\bullet)$ is the mass associated to the Kähler metric. Since T_{V_1} is positive,

$$\mathbf{M}(S_0) > \mathbf{M}(S_1). \tag{3.4}$$

Since S_1 is positive of bidimension (k,k) with rational homology in $H_{2k}(X;\mathbb{Q})$, applying Proposition 2.2, we can iterate the decomposition (3.1) for the positive current S_1, then afterwards iterate it for the similar positives currents $S_2, S_3, \cdots$. With finitely many such iterations, we obtain

$$S_0 = S_N + \sum_{i=1}^{N} T_{V_i} + d\Gamma_N$$

where N is a natural number, and V_i are algebraic cycles with positive rational coefficients. Write it as

$$S_N = S_0 - \left(\sum_{i=1}^{N} T_{V_i} + d\Gamma_N\right). \tag{3.5}$$

Then similarly

$$S_N[\frac{\omega^k}{k!}] = S_0[\frac{\omega^k}{k!}] - \left(\sum_{i=1}^{N} T_{V_i}\right)[\frac{\omega^k}{k!}] \tag{3.6}$$

It implies that

$$\mathbf{M}(S_0) = \mathbf{M}(S_N) + \sum_{i=1}^{N} \mathbf{M}(T_{V_i}). \tag{3.7}$$

Since V_i are positive holomorphic chains, all $\mathbf{M}(T_{V_i})$ are positive. Hence the mass inequality (3.4) is extended to the decreasing sequence

$$\mathbf{M}(S_0) > \mathbf{M}(S_1) > \cdots \mathbf{M}(S_N) > \cdots \tag{3.8}$$

Since the cone of positive currents is closed, the limit $\lim_{N\to\infty} \mathbf{M}(S_N)$ must be zero (otherwise the iteration could continue). By (3.5)

$$\lim_{N\to 0}\left(\sum_{i=1}^{N} T_{V_i} + d\Gamma_N\right) = S_0 \ (\textit{in mass}) \tag{3.9}$$

Notice that by (3.5) and (3.7), it is clear that both currents $\sum_{i=1}^{N} T_{V_i}$ and $d\Gamma_N$ have bounded mass for all N. Thus there are sub-sequences such that

$$\lim_{N_j\to\infty}\sum_{i=1}^{N_j} T_{V_i} = F_\infty \ (\textit{weakly}) \tag{3.10}$$
$$\lim_{N_j\to\infty} d\Gamma_{N_j} = d\Gamma_\infty \ (\textit{weakly}).$$

Note: Γ_∞ is not $\lim_{N\to\infty} \Gamma_N$, but it is obtained through the boundaries $\lim_{N_j\to\infty} d\Gamma_{N_j}$. Once those weak limits are well-defined, we obtain the infinite series for currents

$$S_0 = \sum_{i=1}^{\infty} T_{V_i} + d\Gamma_\infty \ (\textit{in mass})$$

where V_i are algebraic cycles with positive rational coefficients, and Γ_∞ is some current of dimension $2k+1$. By the positivity of the currents T_{V_i} and the formula (3.7), the convergence of (3.10) is the absolute mass-convergence. In cohomological expression, the class u is an infinitely algebraic class with positivity. This completes the proof of Part (1).

(2). In general, we may write

$$u = \sum_{finite\ j} b_j u_j \tag{3.11}$$

where b_j are real numbers which could be all rational if the field $\mathbb{G} = \mathbb{Q}$, and u_j are rational classes of (p,p) type. For each j, we write

$$u_j = a\omega^p + u_j - a\omega^p \tag{3.12}$$

where a is a real number. Since u_j is of (p,p) type, the idea of Demailly in [4], is that for a sufficiently large a, the cohomology class

$$a\omega^p + u_j$$

is represented by a strongly positive current. By Part (1), the class $a\omega^p + u_j$ is infinitely algebraic. So is u_j. This completes the proof of Part (2).

□

Appendix A Lelong's Simple extension

Our strategy is to extend the holomorphic chains to algebraic cycles. This type of extension is first obtained as an extension of currents, which is important notion on its own. Let's give a review first. Let Ω be an open subset of a real manifold Y. Let $t \in \mathscr{D}'(\Omega)$. Any current $\tilde{t}$ on Y that is restricted to t on Ω is called an extension of t. Not all currents in Ω can have extensions, and extensions may not be unique if they exist. A simple extension, denoted by $\tilde{t}_o$, of $t \in \mathscr{D}'(\Omega)$ is a particular extension-by-zero defined by Lelong ([8]). Its functional is $\sum_{j=1}^{\infty} \phi_j t$ where $\{\phi_j\}$ is a partition of unity for an open covering of Ω. The functional is a current if 1) the order of t is 0; 2) the mass of t in $\Omega \cap G$ for any compact set G of Y, denote by $||t||_G$, is finite. (But the simple extension is still dependent of the partition of unity.)

