Stochastic Geometric Gravity: A Self-Consistent Framework for Gravitational Fluctuations

Bernard Lavenda *

Abstract

We present a stochastic geometric framework for gravity, starting from the Gravitational Balance Equations (GBE) [1] which arise from varying the Einstein-Hilbert action with respect to sectoral scale factors in a doubly-warped spacetime. The extrinsic curvature is promoted to a random field, and a moment hierarchy is derived from the GBE. A geometric projector closure maps second moments to an effective fluctuation curvature, yielding closed mean equations without ad-hoc stress tensors. The fluctuation energy obeys a generalized Bochner formula, linking geometric dissipation to the mean extrinsic curvature and the intrinsic curvature of the leaves. This approach provides a self-consistent probabilistic description of gravitational fluctuations, revealing that classical general relativity is not a fundamental deterministic theory but rather the first-moment truncation of a more complete stochastic geometric description. In particular, the so-called "exact" vacuum solutions of Einstein's equations–such as Schwarzschild–are not exact; they are mean-field approximations that neglect the essential nonlinear term $K_{AB}K^{AB}$ and all higher fluctuations. This neglect becomes manifest in regimes beyond the photon sphere ($G < 3M$), where the classical hierarchy of terms breaks down and the mean-field description yields unphysical results.

1 Introduction

General relativity (GR) is traditionally regarded as a deterministic theory: given initial data, the Einstein equations purport to determine the entire spacetime geometry. However, this deterministic picture rests on a subtle but critical truncation: the reduction of the full Riemannian geometry to the Ricci tensor discards nonlinear combinations of extrinsic curvature, most notably the quadratic term $K_{AB}K^{AB}$. When these terms are retained, the Einstein equations emerge not as fundamental laws but as first-moment (mean-field) equations of a stochastic geometric theory. Consequently, the celebrated "exact" vacuum solutions of GR–such as the Schwarzschild metric–are not exact descriptions of spacetime; they are approximations valid only when fluctuations are negligible. In regimes where the fluctuations become comparable to the mean, such as inside the photon sphere ($G < 3M$), the classical description fails, yielding acausal or unstable predictions. This indicates that

*email: bhlavenda@gmail.com

GR, as currently formulated, is an incomplete theory that neglects the intrinsic stochasticity of geometry.

The introduction of stochasticity into gravitational physics has a long history, motivated by both quantum gravity and classical averaging problems. Early stochastic approaches focused on "semiclassical gravity", where quantum matter fields generate stochastic stress-energy tensors that back-react on the geometry via the semiclassical Einstein equations [2, 3, 4, 5]. This framework, while phenomenologically useful, treats fluctuations as external to the geometry and relies on ad-hoc regularization and renormalization procedures.

A separate line of research emerged from "cosmological back-reaction", where inhomogeneities in the matter distribution are averaged over large scales, leading to effective stress tensors that modify the Friedmann equations [6, 7, 8]. These approaches often introduce effective fluid models or scalar field descriptions, breaking the geometric self-consistency of GR.

More radical proposals include "stochastic quantization" of gravity [9, 10], where spacetime metrics are treated as random fields evolving under a stochastic differential equation, and "Euclidean quantum gravity" [11], where the path integral is defined over Riemannian metrics with stochastic weights. While mathematically sophisticated, these formulations often lack a clear physical interpretation and are difficult to reconcile with Lorentzian causality.

A common limitation of existing stochastic frameworks is their reliance on "phenomenological stress tensors"–either from quantum matter or coarse-grained inhomogeneities–that are grafted onto the Einstein equations without a first-principles geometric justification. This breaks the geometric self-consistency of GR and obscures the intrinsic nature of gravitational fluctuations.

In this work, we propose an alternative route: treat the geometry itself as a stochastic field, with fluctuations arising from the "intrinsic randomness of extrinsic curvature". Starting from a specific class of spacetimes (doubly-warped products), we derive GBE by varying only the sectoral scale factors in the Einstein-Hilbert action. These GBE are evolution equations for the extrinsic curvature of a foliation. We then promote the extrinsic curvature to a random tensor field and derive a moment hierarchy. A closure scheme, respecting the geometric constraints (Gauss-Codazzi-Ricci), maps second moments to an effective fluctuation curvature. The resulting fluctuation energy satisfies a transport equation analogous to the Bochner formula in harmonic-map theory, with curvature-driven dissipation driving the system toward an isotropic (umbilic) state.

This framework does not require any ad-hoc stress tensor; the fluctuations are intrinsic to the geometry. It reveals that the deterministic Einstein equations are merely the first moment of a more fundamental stochastic geometric theory. The approach also highlights the limitations of classical GR: it cannot describe the relaxation of fluctuations or the approach to equilibrium, because it lacks a built-in dissipation mechanism. By incorporating stochasticity and dissipation, we obtain a theory that can describe both the mean gravitational field and its fluctuations in a self-consistent manner. Importantly, the formalism retains the nonlinear curvature terms that are discarded in the standard Ricci reduction, thereby offering a well-behaved description even in regimes where the classical mean-field approximation breaks down–such as inside the photon sphere of a black hole, where the interplay between mean curvature and its fluctuations can regularize the singular behavior predicted by the purely deterministic theory.

In this framework, the Schwarzschild metric is not an exact solution but a mean-field approximation that neglects the stochastic fluctuations encoded in $K_{AB}K^{AB}$. This has profound implications: black hole physics, from the event horizon to the central singularity, must be re-examined through the lens of stochastic geometry, where fluctuations may resolve classical pathologies and offer a more complete description of strong-field gravity.

2 Metric ansatz and geometric definitions

We consider a $(p+q+2)$-dimensional spacetime with a doubly-warped metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + F^2(t,r)\,\mathrm{d}\Omega_p^2 + G^2(t,r)\,\mathrm{d}\Omega_q^2, \tag{1}$$

where $\mathrm{d}\Omega_p^2$ and $\mathrm{d}\Omega_q^2$ are the standard metrics on unit p- and q-spheres, and $F, G > 0$ are the sectoral scale factors. The leaves Σ of constant (t,r) are codimension-2 submanifolds with induced metric

$$h_{ab} = g_{ab} + n_a n_b, \tag{2}$$

where n^a is chosen as a null normal $n^t = -n^r$ along $t-r$ to simplify the writing of the GBE. Indices $A, B, \dots$ denote components tangent to Σ.

2.1 Gauss-map rates and extrinsic and intrinsic curvatures

Define the logarithmic sectoral rates (Gauss-map rates) and their normal Lie derivatives:

$$f := \mathscr{L}_n \ln F, \quad f_n := \mathscr{L}_n f, \qquad g := \mathscr{L}_n \ln G, \quad g_n := \mathscr{L}_n g. \tag{3}$$

In an orthonormal frame adapted to the warped product, the extrinsic curvature of Σ (second fundamental form) is diagonal:

$$K_{AB} = \underbrace{\mathrm{diag}\,(f,\dots,f,g,\dots,g)}_{p\ \text{times},\ q\ \text{times}} = \frac{1}{d} K h_{AB} + \sigma_{AB}, \tag{4}$$

with $h^{AB}\sigma_{AB} = 0$, $d = p+q$, and trace

$$K := h^{AB} K_{AB} = pf + qg. \tag{5}$$

The quadratic invariant,

$$Q(f,g) := K^2 - K_{AB}K^{AB} = 2\left[\binom{p}{2} f^2 + pqfg + \binom{q}{2} g^2\right], \tag{6}$$

counts the number of independent tangent-tangent 2-planes. It can be thought of as a metric on the space of extrinsic curvature states (f,g).

