Emergence of Classical Gravity from Continuous Spacetime Dimensions

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Abstract

We develop a geometric framework in which classical gravity emerges from primordial spacetime having *continuous effective dimensions*. Spacetime is modeled as an evolving multifractal structure, analog to the construction of Cantor Dust (CD), where the Hausdorff measure replaces ordinary volume. Two fundamental findings are uncovered, namely, 1) CD is directly tied to Dark Matter phenomenology; 2) Einstein-Hilbert formulation of General Relativity emerges as an effective action of CD.

Key words: Continuous spacetime dimensions, Dark Matter, Cantor Dust, Einstein-Hilbert action.

1. Introduction

The standard formulation of gravity assumes that spacetime is a smooth four-dimensional manifold endowed with a metric structure. Both Newtonian gravity and General Relativity rely fundamentally on the existence of local neighborhoods, differentiable coordinates, and integer-dimensional volume elements. However, a growing body of evidence from the complex dynamics of nonlinear systems, Renormalization Group flows and astrophysical observations suggests that spacetime may exhibit nontrivial structure across scales.

Independently, Dimensional Regularization in Quantum Field Theory has long relied on the formal extension of spacetime dimension to non-integer values. While traditionally regarded as an exclusive calculational device, analysis has shown that dimensional continuation can be invoked as justification for minimal spacetime fractality in Physics [8]. In this context, it is worth noting that the work of Tao [1] has demonstrated that Dimensional

Regularization remains valid when the underlying spacetime is equipped with an appropriate fractal measure.

In this paper, we take this perspective seriously at the classical level. We propose that spacetime itself possesses a continuous effective dimension D, realized geometrically through Cantor-Dust–like constructions. Crucially, such spaces possess no ordinary volume, yet they carry a finite Hausdorff measure. We show that this measure must be identified with gravitational mass, as it is the only quantity to which gravity can be physically coupled.

By reformulating the Poisson equation on a fractal measure space, we derive a nonlocal gravitational Green's function whose scaling depends explicitly on the effective spacetime dimension. No additional Dark Matter fields, modified inertia, or phenomenological force laws are introduced. Instead, gravitational anomalies arise as direct consequences of spacetime geometry itself.

We caution upfront that this report serves exclusively as an introductory study. Interested researchers are encouraged to review, develop or refute the body of ideas detailed here.

Notations and Definitions

1) $D(x)$ is the spacetime dimension, fluctuating around $D_0 = 4$.

2) D is the intrinsic fractal dimension, $1 \leq D \leq 3$.

3) In Riemannian geometry, a *geodesic ball* $B_r(x)$ is the set of points whose geodesic distance from x is less than r; its volume growth encodes curvature through deviations from Euclidean scaling. In this context, Euclidean scaling refers to the power-law growth of geometric volumes with *integer exponent* equal to the dimension of flat space, $V(r) \propto r^n$, and serves as the reference against which curvature and fractal deviations are measured.

2. Mathematical Framework and Field Equations

2.1 Fractal Spacetime and Measure Structure

Following [1-7, 15], we model spacetime as a metric measure space $(\mathcal{M}, d, \mu)$, where $\mathcal{M} \subset \mathbb{R}^n$ is a fractal support, $d(x, y)$ is the embedding metric, and μ is a Hausdorff-type measure. A prototypical realization is provided by Cantor dust, constructed as the Cartesian product of Cantor sets. Such spaces are totally disconnected, self-similar at all scales, and lack differentiable local neighborhoods [1-3].

The defining property of the fractal measure is its scaling behavior:

$$\mu(B_r(x)) \sim r^D, \qquad (1)$$

where $B_r(x)$ is a metric ball of radius r centered at x, and D is the effective (Hausdorff) dimension of spacetime. For $D \neq n$, the ordinary Lebesgue volume vanishes identically,

$$\int_{\mathcal{M}} d^n x = 0, \qquad (2)$$

while the fractal measure remains finite and nontrivial. Consequently, all physical integrals must be defined with respect to $d\mu(x)$.