Lemma A.1. *Let Y be compact. Let $t \in \mathscr{D}'(\Omega)$ be a current that has order* 0 *and $||t||_G$ is finite for any compact G. Then if t is locally rectifiable, so is $\tilde{t}_o$.*

Proof. Since Y is compact, its weak limits are also the mass limits (i.e. strong limits). Let ϕ_j be a partition of unity, associated to the simple extension. For a test form ψ, we have the evaluation

$$\tilde{t}_o[\psi] = \sum_{j=1}^{\infty} t[\phi_j \psi]. \tag{A.1}$$

Since $t \sum_{j=1}^{N} \phi_j$ is well-defined in Y, the weak limit

$$\lim_{N \xrightarrow{w} \infty} t \sum_{j=1}^{N} \phi_j$$

is also well-defined in Y (see (A.1)). Let $\epsilon > 0$ and P_ϵ be the Lipschitzian chains for the approximation of t in Ω. Let $|| \bullet ||$ be the mass in the manifold Y. By Proposition 2, [8]

$$||\tilde{t}_o|| = ||t||,$$

where for $t \in \mathscr{D}'(\Omega)$, the mass is evaluated in the restricted charts as those for

$Y \cap \Omega$ from Y. Then we have the computation

$$||\tilde{t}_o - T_{P_\epsilon}|| = || \lim_{N \xrightarrow{w} \infty} (t - T_{P_\epsilon}) \sum_{j=1}^{N} \phi_j ||$$

(Since the weak limit is the same as the mass limit)

$$= \lim_{N \to \infty} ||(t - T_{P_\epsilon}) \sum_{j=1}^{N} \phi_j ||$$

(since $t - T_{P_\epsilon}$ has order 0)

$$\leq \lim_{N \to \infty} \left|\left| \sum_{j=1}^{N} \phi_j \right|\right|_\infty ||t - T_{P_\epsilon}||$$

$$= ||t - T_{P_\epsilon}||$$

(since t is rectifiable, there is some P_ϵ)

$$\leq \epsilon.$$

Thus $\tilde{t}_o$ is approximated by the same Lipschitzian chains P_ϵ as that for t. Therefore $\tilde{t}_o$ is also locally rectifiable. □

Remark. The simple extension in Harvey-Lawson's result (also in our proof) is the particular one whose original current t is a linear combination of positive closed currents. In this special case, the extension has other important properties. For instance, the simple extension is unique. See [5] or [6] for detail.

Appendix B Demailly's approach

The principle of the proof is transcendental and is related to the work done by many people in the past. The one that is worth to be pointed out is Demailly's work on currents. Demailly's work focused on currents, but our theorem addresses cohomology classes. Thus the difference between classes and currents droves us apart. But our notion of infinite algebraicity is more accessible through Demailly's approximation. In this section, we give the description without any proof. We follow the interpretation in [3].

We first describe the currents in Demailly's idea. Let X be a complex projective manifold. Let $\Sigma(X)$ be the subspace that consists of real closed currents $\mathcal{T} \in \mathscr{D}'(X)$ such that $[\mathcal{T}] \in H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{R}$. Let $\Sigma^+(X)$ be the subset of $\Sigma(X)$ such that those currents are strongly positive ($SPC^p_{\mathbb{Z}}(X)$ by Demailly). Let

$\Lambda(X)$ be the subspace of currents in the form

$$\lim_{i\to\infty} \mathcal{T}_i \qquad with \tag{B.1}$$

$$\mathcal{T}_i = \sum_j \lambda_{ij} T_{V_{ij}} \tag{B.2}$$

where λ_{ij} are real numbers, V_{ij} are irreducible subvarieties of codimension p and the limits are all in the weak topology of $\mathscr{D}'(X)$. A convex cone $\Lambda^+(X)$ of $\Lambda(X)$ is the collection of those currents in (B.1) and (B.2) with non-negative λ_{ij}. We define the statements

$$\mathcal{H}(X): \ \Sigma(X) = \Lambda(X) \tag{B.3}$$

$$\mathcal{H}^+(X): \ \Sigma^+(X) = \Lambda^+(X). \tag{B.4}$$

We call (B.3) and (B.4) Demailly's approximation because he proved

$$\mathcal{H}^+(X) \Rightarrow Corollary\ 1.4 \Leftrightarrow \mathcal{H}(X). \tag{B.5}$$

and asked weather $\mathcal{H}(X)$ implies $\mathcal{H}^+(X)$. It is clear that Demailly's expectation was

$$\mathcal{H}^+(X) \Leftrightarrow Corollary\ 1.4 \Leftrightarrow \mathcal{H}(X).$$

However, in [3], Babaee and Huh provided a counter-example that shows $\mathcal{H}^+$ is false. So, their result shows that $\Sigma^+(X)$ can not be approximated by algebraic cycles with positive real coefficients. This does not go well with $\mathcal{H}(X)$ which asserts that $span(\Sigma^+)$ is approximated by algebraic cycles with real coefficients.

Our Main theorem 1.3 claims that Demailly's approximation is correct for cohomology in the following way

Result B.1. *There is a subset* $\mathcal{P}^+(X) \subsetneq \Sigma^+$ *with*

$$span(\big[\mathcal{P}^+(X)\big]) = H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{R}, \tag{B.6}$$

that can be approximated by algebraic cycles with positive real coefficients.

It indicates that Demailly's approximation is not sufficiently accurate for currents, but it is sufficiently accurate for cohomology. We adjust it with the precision that comes from Harvey-Lawson's work on the boundaries of holomorphic chains. Main theorem claims that the adjustment $\mathcal{P}^+(X)$ is the set of the infinite series of currents $F_\infty = \sum_{i=1}^{\infty} V_i$ as in (3.10), derived from cohomology $H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{G}$ via Harvey-Lawson. So, they are particular types of infinitely algebraic cycles (not classes) with positivity.

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