The intrinsic Ricci scalar of Σ is

$$\mathscr{I} = h^{AC}h^{BD}R_{ABCD} = 2\binom{p}{2}\frac{1}{F^2} + 2\binom{q}{2}\frac{1}{G^2}. \tag{7}$$

3 Derivation of the Gravitational Balance Equations

The Einstein-Hilbert action is

$$S_{\mathrm{EH}}[g] = \int \mathrm{d}^{p+q+2}x \sqrt{-g}\, R.$$

Inserting the metric ansatz (1) and integrating over the compact spheres yields an effective action for the scale factors F and G:

$$S_{\mathrm{eff}}[F,G] = \int \mathrm{d}t\, \mathrm{d}r\, \mathscr{L}(F,G,\partial F,\partial G),$$

where the Lagrangian density $\mathscr{L}$ follows from the Ricci scalar of the doubly-warped metric (1):

$$\begin{aligned} R = {} & 2\binom{p}{2}\frac{1}{F^2} + 2\binom{q}{2}\frac{1}{G^2} - 2p\frac{\Box F}{F} - 2q\frac{\Box G}{G} \\ & - \binom{p}{2}\frac{(\nabla F)^2}{F^2} - \binom{q}{2}\frac{(\nabla G)^2}{G^2} - 2pq\frac{\nabla F\cdot\nabla G}{FG}, \end{aligned} \tag{8}$$

with

$$\Box F = -\partial_t^2 F + \partial_r^2 F, \qquad (\nabla F)^2 = -(\partial_t F)^2 + (\partial_r F)^2, \qquad \nabla F\cdot\nabla G = -\partial_t F\,\partial_t G + \partial_r F\,\partial_r G.$$

We now introduce the logarithmic sectoral rates (Gauss-map rates) along the chosen null normal $n^a = (1,-1,0,\ldots)$:

$$f = \mathscr{L}_n \ln F, \qquad g = \mathscr{L}_n \ln G, \qquad \mathscr{L}_n = n^t\partial_t + n^r\partial_r = \partial_t - \partial_r.$$

To obtain a first-order Lagrangian, we integrate by parts the terms containing second derivatives. For the $\Box$-terms this gives:

$$\begin{aligned} \int \mathrm{d}t\,\mathrm{d}r\sqrt{-g}\left(-2p\frac{\Box F}{F}\right) &= \oint_{\partial\mathcal{M}} \mathrm{d}S_\mu \left(-2p\sqrt{-g}\,\frac{\partial^\mu F}{F}\right) + 2p\int \mathrm{d}t\,\mathrm{d}r\sqrt{-g}\,\frac{(\nabla F)^2}{F^2}, \\ \int \mathrm{d}t\,\mathrm{d}r\sqrt{-g}\left(-2q\frac{\Box G}{G}\right) &= \oint_{\partial\mathcal{M}} \mathrm{d}S_\mu \left(-2q\sqrt{-g}\,\frac{\partial^\mu G}{G}\right) + 2q\int \mathrm{d}t\,\mathrm{d}r\sqrt{-g}\,\frac{(\nabla G)^2}{G^2}. \end{aligned}$$

The boundary integrals over $\partial\mathcal{M}$ (the boundary of the (t,r)-domain) are the surface terms that must vanish by suitable boundary conditions or be retained in the total variation. After this integration the Lagrangian contains only first derivatives of F and G.

A lengthy but straightforward computation of the Euler-Lagrange equations $\delta S_{\mathrm{eff}}/\delta F = 0$ and $\delta S_{\mathrm{eff}}/\delta G = 0$ now yields the GBE:

$$\frac{\delta S_{\mathrm{eff}}}{\delta F} = 0 \implies p\,\mathscr{L}_n f - \big(p(p-1)f^2 + pq\,fg\big) + \mathscr{I}_F = 0, \tag{9}$$

$$\frac{\delta S_{\mathrm{eff}}}{\delta G} = 0 \implies q\,\mathscr{L}_n g - \big(q(q-1)g^2 + pq\,fg\big) + \mathscr{I}_G = 0, \tag{10}$$

where

$$\mathscr{I}_F = 2\binom{p}{2}\frac{1}{F^2}, \qquad \mathscr{I}_G = 2\binom{q}{2}\frac{1}{G^2}.$$

Adding (9) and (10) gives the plane-summed GBE:

$$\mathscr{L}_n K - Q + \mathscr{I} = 0, \tag{11}$$

with

$$K = pf + qg, \qquad \mathscr{I} = \mathscr{I}_F + \mathscr{I}_G, \tag{12}$$

and Q is given by (6).

3.1 Geometric identity relating extrinsic curvature invariant, intrinsic curvature and the wave operators

From the Gauss-Codazzi equations for the codimension-2 leaves Σ with the null normal n one obtains the identity

$$h^{AB} R_{nAnB} = -\mathscr{L}_n K - K_{AB}K^{AB}. \tag{13}$$

For the doubly-warped metric a direct computation gives

$$h^{AB} R_{nAnB} = p\frac{\Box F}{F} + q\frac{\Box G}{G}. \tag{14}$$

Now substitute $\mathscr{L}_n K$ from the GBE (11) to obtain a relation between the wave operators, the intrinsic curvature and the extrinsic-curvature invariants:

$$p\frac{\Box F}{F} + q\frac{\Box G}{G} = \mathscr{I} - K^2. \tag{15}$$

This identity is a direct consequence of the geometric constraints and the equations of motion; it is not an independent equation of motion. It clarifies the role of the nonlinear extrinsic-curvature terms that are omitted in the standard Ricci reduction. In particular, it shows that the combination $K^2 - \mathscr{I}$ is directly related to the wave operators of the scale factors, and hence the neglect of $K_{AB}K^{AB}$ in the reduction from Riemann to Ricci effectively discards this geometric relation.

4 Reduction of the Gravitational Balance Equation to the Null Raychaudhuri Equation

Assume:

- A $(p+q+2)$-dimensional spacetime with a doubly-warped metric (1).
- The null normal n is geodesic and twist-free.
- The leaves Σ of constant (t,r) are codimension-2 submanifolds with induced metric $h_{ab} = g_{ab} + n_a n_b$.

- The extrinsic curvature of Σ is K_{AB}, with trace $K = h^{AB}K_{AB}$.
- The quadratic invariant is $Q = K^2 - K_{AB}K^{AB}$.
- The intrinsic Ricci scalar of Σ is $\mathscr{I} = h^{AC}h^{BD}R_{ABCD}$.

We will now prove that the GBE,

$$\mathscr{L}_n K - Q + \mathscr{I} = 0,$$

reduces to the null Raychaudhuri equation.

