Infinitely algebraic classes

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Abstract

We show that on a complex projective manifold X, for $\mathbb{G} = \mathbb{R}$ or $\mathbb{Q}$, a class in $H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{G}$ is represented by a convergent infinite series of integration currents over algebraic cycles with real coefficients. It implies that a Hodge class is represented by an algebraic cycle with rational coefficients.

Contents

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1 Statements

Definition 1.1. *(Infinitely algebraic) Let X be a complex projective manifold. Let $T_\bullet$ denote the integration current over a chain $\bullet$. Let $\mathbf{M}$ denote the mass of currents. A class $u \in H^{2p}(X;\mathbb{R})$ is infinitely algebraic if it is represented by an absolutely mass-convergent series of currents*

$$\sum_{i=1}^{\infty} r_i T_{V_i} \tag{1.1}$$

where V_i are irreducible subvarieties coupled with real coefficients r_i, and the absolute mass-convergence is defined to be

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \mathbf{M}(T_{V_i}) = 0. \tag{1.2}$$

for any $N' \geq N$. It has positivity if $r_i > 0$ for all i.

So, Infinitely algebraic classes form a subspace and those with positivity form a cone.

Main theorem 1.2. *Let X be a complex projective manifold.*

(1) If $u \in H^{2p}(X;\mathbb{Q})$ is represented by a positive current of bidegree (p,p), then u is an infinitely algebraic class with positivity.

(2) Let $H^{p,p}(X;\mathbb{Z}) = H^{p,p}(X;\mathbb{C}) \cap H^{2p}(X;\mathbb{Z})$. Furthermore if $\mathbb{G}$ is the field $\mathbb{Q}$ or $\mathbb{R}$ and $u \in H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{G}$, then u is infinitely algebraic.

The main theorem implies

Corollary 1.3. *On a complex projective manifold, a Hodge class is represented by an algebraic cycle with rational coefficients.*

Proof. We show that corollary 1.3 follows from Main theorem 1.2. It has two steps: (1) reduce the infinite sum in (1.1) to a finite sum; 2) convert $\mathbb{R}$-coefficients to $\mathbb{Q}$-coefficients.

Step 1: Let $\mathscr{D}'(X)$ be the space of currents. Let $\mathbf{M}$ be the mass on $\mathscr{D}'(X)$, based on the Kähler metric form ω. Let $[\bullet]$ be the homomorphism from the subspace with the closed currents to the cohomology $\sum H^*(X;\mathbb{R})$. Let $C^p_{\mathbb{Z}}$ be the image of the cycle map on the algebraic cycles of codimension p. Let

$$C^p_{\mathbb{R}} := C^p_{\mathbb{Z}} \otimes \mathbb{R}. \tag{1.3}$$

Let $E^p_{\mathbb{R}} \subset C^p_{\mathbb{R}}$ be the cone that consists of positive classes, i.e. those classes $\tau \in C^p_{\mathbb{R}}$ such that $\tau(\phi) \geq 0$ for all the closed real forms with strongly positive (k,k) components $\phi^{k,k} \geq 0$ where $k + p = dim(X)$ (Definition 2.1). Then

$$span(E^p_{\mathbb{R}}) \subset C^p_{\mathbb{R}}. \tag{1.4}$$

On the other hand, since an algebraic cycle is a difference between two effective cycles whose integration currents are strongly positive,

$$span(E^p_{\mathbb{R}}) \supset C^p_{\mathbb{R}}. \tag{1.5}$$

Thus

$$span(E^p_{\mathbb{R}}) = C^p_{\mathbb{R}}. \tag{1.6}$$

Since $C^p_{\mathbb{R}}$ is a subspace of $H^{2p}(X;\mathbb{R})$, it has a finite dimension. So, we can choose finitely many $\mathbb{R}$-coefficient algebraic cycles $\{A^j\}_j$ in $E^p_{\mathbb{R}}$ such that

$$span([A^j]_j) = C^p_{\mathbb{R}}.$$

Next we state a fact on a compact Kähler manifold: due to the Wirtinger's inequality, for a positive bidimensional (k,k) current $\mathcal{T}$, the geometric measure implies that

$$\mathbf{M}(\mathcal{T}) = \mathcal{T}[\frac{\omega^k}{k!}] \tag{1.7}$$

where ω is the Kähler form. This fact has been proved and used at multiple places. For the proof, see Theorem 2.2 and Remark 2.5 in [7].