<u>2.2 Identification of Fractal Measure with Gravitational Mass</u>

In classical gravity, mass is defined operationally as the quantity sourcing the gravitational field through spatial integration. In a fractal spacetime (which, by definition, is endowed with *continuous spacetime dimensions D*), the only admissible generalization of volume integration is via the *fractal measure*. We therefore identify the enclosed gravitational mass as

$$M(r) \equiv \int_{B_r} d\mu(x) \sim r^D. \qquad (3)$$

This identification is <u>*not optional*</u>: since ordinary volume vanishes, the fractal measure is the only quantity capable of sourcing gravity. It follows that the measure plays the physical role traditionally assigned to mass density.

2.3 Gravitational Field Equation on Fractal Spacetime

The Poisson equation of Newtonian gravity is generalized to fractal spacetime by replacing the Laplacian with a fractional or measure-adapted operator $\mathcal{L}_D$:

$$\mathcal{L}_D \Phi(x) = 4\pi G\, \mu(x). \qquad (4)$$

Here, $\mu(x)$ denotes the local density of fractal measure, and $\mathcal{L}_D$ reduces to the standard Laplacian only when $D = 3$. For isotropic fractal geometries, $\mathcal{L}_D$ may be represented by a *fractional Laplacian*,

$$\mathcal{L}_D \equiv (-\Delta)^{D/2-1}. \qquad (5)$$

2.4 Nonlocal Gravitational Green's Function

The gravitational potential is expressed in terms of a Green function $G(x, y)$ defined by

$$\mathcal{L}_D G(x, y) = \delta_\mu(x, y), \qquad (6)$$

where $\delta_\mu(x, y)$ is the Dirac delta distribution with respect to the fractal measure:

$$\int f(y)\, \delta_\mu(x, y)\, d\mu(y) = f(x). \qquad (7)$$

The solution of Eq. (6) yields

$$\Phi(x) = \int G(x, y)\, d\mu(y). \qquad (8)$$

Dimensional analysis fixes the asymptotic form of the kernel:

$$G(x, y) \sim \frac{1}{|\, x - y \,|^{(D-2)}}, \qquad D \neq 2. \qquad (9)$$

For $D = 3$, Newtonian gravity is fully recovered, $G \sim 1/|\, x - y \,|$. For $D < 3$, gravity is enhanced by long-range interactions, while for $D > 3$ it becomes a limited-range (screened) interaction.

2.5 Emergent Force Law and Dimensional Gravity

Gravitational acceleration follows directly from the potential:

$$g(r) = -\nabla\Phi(r) \sim \frac{GM(r)}{r^2} \sim r^{D-2}. \qquad (10)$$

Thus, the force law is determined entirely by the effective spacetime dimension. Unlike MOND or alternative gravitation theories, *no modification* of the gravitational constant or field equations is required [4-7].

2.6 Action Principle on Fractal Spacetime

Only to the extent that the action principle holds as *effective approximation* in fractal spacetime, the gravitational action can be written as

$$S = \int d\mu(x) \left[\frac{1}{16\pi G} R + \mathcal{L}_{\text{matter}}\right]. \qquad (11)$$

Variation with respect to the metric or measure yields field equations in which curvature responds directly to changes in the fractal measure. Even in the absence of conventional matter fields, *spacetime itself carries gravitational mass.*

2.7 Relation to Dimensional Regularization

Tao has shown that dimensional regularization is mathematically consistent when integrals are defined over fractal measure spaces [1, 7]:

$$\int d^D x \;\leftrightarrow\; \int d\mu(x). \qquad (12)$$

In this framework, propagators take the form

$$G(x,y) = \int \frac{d\mu(k)}{\mid k \mid^D} e^{ik\cdot(x-y)}, \qquad (13)$$

demonstrating that nonlocal gravitational kernels arise naturally from the geometry of spacetime rather than from ad-hoc modifications.