4.1 Proof

The proof is carried out in 5 steps:

1. **Express Q in terms of the shear tensor.** Define the traceless part (shear) of the extrinsic curvature:

$$\sigma_{AB} = K_{AB} - \frac{K}{d}h_{AB}, \quad \text{with } d = p + q.$$

Then,

$$K_{AB}K^{AB} = \left(\frac{K}{d}h_{AB} + \sigma_{AB}\right)\left(\frac{K}{d}h^{AB} + \sigma^{AB}\right) = \frac{K^2}{d} + \sigma_{AB}\sigma^{AB},$$

because $h^{AB}\sigma_{AB} = 0$ and $h_{AB}h^{AB} = d$. Hence,

$$Q = K^2 - K_{AB}K^{AB} = \left(1 - \frac{1}{d}\right)K^2 - \sigma_{AB}\sigma^{AB}.$$

2. **Substitute Q into the GBE.**

$$\mathscr{L}_n K - \left[\left(1 - \frac{1}{d}\right)K^2 - \sigma_{AB}\sigma^{AB}\right] + \mathscr{I} = 0,$$

which gives:

$$\mathscr{L}_n K = \left(1 - \frac{1}{d}\right)K^2 - \sigma_{AB}\sigma^{AB} - \mathscr{I}. \tag{16}$$

3. **Use the geometric identity relating $\mathscr{I}$ and R_{nn}.** For a geodesic and twist-free null normal in the doubly-warped metric, we have:

$$h^{AB}R_{nAnB} = \mathscr{I} - K^2.$$

Moreover, under these conditions, $R_{nn} = h^{AB}R_{nAnB}$; consequently,

$$R_{nn} = \mathscr{I} - K^2. \tag{17}$$

4. **Substitute** (17) **into** (16). From (17), $\mathscr{I} = R_{nn} + K^2$, and substituting into (16) there results:

$$\mathscr{L}_n K = \left(1 - \frac{1}{d}\right) K^2 - \sigma_{AB}\sigma^{AB} - (R_{nn} + K^2) = -\frac{1}{d}K^2 - \sigma_{AB}\sigma^{AB} - R_{nn}.$$

5. **Emergence of the Raychaudhuri equation.** Let $\theta = K$ (the expansion of the null congruence) and note that $\mathscr{L}_n K = \frac{d\theta}{d\lambda}$ (since n is the tangent vector to the null geodesics). Then the above equation becomes:

$$\frac{d\theta}{d\lambda} = -\frac{1}{d}\theta^2 - \sigma_{AB}\sigma^{AB} - R_{\mu\nu}n^\mu n^\nu, \tag{18}$$

where $R_{\mu\nu}n^\mu n^\nu = R_{nn}$.

This is exactly the null Raychaudhuri equation for a twist-free congruence. As a special case, we consider the umbilic state.

In the umbilic state, $\sigma_{AB} = 0$ and $K = d \cdot H$ (with $f = g = H$). Then the Raychaudhuri equation simplifies to:

$$\frac{d\theta}{d\lambda} = \left(1 - \frac{1}{d}\right)\theta^2 - \mathscr{I} = Q - \mathscr{I}, \tag{19}$$

with $\theta = d \cdot H$.

4.2 Relation to the ADM formalism

The Hamiltonian constraint of the ADM formalism, when adapted to the present doubly-warped setting, identifies the combination ${}^{(3)}R + K^2 - K_{AB}K^{AB}$ with the energy density. In our notation ${}^{(3)}R = -\mathscr{I}$ and $K^2 - K_{AB}K^{AB} = Q$, so the constraint reads

$$Q - \mathscr{I} = 16\pi G\, \rho_{\mathrm{ADM}}. \tag{20}$$

This constraint should be independent of the evolution equations (9)-(10) and should be imposed on initial data. However, the GBE (11) are evolution equations and because of the form of the Hamilton constraint, it would impose an improper sign condition on the accelerations which would now be given by $\mathscr{L}_n K = 16\pi G\rho_{ADM} \geq 0$.

The GBE and EFE/ADM formalisms are inequivalent: They result from different variational choices (plane-resolved vs. metric, with or without pre-emptive integration-by-parts (IBP)). They are not algebraically equivalent rearrangements.

Nonetheless, the spacetime geometry must be compatible. But we have shown that the formulation of dynamics, the localization of energy, and the matter-geometry correspondence are not invariant under the choice of variational principle.

4.3 The Fundamental Nature of the Gravitational Balance Equations

The reduction to the null Raychaudhuri equation also reveals why the GBE are more fundamental than the standard equations derived from the Riemann tensor. In the classical approach, one often rewrites second derivatives of the scale factors in terms of wave operators and squares of first derivatives:

$$\frac{\Box F}{F} = \mathscr{L}_n f + f^2, \tag{21}$$

$$\frac{\Box G}{G} = \mathscr{L}_n g + g^2. \tag{22}$$

The quadratic terms f^2 and g^2 are precisely the diagonal components of $K_{AB}K^{AB} = pf^2 + qg^2$. In the reduction from the full Riemann curvature to the Ricci tensor, these terms are absorbed into the definition of the Ricci components and are no longer visible as independent dynamical quantities. By contrast, the GBE retain them explicitly in the invariant (6), which drives the evolution of the mean extrinsic curvature.

This retention is not merely a formal difference; it has profound physical implications. The terms $K_{AB}K^{AB}$ represent the intrinsic anisotropic stress of the geometry. In the deterministic limit, they can be neglected only when fluctuations are small, but in regimes such as the photon sphere or near singularities, they become comparable to the mean terms. The GBE, by keeping these terms, provide a more complete description of the geometry even at the classical level. In the stochastic generalization, $K_{AB}K^{AB}$ becomes the umbilic deviation energy $\mathscr{E} = \frac{1}{2}\sigma_{AB}\sigma^{AB}$, which back-reacts on the mean geometry and can regularize classical singularities.

Thus, the GBE are not merely a reformulation of the Einstein equations; they are a more fundamental set of equations that preserve the full nonlinear extrinsic curvature structure. This makes them the natural starting point for a stochastic geometric theory of gravity, where fluctuations are intrinsic to the geometry rather than added as external sources.

5 Critique of the Reduction of Riemann to Ricci and the Neglect of Nonlinear Curvature Sources

One might attempt to derive an equation similar to the GBE from the geometric identities of embedded surfaces. The Ricci equation for a hypersurface with normal n is

$$\mathscr{L}_n K_{ab} = -R_{nanb} + KK_{ab} - 2K_{ac}K^c_b. \tag{23}$$

For our doubly-warped metric, the extrinsic curvature is diagonal and the leaves are codimension-2. However, note that the Ricci equation (23) is typically written for a hypersurface (codimension-1). For a codimension-2 surface, we have two normal vectors. In our case, we are using a null normal n that is tangent to the (t, r)-plane. So we must be cautious when applying the Ricci equation.

The trace of (23) over the leaf indices gives

$$\mathscr{L}_n K + h^{AB}R_{nAnB} - K^2 + 4K_{AB}K^{AB} = 0. \tag{24}$$

However, this equation is not correct in general for our setting. The correct geometric identity for a codimension-2 surface with a null normal can be derived from the Gauss-Codazzi equations. In fact, for a null normal, that is geodesic and without twist, we have equation (13). Substituting (13) into the trace yields

$$\mathscr{L}_n K - \mathscr{L}_n K - K_{AB}K^{AB} - K^2 + 4K_{AB}K^{AB} = 0 \quad \Rightarrow \quad -K^2 + 3K_{AB}K^{AB} = 0.$$

This is a condition on the extrinsic curvature that is not satisfied in general. Therefore, the trace of the Ricci equation does not yield a geometric identity but rather a dynamical equation that coincides with the GBE only when this condition holds.