Let u be a Hodge class, i.e. $u \in H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{Q}$. By Part (2) of Main theorem 1.2, u is infinitely algebraic. So we write a representative

$$\sum_{i=1}^{\infty} r_i T_{V_i} + d\Gamma \tag{1.8}$$

where V_i are irreducible algebraic subvarieties of dimension $k = dim(X) - p$, and Γ is a current. * Let $[V_i] = \sum_{finite\ j} \lambda_i^j [A^j]$ where λ_i^j are real numbers. Since $[V_i]$ and $[A^j]$ are all the positive classes, λ_i^j must be all non-negative. The evaluation

$$T_{V_i}[\frac{\omega^k}{k!}] = \sum_{finite\ j} \lambda_i^j T_{A^j}[\frac{\omega^k}{k!}]. \tag{1.9}$$

implies

$$\mathbf{M}(T_{V_i}) = \sum_{finite\ j} \lambda_i^j \mathbb{M}(T_{A^j}), \tag{1.10}$$

where $\mathbb{M}(T_{A^j}) \geq 0$ is the evaluation $T_{A^j}[\frac{\omega^k}{k!}]$ which is positive because the class $[A^j]$ is positive and the form $\frac{\omega^k}{k!}$ is strongly positive.† On the other hand, by the absolute mass-convergence of (1.1), we have

$$\lim_{N\to\infty} \sum_{i=N}^{N'} |r_i| \mathbf{M}(T_{V_i}) = 0. \tag{1.11}$$

*We should note that the convergence is the absolute mass-convergence.

†By Proposition 3.3, [6], there is a positive current T_j for each A^j such that $\mathbb{M}(A^j) = \mathbf{M}(T_j)$.

where $N' \geq N$. Plugging (1.10) into (1.11), we obtain

$$\sum_{finite\ j} \mathbb{M}(T_{A^j})\left(\lim_{N\to\infty}\sum_{i=N}^{N'}|r_i|\lambda_i^j\right) = 0, \tag{1.12}$$

Since $\mathbb{M}(T_{A^j})$ and λ_i^j are all non-negative, for each j

$$\lim_{N\to\infty}\sum_{i=N}^{N'}|r_i|\lambda_i^j = 0. \tag{1.13}$$

Then by the absolute convergence (1.13), for each j, the number

$$a_j := \sum_{i=1}^{\infty} r_i\lambda_i^j \tag{1.14}$$

exists.

So far, we have been dealing with the absolute mass-convergence of the currents. Now we work with the convergence in cohomology group - the real vector space. Due to the finite Betti number, its convergence is determined by the convergence of the real numbers on each axis. Precisely, we see that the (1.8) implies that u, which is

$$\left[\sum_{i=1}^{\infty} r_i V_i\right], \tag{1.15}$$

is approached by the cycle classes with real coefficients,

$$\left[\sum_{i=1}^{N} r_i V_i\right] = \sum_{finite\ j}\left(\sum_{i=1}^{N} r_i\lambda_i^j\right)[A^j],\ as\ N\to\infty. \tag{1.16}$$

(since $[\bullet]$ is continuous due to the cup-product). Notice that the cohomology class (1.16), by the (1.14), also converges to a cycle class with real coefficients written as

$$u = \sum_{finite\ j} a_j[A^j]; \tag{1.17}$$

$$a_j = \sum_{i=1}^{\infty} r_i\lambda_i^j. \tag{1.18}$$

So, u is a cycle class with real coefficients.

Step 2: Next we convert it to $\mathbb{Q}$-coefficients. For any closed subset $W \subset X$, the subgroup

$$ker\left(H^i(X;\mathbb{Q}) \to H^i(X\backslash W;\mathbb{Q})\right) \tag{1.19}$$

will be denoted by $H^i_{(W)}(X;\mathbb{Q})$ where ker stands for the kernel of the restriction map. A class $\alpha \in H^i(X;\mathbb{Q})$ is said to be class-supported on W if $\alpha \in H^i_{(W)}(X;\mathbb{Q})$. In the other way, we say a class is current-supported on W if it has a representative of a closed current supported on W. Through de Rham's homology of currents, a current-support implies the class-support. Recall (1.17)

$$u = \sum_{finite\ j} a_j[A^j] \tag{1.20}$$

where a_j are real and A^j are algebraic cycles with real coefficients. If we let $V = \cup_j |A^j|$ be the algebraic set, u is current-supported on V, then u is also class-supported on V. Let

$$\widetilde{V} \quad \stackrel{J}{\to} \quad V \quad \stackrel{I}{\hookrightarrow} \quad X$$