2.8 Stochastic and Nonlocal Effects

Inherent fluctuations in the fractal measure,

$$\mu(x) \to \mu(x) + \delta\mu(x), \qquad (14)$$

lead to correlated gravitational force fluctuations:

$$\langle F(x)F(y)\rangle \sim \nabla_x \nabla_y G(x,y). \qquad (15)$$

These correlations generate effective self-interactions and transport phenomena without introducing new particle degrees of freedom. These observations carry key consequences in modeling Dark Matter and primordial gravity [4-7].

3. Phenomenological Implications

3.1 Galactic Rotation Curves

In the present framework, the gravitational acceleration produced by a fractal spacetime of effective dimension D scales as [4-7]

$$g(r) \sim r^{D-2}. \qquad (16)$$

The circular velocity of a test particle orbiting at radius r,

$$v_c^2(r) = r\, g(r), \qquad (17)$$

scales as

$$v_c(r) \sim r^{(D-1)/2}. \qquad (18)$$

For a background having $D \approx 1$ or $D \approx 2$ (filamentary Cantor Dust or sheet − like fractal distribution), the velocity becomes approximately constant, reproducing the observed flat rotation curves of spiral galaxies without invoking Dark Matter halos. Importantly, this behavior arises from the geometric scaling of the fractal measure rather than from modifications of inertia or phenomenological interpolation functions. The effective dimension may vary weakly with scale, allowing smooth interpolation between Newtonian dynamics in the inner regions and flat rotation curves at large radii. Similar findings extend to the *baryonic Tully-Fisher relation* [4-7].

3.2 Gravitational Lensing

Gravitational lensing depends on the integrated gravitational potential along the line of sight. In fractal spacetime, the potential is given by

$$\Phi(x) = \int G(x, y)\, d\mu(y), \qquad (19)$$

with the nonlocal kernel $G(x, y) \sim | x - y |^{(D-2)}$.

Because fractal measure acts as gravitational mass, lensing probes the same geometric quantity responsible for dynamical effects. The deflection angle α scales as

$$\alpha \sim \int \nabla_{\perp}\Phi \, dl \sim \int d\mu, \qquad (20)$$

implying enhanced lensing relative to visible matter alone. This may naturally explain the observed coincidence between dynamical mass inferred from rotation curves and lensing mass, without introducing collisionless Dark Matter particles. Since lensing depends only on the measure distribution, the framework may predict consistent lensing signals even in systems with low baryonic content.

3.3 Emergent Self-Interacting Dark Matter Phenomenology

Fluctuations of the fractal measure induce correlated force fluctuations:

$$\langle F(x)F(y)\rangle \sim \nabla_x \nabla_y G(x,y). \qquad (21)$$

These correlations lead to effective momentum exchange between trajectories moving through spacetime, giving rise to transport phenomena analogous to *self-interacting dark matter* (SIDM) [4].

The effective scattering rate scales with the variance of measure fluctuations,

$$\Gamma_{\text{eff}} \sim \int \langle \delta\mu(x)\, \delta\mu(y) \rangle\, G^2(x, y)\, d\mu(y), \qquad (22)$$

yielding cross sections that are:

- velocity-dependent,
- enhanced in low-velocity environments,
- suppressed in high-velocity systems.

These properties align with SIDM phenomenology inferred from dwarf galaxies, galaxy clusters, and halo cores, while avoiding the introduction of new particle species. In this picture, SIDM behavior is not fundamental but emergent, arising from fractal geometry rather than microscopic interactions.

3.4 Summary of Potential Observational Signatures

When fully developed, our framework can likely predict:

1. Flat or slowly rising rotation curves without Dark Matter halos,
2. Gravitational lensing consistent with dynamical mass estimates,
3. Velocity-dependent self-interaction effects emerging from geometry,
4. Smooth transitions between regimes controlled by effective dimension $D(r)$.

These signatures may open clear observational avenues for testing gravity as an emergent consequence of continuous spacetime dimensions.

A full report on these observations and similar findings is planned to be provided in [4].