In contrast, the GBE (11) derived from the action does not require such a condition. It is a genuine equation of motion. Therefore, the GBE cannot be derived from the Ricci equation alone; it must come from the variational principle.

Moreover, there is a common error in the literature when reducing the Riemann tensor to the Ricci tensor. The component $R_{nn} = g^{\mu\nu}R_{n\mu n\nu}$ includes contributions from all directions, not just the leaf directions. Specifically,

$$R_{nn} = g^{tt}R_{ntnt} + g^{rr}R_{nrnr} + h^{AB}R_{nAnB}.$$

For the doubly warped metric, with null normal $n = \partial_t - \partial_r$, we have:

$$R_{nAnB} = R_{tAtB} - R_{tArB} - R_{rAtB} + R_{rArB}.$$

Assuming no cross terms $R_{tArB} = 0$ due to the warped product structure, we get:

$$h^{AB}R_{nAnB} = h^{AB}(R_{tAtB} + R_{rArB}) = p\frac{\Box F}{F} + q\frac{\Box G}{G}.$$

The quadratic invariant is

$$K_{AB}K^{AB} = p\frac{(\dot{F} - F')^2}{F^2} + q\frac{(\dot{G} - G')^2}{G^2}.$$

When deriving an equation resembling the GBE (11) from the Ricci-Codazzi equation (23) through contraction, we get additional terms from these mixed components that do not appear in the variational derivation.

This neglect of nonlinear terms such as $K_{AB}K^{AB}$ in the reduction from the full Riemann to the Ricci tensor is not merely a technical oversight; it reflects a deeper limitation in how classical GR 'packages' curvature. In "exact" solutions such as Schwarzschild, terms like G''/G are often assumed to dominate over nonlinear terms G'^2/G^2, but this hierarchy breaks down precisely at regimes such as the photon sphere $G = 3M$, where $G'' \sim G'^2/G$.

Consequently, solutions that appear exact in the linearized Ricci formalism are, in fact, only approximate when the full nonlinear geometry is considered [12, 13, 14, 15]. This has direct implications for back-reaction and averaging problems in inhomogeneous cosmology [6, 7, 8], and underscores the need for a stochastic geometric framework in which such terms are retained.

In the stochastic framework, this term becomes crucial as it contributes to the back-reaction of fluctuations. The correct treatment requires keeping all nonlinear terms, as is standard in the GR literature on averaging and back-reaction [6, 7, 8].

In summary, the GBE are derived solely from the Einstein-Hilbert action and cannot be obtained from the Ricci equation without imposing additional conditions. The geometric identities must be carefully distinguished from the equations of motion. This distinction is especially important in the stochastic generalization, where the fluctuations are intrinsic to the geometry.

6 Stochastic extrinsic curvature and moment hierarchy

We promote K_{AB} to a random tensor field on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For each leaf point x on a fixed foliation Σ, the extrinsic curvature is a random variable

$$K_{AB}(x;\omega) = \frac{1}{d} K(x;\omega)\, h_{AB}(x) + \sigma_{AB}(x;\omega), \tag{25}$$

with $\mathbb{E}[\sigma_{AB}] = 0$. The traceless part σ_{AB} represents the anisotropic fluctuations.

The averaged GBE are obtained by taking the expectation $\mathbb{E}[\cdot]$ of (9) and (10):

$$p\,\bar{f}_n - 2\binom{p}{2}\,\mathbb{E}[f^2] - pq\,\mathbb{E}[fg] + 2\binom{p}{2}\frac{1}{F^2} = 0, \tag{26}$$

$$q\,\bar{g}_n - 2\binom{q}{2}\,\mathbb{E}[g^2] - pq\,\mathbb{E}[fg] + 2\binom{q}{2}\frac{1}{G^2} = 0, \tag{27}$$

where $\bar{f}_n = \mathbb{E}[\mathscr{L}_n f]$. The second moments $\mathbb{E}[f^2]$, $\mathbb{E}[g^2]$, $\mathbb{E}[fg]$ require closure.

6.1 Geometric constraints

The Gauss-Codazzi-Ricci identities impose constraints on the fluctuations. In particular, the Codazzi equation yields the divergence constraint

$$\nabla^B\big(K_{AB} - h_{AB}K\big) = R_{An} := J_A, \tag{28}$$

where J_A is a normal flux, which must hold pointwise for each realization. This can be written as:

$$J_A = \big(\tfrac{1}{d} - 1\big)\nabla_A K + \nabla^B \sigma_{AB}. \tag{29}$$

In the state of umbilicity, $\sigma_{AB} = 0$ and $\nabla_A K = d\nabla_A H = 0$ implying that an umbilic leaf must be one of constant mean curvature (CMC). This equation is crucial for the dynamics of fluctuations and will lead to a spatial flux in the fluctuation energy balance.

7 Two fluctuation energies

From the decomposition (4), we identify two distinct fluctuation energies:

7.1 Gradient energy

The gradient energy density associated with fluctuations in the trace of extrinsic curvature is defined as:

$$\mathscr{E}_{\text{grad}} = \frac{1}{2}\nabla_A K \nabla^A K. \tag{30}$$

This energy measures the inhomogeneity in the mean curvature across the leaf.

7.2 Umbilic deviation energy

The umbilic deviation energy density is defined as:

$$\mathscr{E} = \frac{1}{2}\sigma_{AB}\sigma^{AB}, \tag{31}$$

which quantifies the deviation from the umbilic (isotropic) state. Using the explicit form of σ_{AB} in the doubly-warped geometry,

$$\sigma_{AB} = \frac{f-g}{d} \times \begin{cases} q & \text{for } A = B \in p\text{-sector}, \\ -p & \text{for } A = B \in q\text{-sector}, \\ 0 & \text{otherwise}, \end{cases} \tag{32}$$

we find

$$\mathscr{E} = \frac{1}{2}\,\frac{pq}{p+q}\,\mathbb{E}[(f-g)^2]. \tag{33}$$

The factor $\frac{pq}{p+q}$ is (up to a factor of 2) the harmonic mean of the sector dimensions p and q. Its appearance, together with the weighted arithmetic mean in the trace $K = pf + qg$ and the geometric-mean-like cross-term fg in the quadratic invariant Q (equation (6), reflects a natural hierarchy of averages that emerge from the Gauss–Codazzi projections. This consistent presence of different means underscores the statistical character of the extrinsic-curvature description and further motivates treating the geometry as a random field.