be the composite such that J is a smooth resolution and I is the inclusion. Since the codimension condition

$$deg(u) - 2cod(V) \geq 0$$

is satisfied, we apply Deligne's corollary 8.2.8, [2] which addresses the class-support. Precisely it states that the Gysin map

$$(I \circ J)_! : H^0(\widetilde{V};\mathbb{Q}) \to H^{2p}_{(V)}(X;\mathbb{Q}) \tag{1.21}$$

is surjective. Then a pre-image $\tilde{u}$ of u is a cohomological class of degree 0 on the complex manifold $\widetilde{V}$. So, $\tilde{u}$ must be represented by a rational linear combinations of irreducible components of $\widetilde{V}$. Since J is a complex analytic map from $\widetilde{V}$ onto V, $u = (I \circ J)_!(\tilde{u})$ is represented by a rational, linear combination of irreducible components of V. The proof is completed. □

We organize the rest of paper as follows. In section 2, we turn Harvey-Lawson's construction in relative homology to that in absolute homology. In section 3, we iterate the result in Section 2 infinitely many times to prove Main theorem 1.2. In appendix A, we prove a lemma about current's extension that is crucial for the proof in Section 2. In Appendix B, we look into our idea from Demailly's point of view.

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2 Harvey-Lawson's approach

The technique we'll use is derived from Harvey-Lawson's geometric result on the boundary of a homological chain. It is important to notice that they also extended the result to relative homology where our application is focused on.

Definition 2.1. *(Harvey-Lawson [6]) Let X be a compact Kähler manifold. Let M be an oriented compact real analytic submanifold of dimension $2k-1$. A class $\tau \in H_{2k}(X, M; \mathbb{R})$ is called positive if $\tau(\phi) \geq 0$ for all closed, real 2k-forms ϕ whose (k, k) component $\phi^{k,k}$ is strongly positive.*

Our strategy is to turn Harvey-Lawson's result in relative homology to that in absolute homology.

Proposition 2.2. *Let X be a complex projective manifold. Let $\tau \in H_{2k}(X; \mathbb{Q})$ be positive. Then there exist an algebraic cycle V_r with positive rational coefficients and closed positive current S_r of bidimension (k, k) such that τ is represented by $T_{V_r} + S_r$.*

Proof. Let $\tau \in H_{2k}(X; \mathbb{Q})$ be positive. Let N be a positive integer such that $N\tau \in H_{2k}(X; \mathbb{Z})/tors$ is non-zero. Let M be any real analytic compact oriented submanifold of dimension $2k - 1$. We denote $H_{2k}(X, M; \mathbb{Z})/tors$ by $\overline{H}_{2k}(X, M; \mathbb{Z})$. Let

$$\pi : (X, \varnothing) \to (X, M)$$

be the inclusion map for the pairs ‡ . Then $\pi_*(N\tau)$ is in the relative homology $\overline{H}_{2k}(X, M; \mathbb{Z})$ with

$$\partial(\pi_*(N\tau)) = 0 \in i_*(\mathscr{D}'(M))$$

where $i : M \hookrightarrow X$ is the inclusion, and the boundary $\partial(\pi_*(N\tau))$ is a well-defined chain as explained in [6]. By Theorem 3.4, [6], for such $\pi_*(N\tau) \in \overline{H}_{2k}(X, M; \mathbb{Z})$ there exist a positive holomorphic chain $\widetilde{V}$ in the open submanifold $X \backslash M$, and a closed positive current S of bidimension (k, k) on X such that for the simple extension $\mathcal{T}$ of $T_{\widetilde{V}}$ to X, the current $\mathcal{T} + S$, through the inclusion π, represents the relative class $\pi_*(N\tau)$ in

$$\overline{H}_{2k}(X, M; \mathbb{Z})$$

with $\partial(\pi_*(N\tau)) = d\pi_*(\mathcal{T})$. Since $\partial(\pi_*(N\tau)) = 0$ (as a chain), $\mathcal{T}$ is closed as a current in X. Furthermore, since $T_{\widetilde{V}}$ is a locally rectifiable current in the open manifold $X \backslash M$, by Lemma A1 in Appendix, its simple extension $\mathcal{T}$ in X is also locally rectifiable §. Hence we have the closed current $\mathcal{T} \in \mathcal{R}^{loc}_{k,k}(X)$-the set of bidimension (k, k) currents that are locally rectifiable. The main theorem of [1] says that $\mathcal{T}$ is a holomorphic chain (i.e. as a current of integration) in X. According to Chow's theorem, this holomorphic chain is an algebraic cycle. Let's denote it by V. So $\mathcal{T} = T_V$. Notice that $N\tau$ is represented by $T_V + S + \mathcal{T}_{tor}$ in $H_{2k}(X; \mathbb{Z})$ with a torsion $\mathcal{T}_{tor}$. So, τ is represented by

$$\frac{T_V}{N} + \frac{S}{N}$$

‡Harvey-Lawson's work is about relative homology only. Therefore it is necessary for our application to use π to distinguish the absolute homology and the relative homology.