4. Conclusions

We have shown that classical gravity can be derived from spacetime endowed with continuous, non-integer effective dimensions. In such geometries, exemplified by Cantor Dust (CD) constructions, ordinary volume vanishes and is replaced by a fractal measure. This measure is *not a mathematical artifact* but a physical quantity that must be identified with gravitational mass.

The resulting gravitational field equations are intrinsically *nonlocal* and *stochastic*, with Green's functions determined by the fractal dimension of spacetime. Deviations from Newtonian gravity emerge naturally when the effective dimension differs from 3, providing a geometric explanation for phenomena traditionally attributed to Dark Matter. Importantly, this framework does not modify the fundamental principles of gravity but generalizes the notion of spacetime on which they act.

Our results establish a direct link between Dimensional Regularization, fractal geometry, and classical gravity. They suggest that gravitational dynamics may ultimately be governed not by additional matter components but by the dimensional structure of spacetime itself. This perspective opens a new avenue for understanding gravitational anomalies as emergent phenomena rooted in geometry rather than particle physics.

APPENDIX A

On the Statistical Emergence of the Einstein–Hilbert Action from Fractal Measures

Near the Planck scale, spacetime is endowed with a Cantor Dust (CD) structure defined by:

- a support set $\mathcal{M}$,
- a distance $d(x, y)$,
- a fractal measure $d\mu(x)$.

There is no smooth metric field $g_{\mu\nu}$ and no curvature tensor. By construction, the fundamental action of spacetime is purely measure-based, which means that:

$$S_0 = \int d\mu(x) \qquad \text{(A1)}$$

Consider a coarse-graining scale ℓ such that:

- ℓ is much larger than the Cantor scale,
- ℓ is much smaller than astrophysical scales.

Define coarse-grained cells $B_\ell(x)$. The average measure over such cells is

$$\langle d\mu \rangle_\ell = v_\ell(x)\, d^4x \qquad \text{(A2)}$$

This defines a *smooth density field* $v_\ell(x)$. Since, at macroscopic scales commensurate with ℓ, physics probes only averaged quantities, we define the following metric determinant

$$\boxed{\sqrt{-g_\ell(x)} \equiv v_\ell(x)} \qquad \text{(A3)}$$

Note that (A3) *is not* an assumption — it is dictated by the following observations:

- All classical actions depend on the measure,
- Only a metric determinant plays this role in continuum theories.

Note that (A3) has the properties of an *emergent statistical metric*.

In Riemannian geometry, the volume of a geodesic ball satisfies:

$$\mathrm{Vol}(B_r) = \omega_4 r^4 \left[1 - \frac{R(x)}{6} r^2 + O(r^4)\right] \qquad \text{(A4)}$$

where $R(x)$ represents the Ricci curvature [11-14]. In fractal spacetime one has instead:

$$\mathrm{Vol}(B_r) = \int_{B_r} d\mu \sim r^{D(x)} \qquad \text{(A5)}$$

Expanding (A5) for slowly varying $D(x)$ yields

$$r^{D(x)} = r^4 \exp\left[(D(x) - 4) \ln\, r\right] \qquad \text{(A6)}$$

Comparing (A4) and (A6), leads to the identification

$$\boxed{R(x) \;\propto\; -\nabla^2 D(x)} \qquad \text{(A7)}$$

In this picture, *Ricci curvature emerges as statistical effect of dimensional fluctuations.* The number of microscopic spacetime configurations inside B_r takes the form

$$\Omega(r) \sim \exp\left(\alpha\, r^{D(x)}\right) \qquad \text{(A8)}$$

with α a scaling constant. Now, define a geometric entropy:

$$S_{\text{geom}} = \ln\, \Omega \sim r^{D(x)} \qquad \text{(A9)}$$

The action governing geometry can therefore be considered as being of an *entropic nature,*

$$S_{\text{eff}} \sim \int d^4x\, \sqrt{-g}\, \mathcal{F}[D(x), \partial D(x)] \qquad \text{(A10)}$$