8 Bochner formula for the gradient energy

The Bochner formula for the gradient energy $\mathscr{E}_{\text{grad}}$ involves the Laplacian of the energy density. Starting from the definition, we compute:

$$\begin{aligned} \Delta\mathscr{E}_{\text{grad}} &= \nabla^A\nabla_A\Big(\frac{1}{2}\nabla_B K \nabla^B K\Big) \\ &= \nabla^A\big(\nabla_A\nabla_B K \nabla^B K\big) \\ &= \nabla^A\nabla_B K \nabla_A\nabla^B K + \nabla^B K \nabla^A\nabla_A\nabla_B K. \end{aligned}$$

Using the commutation of covariant derivatives and the contracted Ricci identity, we obtain:

$$\Delta\mathscr{E}_{\text{grad}} = \nabla^A\nabla_B K \nabla_A\nabla^B K + \nabla^A K \Delta\nabla_A K + \mathrm{Ric}_{AB}\,\nabla^A K \nabla^B K. \tag{34}$$

The second term can be rewritten using the evolution equation for $\nabla_A K$ derived from the GBE. After a detailed calculation (see Appendix B), we arrive at the Bochner formula:

$$\Delta\mathscr{E}_{\text{grad}} = |\nabla^2 K|^2 + \frac{p-1}{F^2}\sum_{i=1}^{p}(\nabla_i K)^2 + \frac{q-1}{G^2}\sum_{j=1}^{q}(\nabla_j K)^2 + \text{lower order terms}. \tag{35}$$

The Bochner formula for the gradient energy relates the Laplacian of the energy to the Hessian of K and the intrinsic Ricci curvature of Σ. The term $|\nabla^2 K|^2$ is non-negative and tends to disperse gradients, while the Ricci term $\mathrm{Ric}_{AB}\nabla^A K \nabla^B K$ can be positive or negative.

For stability, we require $\Delta\mathscr{E}_{\rm grad} > 0$, which is ensured if the Ricci curvature is positive. Conversely, negative Ricci curvature can lead to instability by amplifying gradients. The vanishing of $\mathscr{E}_{\rm grad}$ corresponds to CMC leaves, indicating homogeneity of the mean extrinsic curvature across Σ. The same intrinsic curvatures that appear in the Ricci term,

$$\mathrm{Ric}_{AB}\,\nabla^A K \nabla^B K = \frac{p-1}{F^2}\sum_{i=1}^{p}(\nabla_i K)^2 + \frac{q-1}{G^2}\sum_{j=1}^{q}(\nabla_j K)^2, \tag{36}$$

reappear in the dissipation term in the transport equation for the umbilic energy.

9 Transport equation for the umbilic deviation energy

To obtain the evolution equation for the umbilic deviation energy $\mathscr{E}$, we derive the exact equation for the second moment $\mathbb{E}[\delta K_{AB}\delta K^{AB}]$ from the stochastic GBE. We work in an umbilic background $K^{(0)}_{AB} = \frac{K}{d}h_{AB}$ and consider fluctuations $\delta K_{AB} = K_{AB} - K^{(0)}_{AB}$ with $\mathbb{E}[\delta K_{AB}] = 0$.

The linearized fluctuation equations (in an umbilic background) are

$$p\,\mathscr{L}_n\delta f = \binom{p}{2}\big(2H\delta f\big) + pq\big(H\delta g + \delta f H\big) - \binom{p}{2}\frac{2}{F^3}\,\delta F, \tag{37}$$

$$q\,\mathscr{L}_n\delta g = \binom{q}{2}\big(2H\delta g\big) + pq\big(H\delta f + \delta g H\big) - \binom{q}{2}\frac{2}{G^3}\,\delta G, \tag{38}$$

where H is the common value of f and g in the background.

Because the background is umbilic, δK_{AB} is traceless and diagonal, so

$$\delta K_{AB}\,\delta K^{AB} = p(\delta f)^2 + q(\delta g)^2.$$

Applying $\mathscr{L}_n$ and using (33) gives, after taking expectations,

$$\begin{aligned}\mathscr{L}_n\mathscr{E} = \binom{p}{2}H\,\sigma_f^2 + pqH\,\sigma_{fg} - \binom{p}{2}\frac{\mathbb{E}[\delta f\,\delta F]}{F^3} \\ + \binom{q}{2}H\,\sigma_g^2 + pqH\,\sigma_{fg} - \binom{q}{2}\frac{\mathbb{E}[\delta g\,\delta G]}{G^3} + \cdots\,,\end{aligned} \tag{39}$$

where $\sigma_f^2 := \mathbb{E}[(\delta f)^2]$, $\sigma_g^2 := \mathbb{E}[(\delta g)^2]$, $\sigma_{fg} := \mathbb{E}[\delta f\,\delta g]$.

To close (39), we relate the mixed moments $\mathbb{E}[\delta f\,\delta F]$ and $\mathbb{E}[\delta g\,\delta G]$ to the covariances. From $f = \mathscr{L}_n \ln F$, we have to linear order $\delta f = \mathscr{L}_n(\delta F/F_0)$. Assuming statistically stationary fluctuations on Σ, a short-scale approximation gives

$$\mathbb{E}[\delta f\,\delta F] \approx -\ell_c^2\,\sigma_f^2, \qquad \mathbb{E}[\delta g\,\delta G] \approx -\ell_c^2\,\sigma_g^2, \tag{40}$$

where ℓ_c is a correlation length intrinsic to the leaf. Substituting into (39) yields

$$\mathscr{L}_n\mathscr{E} = -\left(\alpha_1 H - \alpha_2\ell_c^{-2}\right)\,\mathscr{E} = -2\gamma\mathscr{E}, \tag{41}$$

with constants α_1, α_2 depending on p, q. Note that α_1 can be both positive or negative depending on the specific solution of the linearized system.

10 Gauss' principle and phenomenological dissipation

The GBE are inherently dissipationless, being derived from a Hamiltonian action principle. With the goal to obtain a dissipative structure, we introduce a phenomenological relaxation law for the traceless extrinsic curvature:

$$\mathscr{L}_n\sigma_{AB} = -\gamma\sigma_{AB} + \Xi_{AB}, \tag{42}$$

where γ is a relaxation rate, and Ξ_{AB} is modeled as white noise. Contracting with σ^{AB}, taking the expectation value, and using $\mathscr{E} = \frac{1}{2}\mathbb{E}[\sigma_{AB}\sigma^{AB}]$ give

$$\mathscr{L}_n\mathscr{E} = -2\gamma\mathscr{E}. \tag{43}$$

The damping rate γ is not arbitrary; it is determined by the geometry via Gauss' principle of least constraint. We consider the functional that measures the total umbilic deviation energy:

$$\mathscr{L}[\sigma] = \int_\Sigma \sigma_{AB}\sigma^{AB}\mathrm{d}V_h.$$

We wish to minimize this functional subject to the constraint that the fluctuations satisfy the linearized GBE. Following the Onsager [16]-Lavenda [17] corrected-variational principle, we introduce a Lagrange multiplier tensor λ^{AB} and form the augmented functional:

$$I[\sigma,\lambda^{AB}] = \int_\Sigma \Big\{\sigma_{AB}\sigma^{AB} + \lambda^{AB}\big[(\mathscr{L}_n\sigma_{AB}+\gamma\sigma_{AB})(\mathscr{L}_n\sigma^{AB}+\gamma\sigma^{AB})\big]\Big\}\mathrm{d}V_h.$$

Expanding the quadratic term and setting $\lambda\gamma^2 = -1$ to eliminate the quadratic term in σ, we obtain

$$I = \int_\Sigma \lambda\Big[2\gamma\mathscr{L}_n\sigma_{AB}\sigma^{AB} + \mathscr{L}_n\sigma_{AB}\mathscr{L}_n\sigma^{AB}\Big]\mathrm{d}V_h.$$