§This can also be confirmed by Lemma 3.13, chapter 6, [9].

in $H_{2k}(X;\mathbb{Q})$ such that $\frac{T_V}{N}$ is an algebraic cycle with positive rational coefficients and $\frac{S}{N}$ is a positive current of bidimension (k,k). We complete the proof.

□

Remark. Harvey-Lawson's original work used the weak positivity for the $\phi^{k,k} \geq 0$ (as that in Definition 2.1). For a convenience of the iteration below, we switch it to the strong positivity. This technical altering does not change the proof in the reference [6].

Proposition 2.2 contains two steps: (1) construct the holomorphy; (2) extend the holomorphy. Harvey-Lawson's method constructs the holomorphy which is paused at the lower dimensional submanifold M. The following is a trivial example showing how to extend the holomorphy across M.

Example 2.3. *Let X be a compact Kähler manifold, $V \subset X$ be a compact complex submanifold of the complex dimension k. Let $M \subset V$ be a $2k-1$ dimensional sphere that bounds a $2k$ dimensional Euclidean ball $B \simeq \mathbb{R}^{2k} \subset V$, i.e. $M = \partial B$. Let $\tau = [V] \in H_{2k}(X;\mathbb{Z})$. Let $\pi : (X, \varnothing) \to (X, M)$ be the inclusion map. Then $\pi_*(\tau) \in H_{2k}(X, M;\mathbb{Z})$. In the open manifold $X\backslash M$, the holomorphic chain $\widetilde{V}$ for this $\pi_*(\tau)$ (as that in Proposition 2.2) is the sum of two holomorphic chains $B + V\backslash\overline{B}$. The simplex extension (extension-by-0) of each holomorphic piece, has a boundary with the same un-oriented manifold M but the orientation opposite to each other, i.e. TM is maximally complex everywhere. The simple extension as the sum of the currents is*

$$T_B + T_{V\backslash B},$$

which is equal T_V (the key requirement for the holomorphic extension, in a general case, is that the boundary of the current $T_B + T_{V\backslash B}$ is zero. Of course, this is obvious in our trivial case.). Hence through the simple extension of currents, we extend $\widetilde{V}$ across M to obtain the subvariety V. With $S = 0$, τ is represented by the current $T_V + S$.

3 Proof

Proof of Main theorem 1.2: (1) First we assume that the non-zero rational class u is represented by a closed positive current S_0 of bidegree (p,p). Then the class $[S_0]$ by the definition is positive. Let $k = dim(X) - p$. By Proposition 2.2, there exist an algebraic cycle $\tilde{V}_1$ with positive rational coefficients, and a positive closed current S_1 of bidimension (k,k) such that for $V_1 := T_{\tilde{V}_1}$, $[S_0]$ is represented by the current $S_1 + V_1$, i.e.

$$S_0 = S_1 + V_1 + d\Gamma_1 \tag{3.1}$$

where Γ_1 is a current of dimension $2k+1$. Let ω be the Káhler form. We obtain

$$S_0[\frac{\omega^k}{k!}] = S_1[\frac{\omega^k}{k!}] + V_1[\frac{\omega^k}{k!}] + (d\Gamma_1)[\frac{\omega^k}{k!}]. \tag{3.2}$$