Expanding to second order in gradients:

$$\mathcal{F} = \Lambda + \beta(\partial_\mu D(x))(\partial^\mu D(x)) + \cdots \qquad \text{(A11)}$$

and using (A7), it can be shown that the gradient term maps to curvature as in:

$$(\partial(\delta D))^2 \longrightarrow R \qquad \text{(A12)}$$

Combining the above, the effective macroscopic action echoes the Einstein-Hilbert (EH) action,

$$\boxed{S_{\mathrm{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}(R - 2\Lambda)} \qquad \text{(A13)}$$

in which Λ encodes the mean fractal density and curvature arises exclusively from dimensional parameters. Varying the effective action (A13) with respect to $g_{\mu\nu}$ yields:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}^{(\mathrm{matter})} \qquad \text{(A14)}$$

Because the underlying theory is *fractal* and *nonlocal*:

- higher-order terms of (A13) must be accounted for,
- fractional Laplacians emerge,
- nonlocal kernels correct Einstein gravity.

At least in principle, this explains Dark Matter–like behavior, and scale-dependent gravity.

The reader is directed to Appendix B for a step-by-step justification of (A7) and (A10).

Summing up, the EH action emerges as the coarse-grained, entropic effective action of a fractal spacetime endowed with a nontrivial measure. In this formulation, curvature encodes fluctuations of spacetime dimensionality.

One must keep in mind, however, that there are several conceptual limitations of the derivation outlined in this Appendix. For instance,

1) Lagrangian description of phenomena occurring near the Planck scale is prone to break down somewhere below this scale [9],

2) Following the points raised in [10], embedding the derivation in a Quantum Gravity (QG) context is likely incorrect.

It is worth noting that, while scale-dependent dimensions have been extensively investigated in many QG studies, the identification of curvature and dimensional deviations *is unrelated* to QG and represents our original contribution.

APPENDIX B: On the Derivation of (A7) and (A10)

Consider the effective action (A10):

$$S_{\text{eff}} \sim \int d^4x \sqrt{-g}\, \mathcal{F}[D(x), \partial D(x)] \qquad \text{(A10)}$$

At the "microscopic" Cantor Dust level, the *only* fundamental structure is the measure $d\mu(x)$. There is no metric, no curvature tensor and no preferred coordinates. After coarse graining:

$$d\mu(x) \;\longrightarrow\; v(x)\, d^4x$$

with

$$v(x) \equiv \sqrt{-g(x)}.$$

Thus, the only emergent macroscopic field is $D(x)$ (or equivalently $g_{\mu\nu}(x)$), and no other degrees of freedom exist at this level. Next, we require (A10) to satisfy three constraints, namely:

(a) General covariance

Because Cantor Dust has no preferred coordinates, the coarse-grained theory must be invariant under diffeomorphisms. This forces the introduction of the measure factor $\sqrt{-g}$.

(b) Locality at macroscopic scales

A reasonable hypothesis is that, although the microscopic theory is nonlocal, coarse graining over many cells suppresses nonlocality. The effective action (A10) must therefore admit a *local derivative expansion*.

(c) Statistical isotropy and homogeneity

Cantor Dust is *statistically* isotropic. Therefore:

- it contains no vector terms,

- it has no odd derivatives,
- it can only include scalar combinations.

At leading order, the only scalar built from $D(x)$ is either $D(x)$ itself or the product $\left(\partial_\mu D(x)\, \partial^\mu D(x)\right)$. No other terms exist until higher derivatives are added. By analogy with the *Landau-Ginzburg functional*, the most general leading-order action is:

$$\mathcal{F}[D(x), \partial D(x)] = V[D(x)] + K[D(x)][\partial D(x)]^2 + O(\partial^4) \qquad \text{(B1)}$$

The system has a preferred mean dimension $D_0 \approx 4$ and fluctuations away from this value are entropically suppressed. Thus, the first term in (B1) assumes the form,

$$V(D) \simeq \Lambda + \frac{m^2}{2}(D(x) - D_0)^2 + \cdots$$

where Λ denotes the *average spacetime density* (echoing the contribution of cosmological constant) and the *quadratic term* is the analogue of the mass term in the Landau–Ginzburg theory. Introducing this term in the expression of $V[D(x)]$ is dictated by,

- smoothness of the coarse-grained geometry,
- central-limit suppression of large gradients,
- statistical independence of distant Cantor cells.