Using $\mathscr{L}_n\sigma_{AB}\sigma^{AB} = \frac{1}{2}\mathscr{L}_n(\sigma_{AB}\sigma^{AB})$, this becomes

$$I = \int_\Sigma \lambda\Big[\gamma\mathscr{L}_n(\sigma_{AB}\sigma^{AB}) + \mathscr{L}_n\sigma_{AB}\mathscr{L}_n\sigma^{AB}\Big]\mathrm{d}V_h.$$

We have thus converted the constrained variational principle of least dissipation of energy,

$$\int_\Sigma \Phi(\mathscr{L}_n\sigma^{AB},\mathscr{L}_n\sigma_{AB})\mathrm{d}V_h = \min,$$

into a free variational problem through the addition of the Lie derivative of the energy,

$$\int_\Sigma (\Phi + \gamma\mathscr{L}_n\mathscr{E})\mathrm{d}V_h = \mathrm{ext.}$$

Varying with respect to the flux $\mathscr{L}_n\sigma^{AB}$ while holding σ^{AB} fixed yields

$$\delta I = 2\lambda\int_\Sigma \big(\gamma\sigma_{AB} + \mathscr{L}_n\sigma_{AB}\big)\,\delta(\mathscr{L}_n\sigma^{AB})\,\mathrm{d}V_h = 0,$$

for arbitrary variations $\delta(\mathscr{L}_n\sigma^{AB})$. Hence we obtain the relaxation law

$$\mathscr{L}_n\sigma_{AB} + \gamma\sigma_{AB} = 0, \tag{44}$$

which, upon contracting with σ^{AB} and taking the expectation value, reproduces (43). The damping rate γ is not fixed by this variational principle; it is instead determined by the geometric and statistical properties of the fluctuations. From the linearized GBE and the short-scale approximation we find

$$\gamma = \tfrac{1}{2}(\alpha_2\ell_c^{-2} - \alpha_1 H). \tag{45}$$

Thus, the dissipation is curvature-driven and can be positive or negative depending on the sign of H.

Thus, while the fundamental GBE are conservative, the phenomenological relaxation law justified by Gauss' principle provides a dissipation mechanism that drives the system toward the isotropic umbilic state.

11 Derivation of the Parabolic Law from the Dissipation Function

The extrinsic curvature K_{AB} has been promoted to a random tensor field. It is decomposed into its trace $K = h^{AB}K_{AB}$ and traceless anisotropic part $\sigma_{AB} = K_{AB} - \frac{K}{d}h_{AB}$. The Codazzi constraint (28) relates the divergence of σ_{AB} to a normal curvature flux J_A:

$$J_A = \left(\frac{1}{d} - 1\right)\nabla_A K + \nabla^B\sigma_{AB}. \tag{46}$$

The *dissipation function* is defined as the energy associated with this flux:

$$\mathscr{D} = \frac{1}{2}\, J_A J^A. \tag{47}$$

The umbilic state, characterized by $\sigma_{AB} = 0$ and $\nabla_A K = 0$, corresponds to $\mathscr{D} = 0$. Our goal is to derive an effective parabolic law for the relaxation toward this state.

11.1 Bochner Formula for the Dissipation Function

On a Riemannian leaf Σ, the Bochner identity for the vector field J_A gives:

$$\Delta\mathscr{D} = \nabla_A J_B\,\nabla^A J^B \;+\; R_{AB}J^AJ^B \;-\; |\nabla_A J^A|^2 \;+\; \nabla_B\big(J^B\nabla_A J^A\big), \tag{48}$$

where $\Delta = \nabla^A\nabla_A$ is the Laplacian on Σ, and R_{AB} is the Ricci curvature of Σ. The first term on the right-hand side is non-negative and represents a "stiffness" of the flux field. The second, curvature, term can be positive or negative, while the penultimate, divergence-squared, term tends to oppose dissipation. The last term $\nabla_B\big(J^B\nabla_A J^A\big)$ is a total divergence; under suitable boundary conditions (e.g., Σ closed, or Dirichlet/Neumann conditions that make the boundary integral vanish) it does not affect the interior dissipation inequality, and is typically omitted when deriving parabolic estimates via the maximum principle.

11.2 Time Evolution of the Dissipation Function

The time derivative of $\mathscr{D}$ follows from the definition (47):

$$\partial_t \mathscr{D} = J^A \, \partial_t J_A. \tag{49}$$

To evaluate $\partial_t J_A$ we use the stochastic GBE and the phenomenological relaxation law for the anisotropic stress. Linearizing around an umbilic background (where $J_A^{(0)} = 0$) and neglecting commutator terms and gradients of the mean curvature (higher order in fluctuations), we obtain the dominant contribution from the relaxation of σ_{AB}:

$$\partial_t \sigma_{AB} = -\gamma \, \sigma_{AB} + \Xi_{AB}, \tag{50}$$

with $\gamma > 0$ a curvature-driven damping rate and Ξ_{AB} a stochastic noise. Substituting into the time derivative of (46) and neglecting the gradient of $\partial_t K$ (which is of higher order) yields:

$$\partial_t J_A \approx -\gamma \, \nabla^B \sigma_{AB} \approx -\gamma \, J_A, \tag{51}$$

where the last approximation uses the fact that, to linear order, the term $\left(\frac{1}{d} - 1\right) \nabla_A K$ is subdominant compared to $\nabla^B \sigma_{AB}$. Consequently,

$$\partial_t \mathscr{D} \approx -\gamma \, J_A J^A = -2\gamma \, \mathscr{D}. \tag{52}$$

11.3 Combined Parabolic Evolution

Combining the time evolution (52) with the Bochner identity (48), we obtain the effective parabolic law for the dissipation function:

$$(\partial_t - \Delta) \, \mathscr{D} = -2\gamma \, \mathscr{D} \; - \underbrace{\nabla_A J_B \nabla^A J^B}_{\geq 0} - R_{AB} J^A J^B \; + \; |\nabla_A J^A|^2 \; + \; \text{(boundary terms)}. \tag{53}$$

For the system to relax toward the umbilic state ($\mathscr{D} \to 0$), the right-hand side must be negative when $\mathscr{D} > 0$. The first term $-2\gamma\mathscr{D}$ provides explicit, curvature-driven damping. The geometric terms (the last three terms on the right) constitute the Bochner Laplacian contribution $\Delta\mathscr{D}$. Notably, the first geometric term $-\nabla_A J_B \nabla^A J^B$ is non-positive, thus always promoting dissipation. The sign of the curvature term $-R_{AB} J^A J^B$ depends on the Ricci curvature of the leaf Σ: positive Ricci curvature enhances dissipation, while negative Ricci curvature opposes it.

Equation (53) can be written in a more illuminating form by recognizing the combination of the last three terms as related to the quadratic form $Q(J, J)$ derived from the Bochner identity for a vector field. Dropping the explicit damping term, we obtain the inequality:

$$(\partial_t - \Delta) \, \mathscr{D} \leq -\nabla_A J_B \nabla^A J^B + Q(J, J) \leq 0, \tag{54}$$

where

$$Q(J, J) := |\nabla_A J^A|^2 - R_{AB} J^A J^B \tag{55}$$

is the flux analog of (6) expressed in terms of the extrinsic curvature.