Since $\tilde{V}_1$ has positive coefficients, V_1 is a positive current. Hence S_0, S_1, V_1 are all positive currents of bidimension (k, k). By the mass formula used in the proof of Corollary 1.3, (3.2) can be written as

$$\mathbf{M}(S_0) = \mathbf{M}(S_1) + \mathbf{M}(V_1) \tag{3.3}$$

where $\mathbf{M}(\bullet)$ is the mass associated to the Kähler metric. Since V_1 is positive,

$$\mathbf{M}(S_0) > \mathbf{M}(S_1). \tag{3.4}$$

Since S_1 is positive with homology in $H_{2k}(X;\mathbb{Q})$, applying Proposition 2.2, we can iterate the decomposition (3.1) for the positive current S_1, then afterwards iterate it for the derived positives currents $S_2, S_3, \cdots$ that follow. With finitely many such iterations, we obtain

$$S_0 = S_N + \sum_{i=1}^{N} V_i + d\Gamma_N \tag{3.5}$$

where N is a natural number, and V_i are algebraic cycles with positive rational coefficients. Then similarly

$$\begin{aligned} S_0[\frac{\omega^k}{k!}] &= S_N[\frac{\omega^k}{k!}] + \left(\sum_{i=1}^{N} V_i\right)[\frac{\omega^k}{k!}] \\ &= S_N[\frac{\omega^k}{k!}] + \sum_{i=1}^{N} V_i[\frac{\omega^k}{k!}]. \end{aligned} \tag{3.6}$$

It implies that

$$\mathbf{M}(S_0) = \mathbf{M}(S_N) + \sum_{i=1}^{N} \mathbf{M}(V_i). \tag{3.7}$$

The mass inequality (3.4) is extended to the decreasing sequence

$$\mathbf{M}(S_0) > \mathbf{M}(S_1) > \cdots \mathbf{M}(S_N) > \cdots \tag{3.8}$$

Since the cone of positive currents is closed, the limit $\lim_{N\to\infty} \mathbf{M}(S_N)$ must be zero (otherwise the iteration could continue). So we obtain the infinite series for currents

$$S_0 = \sum_{i=1}^{\infty} V_i + d\Gamma_\infty \tag{3.9}$$

where V_i are integration currents over algebraic cycles with positive rational coefficients, and Γ_∞ is some current of dimension $2k+1$ (may not be the limit of Γ_N). By the positivity of the currents T_{V_i}, the convergence of (3.9) is the absolute mass-convergence. Hence u is an infinitely algebraic class with positivity. This completes the proof of Part (1).

(2). In general, we write

$$u = \sum_{finite\ j} b_j u_j \tag{3.10}$$

where b_j are real numbers and u_j are rational classes, and of (p,p) over $\mathbb{C}$. Let Ψ_j be a current of bidegree (p,p) that represents u_j. We write

$$\Psi_j = aT_L + \Psi_j - aT_L \tag{3.11}$$

where a is a real number and L is the p-codimensional plane section. Since u_j is of (p,p) type, the idea of Demailly in [4], is that for a sufficiently large a, the closed current

$$aT_L + \Psi_j$$

is positive. By Part (1), the cohomology class of $aT_L + \Psi_j$ is infinitely algebraic. So is u_j. This completes the proof of Part (2), i.e. for general u.

□

Appendix A Simple extension of locally rectifiable currents

The extension of currents is an important notion. Let Ω be an open subset of a complex manifold Y. Let $t \in \mathscr{D}'(\Omega)$. Any current $\tilde{t}$ on Y that is restricted to t on Ω is called an extension of t. Not all currents in Ω can have extensions, and extensions may not be unique if they exist. A simple extension of $t \in \mathscr{D}'(\Omega)$ is an extension-by-zero, but a particular one defined by Lelong. Its existence needs two requirements: 1) the order of t is 0; 2) the local mass of t is bounded. These two conditions allow t to be extended to 0 outside uniquely (see [8] for the general definition and [5] for the particular case in this paper.).

Lemma A.1. *Let Y be compact. Let $t \in \mathscr{D}'(\Omega)$ be a current that has a simple extension, denoted by $\tilde{t}_o$. Then if t is locally rectifiable, so is $\tilde{t}_o$.*

Proof. Since Y is compact, its weak limits are also the mass limits (i.e. the type of strong limits). Let ϕ_j be a partition of unity for a countable open covering of Ω. By the argument in [8], for a test form ψ, we have the evaluation

$$\tilde{t}_o[\psi] = \sum_{j=1}^{\infty} t[\phi_j \psi] \tag{A.1}$$

which is independent of the choice of the partition of unity ϕ_j. Since $t\sum_{j=1}^{N}\phi_j$ is well-defined in Y, the weak limit

$$\lim_{N \xrightarrow{weakly} \infty} t\sum_{j=1}^{N}\phi_j$$

is also well-defined in Y (see (A.1)). Let $\epsilon > 0$ and P_ϵ be the Lipschitzian chains for the approximation of t in Ω. Let $\mathbf{M}_Y$ be the mass on Y, which is also the mass $\mathbf{M}_\Omega$ for currents on Ω, that have the extension by zero to X. We have the computation