The takeaway point here is that the effective action (A10) is the most general diffeomorphism-invariant, local functional governing the coarse-grained statistics of a fractal measure. Its structure is uniquely fixed by symmetry and the requirement that spatial variations of the effective dimension be entropically suppressed. The Einstein–Hilbert (EH) action emerges as the leading term in this expansion, with higher-order corrections encoding residual nonlocality of the microscopic Cantor Dust geometry.

To justify the derivation of (A7) from (A4) and (A6) we proceed as follows:

According to [11-14], equation (A4),

$$\mathrm{Vol}(B_r) = \omega_4 r^4 \left[1 - \frac{R(x)}{6} r^2 + O(r^4)\right]$$

is a geometric identity in smooth Riemannian geometry reflecting how curvature modifies the volume growth of small geodesic balls. Taking the logarithm

$$\ln \mathrm{Vol}(B_r) = \ln \omega_4 + 4\ln r + \ln\left(1 - \frac{R}{6}r^2 + \cdots\right)$$

and expanding for small r gives

$$\ln\left(1 - \frac{R}{6}r^2\right) \simeq -\frac{R}{6}r^2$$

and so:

$$\boxed{\ln \mathrm{Vol}(B_r) \simeq \mathrm{const} + 4\ln r - \frac{R(x)}{6}r^2 + O(r^4)} \qquad \text{(B2)}$$

Recall the definition

$$\mathrm{Vol}(B_r) \sim r^{D(x)}$$

and take the logarithm

$$\ln \mathrm{Vol}(B_r) = D(x) \ln r$$

Now write:

$$D(x) = 4 + \delta D(x), \qquad |\, \delta D \,| \ll 1$$

Then:

$$\ln \mathrm{Vol}(B_r) = 4 \ln r + \delta D(x) \ln\, r \qquad \text{(B3)}$$

In a coarse-grained theory, $D(x)$ is not evaluated at a point but averaged over the ball $B_r(x)$:

$$\delta D(x) \;\rightarrow\; \langle \delta D(x) \rangle_{B_r}$$

For a smooth field $D(x)$, a Taylor expansion gives:

$$\langle \delta D(x) \rangle_{B_r} = \delta D(x) + \frac{r^2}{12} \nabla^2 \delta D(x) + O(r^4) \qquad \text{(B4)}$$

Substituting (B4) into (B3) yields

$$\ln\, \mathrm{Vol}(B_r) \simeq 4 \ln\, r + \left[\delta D(x) + \frac{r^2}{12} \nabla^2 \delta D(x) \right] \ln\, r$$

or, isolating the leading r^2 correction:

$$\ln\,\mathrm{Vol}(B_r) \simeq 4\ln\,r + \delta D(x)\ln\,r + \frac{r^2}{12}\,[\nabla^2 \delta D(x)]\,\ln\,r \qquad \text{(B5)}$$

Comparing (B2) and (B5) leads to the identification:

$$\frac{R(x)}{6} r^2 \quad <=> \quad \frac{r^2}{12}\,[\nabla^2 \delta D(x)]\; ln\;r$$

At the coarse-graining scale, $\ln r$ is a slowly varying constant fixed by the averaging operation. Absorbing numerical factors into normalization, we are led to

$$\boxed{R(x)\;\propto\;-\,\nabla^2 \delta D(x)} \qquad \text{(B6)}$$

The minus sign reflects the fact that:

- a positive curvature *suppresses* volume growth,
- a positive $\nabla^2 \delta D$ *enhances* effective dimensionality locally.

In summary, comparing the small-radius expansion of geodesic ball volumes in Riemannian geometry with the coarse-grained scaling of fractal measure shows that dimensional deviations from integer dimensionality map to the behavior of scalar curvature.

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