For the dissipation function to decay, we require $(\partial_t - \Delta)\,\mathscr{D} \leq 0$, which is equivalent to the condition

$$\nabla_A J_B \nabla^A J^B \geq Q(J, J),$$

since the Hessian term $\nabla_A J_B \nabla^A J^B$ is non-negative and appears with a negative sign in (54). Thus, the stability of the umbilic state is governed by the competition between the Hessian (which provides rigidity) and the quadratic flux invariant form (55). Only when the Hessian term dominates $Q(J, J)$ is there a guarantee to the monotonic decay of $\mathscr{D}$.

In the short-correlation-length approximation, the Hessian term dominates, ensuring the overall negativity of the right-hand side of (53).

12 Analogy Between Harmonic Map Flow and Stochastic Geometric Gravity

In the classical Eells–Sampson harmonic map flow [18], the Bochner formula for the energy density contains a term involving the Riemann curvature of the *target* manifold. If the target has non-positive sectional curvature, that term is non-negative, leading to a subharmonic inequality that ensures dissipation and global existence. In the present stochastic geometric framework, the target spaces (the space of scale factors and the space of extrinsic-curvature pairs) are flat, so the Eells–Sampson curvature condition is trivially satisfied. The dissipation does not arise from target curvature; it is introduced phenomenologically through the damping rate γ in the relaxation law (42).

The Bochner formula (48) separates the spatial diffusion (the positive Hessian term $\nabla_A J_B \nabla^A J^B$) from the curvature coupling $R_{AB} J^A J^B$, where R_{AB} is the Ricci curvature of the *domain*—the leaf Σ. This domain curvature can be positive or negative, and thus can either enhance or oppose dissipation. In particular, in hyperbolic regions of the leaf ($R_{AB} < 0$), the curvature term in (53) can become destabilizing, potentially slowing down the relaxation or even triggering fluctuation growth if γ is too small. Hence, the stochastic geometric framework unifies both stabilizing and destabilizing curvature effects within a single parabolic relaxation law.

12.1 Why is J_A the "mapping field" and not K_{AB}

In the Eells-Sampson theorem, the map ϕ is the fundamental field, and its tension $\tau(\phi)$ is the derivative of the energy functional. In stochastic geometric gravity, the fundamental geometric field is the extrinsic curvature K_{AB}, which describes how the leaf Σ is embedded in spacetime. However, the quantity that plays the role of the **tension** (the gradient of the energy) is **not** K_{AB} itself, but the flux J_A because:

Eells–Sampson Harmonic Map Flow	**Stochastic Geometric Gravity**
Field: Map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds.	**Field:** Extrinsic curvature K_{AB} of codimension-2 foliation $\Sigma \subset \mathcal{M}$.
Equilibrium: Harmonic map $\tau(\phi) = 0$ (vanishing tension).	**Equilibrium:** Umbilic leaf $\sigma_{AB} = 0$, $\nabla_A K = 0$ (isotropic, constant mean curvature).
Deviation measure: Tension field $\tau(\phi) = \Delta\phi - \mathrm{tr}_g(\nabla d\phi)$.	**Deviation measure:** Curvature flux $J_A = \left(\frac{1}{d} - 1\right)\nabla_A K + \nabla^B \sigma_{AB}$.
Evolution law: Parabolic gradient flow $\partial_t \phi = -\tau(\phi)$.	**Evolution law:** Exponential relaxation $\partial_t J_A \approx -\gamma J_A$ (linearized, short-correlation regime).
Lyapunov functional: Dirichlet energy $\mathscr{E}(\phi) = \frac{1}{2}\int_M |d\phi|^2 dV_g$.	**Lyapunov functional:** Dissipation function $\mathscr{D} = \frac{1}{2}\int_\Sigma J_A J^A dV_h$.
Driving force: Negative tension $-\tau(\phi)$ minimizes $E(\phi)$.	**Driving force:** Damped flux $-\gamma J_A$ minimizes $\mathscr{D}$ via Gauss' principle.
Spatial smoothing: Laplacian $\Delta\phi$ in $\tau(\phi)$ provides diffusion.	**Spatial smoothing:** Bochner identity $\Delta\mathscr{D}$ provides geometric diffusion via Ricci curvature.

Table 1: Comparison of Eells–Sampson Harmonic Map Flow and Stochastic Geometric Gravity Framework

- J_A is derived from the Codazzi constraint, which is a geometric identity linking K_{AB} to the normal curvature.
- J_A vanishes precisely in the equilibrium state (umbilic leaf), just as $\tau(\phi) = 0$ for a harmonic map.
- The dissipation function $\mathscr{D} = \frac{1}{2} J_A J^A$ is a positive-definite functional that decreases in time, analogous to the Dirichlet energy $\mathscr{E}(\phi)$.
- The evolution law $\partial_t J_A \approx -\gamma J_A$ is a relaxation equation for the "tension" J_A, driving it to zero.

Thus, K_{AB} is the analogue of the map ϕ, while J_A is the analogue of the tension $\tau(\phi)$. The mapping is essentially the embedding of Σ, described by K_{AB}, but the dynamics toward equilibrium are governed by the relaxation of J_A.

13 The Photon Sphere as the Globally Unique Umbilic Equilibrium

In the stochastic geometric framework, the classical Schwarzschild metric is a mean-field approximation. The photon sphere emerges as a solution of the mean-field GBE (10),

$$\mathscr{L}_n g - g^2 + \frac{1}{G^2} = 0 \Longrightarrow 1 - \frac{3M}{G} = 0. \tag{56}$$

At the photon sphere, the extrinsic curvature becomes isotropic (umbilic). For the Schwarzschild metric in the doubly-warped form with $p = 1$, $q = 2$, and areal radius G, the umbilic condition $f = g$ at $G = 3M$ yields the common value

$$f = g = \frac{1}{3\sqrt{3}\,M}.$$

This state minimizes the umbilic deviation energy $\mathscr{E}$ and is the unique, globally attracting equilibrium state of the stochastic relaxation dynamics.

The relaxation law (43) drives the system toward this umbilic state. The damping rate (45) is curvature-driven and, at the photon sphere, becomes a constant determined by the background geometry. The parabolic part of the evolution (Bochner term $\Delta\mathscr{D}$) provides spatial smoothing, ensuring that the geometry flows smoothly toward the photon sphere. Thus, the stochastic geometric framework predicts that gravitational fluctuations will irreversibly relax to the umbilic configuration exactly at the photon sphere.

Eells–Sampson Theorem (Harmonic Map Flow)	**Stochastic Geometric Gravity**
Field: Map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds.	**Field:** Extrinsic curvature K_{AB} of codimension-2 foliation $\Sigma \subset \mathcal{M}$.
Equilibrium: Harmonic map $\tau(\phi) = 0$ (vanishing tension).	**Equilibrium:** Umbilic leaf $\sigma_{AB} = 0$, $\nabla_A K = 0$ (isotropic, constant mean curvature).
Deviation measure: Tension field $\tau(\phi) = \Delta\phi - \mathrm{tr}_g(\nabla d\phi)$.	**Deviation measure:** Curvature flux $J_A = (\frac{1}{d} - 1)\nabla_A K + \nabla^B \sigma_{AB}$.
Evolution law: Parabolic gradient flow $\partial_t \phi = -\tau(\phi)$.	**Evolution law:** Exponential relaxation $\partial_t J_A \approx -\gamma J_A$ (linearized, short-correlation regime).
Lyapunov functional: Dirichlet energy $\mathscr{E}(\phi) = \frac{1}{2}\int_M \lvert d\phi\rvert^2 dV_g$.	**Lyapunov functional:** Dissipation function $\mathscr{D} = \frac{1}{2}\int_\Sigma J_A J^A dV_h$.
Driving force: Negative tension $-\tau(\phi)$ minimizes $E(\phi)$.	**Driving force:** Damped flux $-\gamma J_A$ minimizes $\mathscr{D}$ via Gauss' principle.
Spatial smoothing: Laplacian $\Delta\phi$ in $\tau(\phi)$ provides diffusion.	**Spatial smoothing:** Bochner identity $\Delta\mathscr{D}$ provides geometric diffusion via Ricci curvature.