$$\begin{aligned}
&\mathbf{M}_Y(\tilde{t}_o - T_{P_\epsilon}) = \mathbf{M}_Y\left(\lim_{N \xrightarrow{weakly} \infty} (t - T_{P_\epsilon})\sum_{j=1}^{N}\phi_j\right) \\
&\text{(Since the weak limit is the same as the mass limit)} \\
&= \lim_{N\to\infty}\mathbf{M}_\Omega\left((t - T_{P_\epsilon})\sum_{j=1}^{N}\phi_j\right) \\
&\text{(since } t - T_{P_\epsilon} \text{ has order 0)} \\
&\leq \lim_{N\to\infty}\left\|\sum_{j=1}^{N}\phi_j\right\|_\infty \mathbf{M}_\Omega(t - T_{P_\epsilon}) \\
&= \mathbf{M}_\Omega(t - T_{P_\epsilon}) \\
&\text{(since } t \text{ is rectifiable, there is some } P_\epsilon) \\
&\leq \epsilon.
\end{aligned}$$

Thus $\tilde{t}_o$ is approximated by the same Lipschitzian chains P_ϵ as that for t. Therefore $\tilde{t}_o$ is also locally rectifiable. □

Appendix B Demailly's approach

The principle of the proof is transcendental and is related to the work done by many people in the past. The one that is worth to be pointed out is Demailly's work on currents. Demailly's work focused on currents, but our theorem addresses cohomology classes. Thus the difference between classes and currents droves us apart. But we made an adjustment of it. In this section, we give the description without any proof. We follow the interpretation in [3].

Let X be a complex projective manifold. We first organize the currents in Demailly's idea. Let $\Sigma(X)$ be the subspace that consists of real closed currents $\mathcal{T} \in \mathscr{D}'(X)$ such that $[\mathcal{T}] \in H^{p,p}(X;\mathbb{Z}) \otimes \mathbb{R}$. Let $\Sigma^+(X)$ be the subset of

$\Sigma(X)$ such that those currents are strongly positive ($SPC^p_{\mathbb{Z}}(X)$-the notion by Demailly). Let $\Lambda(X)$ be the subspace of currents in the form

$$\lim_{i\to\infty} \mathcal{T}_i \qquad with \tag{B.1}$$

$$\mathcal{T}_i = \sum_j \lambda_{ij} T_{V_{ij}} \tag{B.2}$$

where λ_{ij} are real numbers, V_{ij} are irreducible subvarieties of codimension p and the limits are all in the weak topology of $\mathscr{D}'(X)$. Let $\Lambda^+(X)$ be the subset such that those coefficients λ_{ij} in (B.2) are all positive. We define the statements

$$\mathcal{H}(X):\ \Sigma(X) = \Lambda(X) \tag{B.3}$$

$$\mathcal{H}^+(X):\ \Sigma^+(X) = \Lambda^+(X). \tag{B.4}$$

Demailly proved

$$\mathcal{H}^+(X) \Rightarrow Corollary\ 1.3 \Leftrightarrow \mathcal{H}(X). \tag{B.5}$$

and asked weather $\mathcal{H}(X)$ implies $\mathcal{H}^+(X)$.

However, in [3], Babaee and Hub provided a counter-example that shows $\mathcal{H}^+$ is false. So, their result shows that $\Sigma^+(X)$ can not be approximated by algebraic cycles with positive real coefficients. This does not go well with Demailly's result which asserts that $span(\Sigma^+)$ is approximated by algebraic cycles.

The Main theorem 1.2 explains the situation as follows

Result B.1. *There is a subset* $\mathcal{P}^+(X) \subsetneq \Sigma^+$ *with*

$$span(\big[\mathcal{P}^+(X)\big]) = H^{p,p}(X;\mathbb{Z})\otimes\mathbb{R}, \tag{B.6}$$

that can be approximated by algebraic cycles with positive rational coefficients.

It indicates that convex cone approximation is not sufficiently precise. We adjust it with the precision that comes from Harvey-Lawson's work on the boundaries of holomorphic chains. Main theorem claims that the adjustment $\mathcal{P}^+(X)$ is the set of the infinite series of currents $\sum_{i=1}^{\infty} V_i$ as in (3.9), derived from cohomology $H^{p,p}(X;\mathbb{Z})\otimes\mathbb{G}$ via Harvey-Lawson. So, they are particular types of infinitely algebraic cycles (not classes) with positivity.

This paper has no conflict of interest.

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