Table 2: Comparison of the Eells–Sampson harmonic map flow theorem and the stochastic geometric gravity framework.

14 The Bochner Formula and Irreversible Gravitational Fluctuations

In the stochastic geometric framework, the Bochner formula plays a central role in relating the geometry of the foliation to the dissipation of fluctuation energy. This section details the two harmonic-map structures that appear, clarifies the metrics involved, and explains how positive dissipation–rather than non-positive target curvature–leads to an irreversible diffusion inequality. The special case $p = 1$ (a circular extra dimension) is highlighted, and the connection with the photon-sphere condition in the asymptotic limit $t \to \infty$ is discussed.

14.1 Target metric for the Base Map

Whereas the metric (1) is written in synchronous (Gaussian normal) coordinates, we can perform a coordinate transformation to null coordinates and bring the doubly-warped metric into the form:

$$ds^2 = -2A\,dt\,dr + F(t,r)^2\,d\Omega_p^2 + G(t,r)^2\,d\Omega_q^2 \tag{57}$$

with $A = \frac{1}{2}$. Allowing $A(t,r)$ to be a non-constant function would correspond to a more general metric ansatz allowing for a wider class of spacetimes within the doubly warped product structure. Physically, it can be interpreted as a lapse-like function controlling the scaling between null directions defined by dt and dr. However, for simplicity, we will retain $A = \frac{1}{2}$ so that it reduces (up to a coordinate relabeling) to the original form (1).

Equation (57) (spacetime metric) is the domain metric for a harmonic map

$$\phi_1 : (t,r) \to N_1 = \{(u,v)\},$$

where N_1 is equipped with the flat metric:

$$ds^2_{N_1} = \binom{p}{2} du^2 + \binom{q}{2} dv^2 + pq\,du\,dv, \tag{58}$$

where

$$u = \ln F, \qquad v = \ln G. \tag{59}$$

This flat metric is the reason why the Base Map has a trivial target curvature while the dissipative dynamics come from the curvature of the domain leaf Σ, a phenomenological damping rate γ.

14.2 Target metric for the Gauss Map

The target space of the Gauss map from a leaf Σ to the space of extrinsic-curvature pairs (f,g) is endowed with the same constant quadratic form as in (58), but now expressed in the variables (f,g):

$$ds^2_{N_2} = \binom{p}{2} (df)^2 \;+\; pq\,df\,dg \;+\; \binom{q}{2} (dg)^2. \tag{60}$$

Both target metrics (58) and (60) are flat because their coefficients are constants (combinatorial numbers depending only on p,q). The flatness of the target spaces means that the Eells–Sampson condition (non-positive target curvature) is trivially satisfied, but it plays no role in the dissipation mechanism. Instead, the dissipative dynamics arise from the curvature of the domain Σ (the leaf) and the phenomenologically introduced, geometry-determined damping rate γ.

Map 1: Base map	Map 2: Gauss map
Domain: Lorentzian base (t, r)	**Domain:** Riemannian leaf Σ (product of spheres)
Target: $(u, v) = (\ln F, \ln G)$	**Target:** $(f, g) = (\mathscr{L}_n u, \mathscr{L}_n v)$
Target metric: $ds^2 = \binom{p}{2} du^2 + pq\, du\, dv + \binom{q}{2} dv^2$	**Target metric:** $ds^2 = \binom{p}{2} df^2 + pq\, df\, dg + \binom{q}{2} dg^2$
Dynamics: Conservative (GBE) – wave-like	**Dynamics:** Dissipative – relaxation with rate γ
Source of dissipation: None (conservative system)	**Sources of dissipation:** 1. Damping rate γ (geometry-driven) 2. Ricci curvature of Σ in Bochner formula

Table 3: The two harmonic-map structures in the stochastic geometric framework.

14.3 Loss of a degree of freedom when $p = 1$

When $p = 1$ the sphere S^p reduces to a circle. In the metric (58) the coefficient $\binom{p}{2}$ vanishes, so the du^2 term is absent. Although the metric remains non-degenerate (its determinant is $-q^2$), the disappearance of the du^2 term means that the harmonic map from the base to (u, v) loses a "radial" direction associated with the scale factor F. Geometrically, this reflects the fact that a circle has only one independent metric component, so the full doubly-warped metric possesses one fewer dynamical degree of freedom.

Recall that for $p = 1$, the circle has zero intrinsic Ricci scalar ($\mathscr{I}_F = 0$). The inverse square of its radius, $1/F^2$, appears only in extrinsic curvature expressions (e.g., in the second fundamental form), not in the intrinsic curvature of Σ.

14.4 Difference between the metric Q and the Hessian of r

The metric Q in (63) is a quadratic form on the space of extrinsic-curvature rates (f, g). It serves as the target metric for the Gauss map. In contrast, the Hessian of the radial coordinate r on the leaf Σ is a tensor that measures the second derivatives of r along the spherical directions:

$$\mathrm{Hess}(r) = F'F\, d\Omega_p^2 + G'G\, d\Omega_q^2. \tag{61}$$

Here primes denote derivatives with respect to r. While Q is an invariant of the extrinsic curvature, Hess(r) is a purely intrinsic object that describes how the spheres are "bent" in the ambient spacetime. The two quantities are geometrically distinct; Q belongs to the fluctuation dynamics, whereas Hess(r) appears in the decomposition of the full Riemann tensor.

14.5 Domain Curvature vs. Target Curvature: The Source of Dissipation

A potential source of confusion arises from the fact that both harmonic-map structures involve *flat target metrics*, yet the dissipation dynamics are curvature-coupled. The resolution lies in the distinction between the curvature of the *target* and the curvature of the *domain* in the Bochner formula.

For Map 2 (the Gauss map from Σ to the space of extrinsic-curvature pairs (f, g)), the target metric Q is flat. However, the *domain* Σ—a product of spheres with radii F and G—has non-vanishing intrinsic Ricci curvature. The Bochner identity (48) for the dissipation function $\mathscr{D}$ contains the term $R_{AB}J^A J^B$, where R_{AB} is precisely the Ricci tensor of the leaf Σ. This term does not vanish; it is given explicitly by (36). Thus, even though the target of the harmonic map is flat, the curvature of the *embedding leaf* Σ enters the dissipation inequality and can either enhance or oppose the relaxation, depending on the signs of $p-1$ and $q-1$ and the gradients of K.

Consequently, the Eells-Sampson condition (non-positive target curvature) is trivially satisfied but irrelevant for the dissipation mechanism in stochastic geometric gravity. Instead, irreversibility is introduced through the phenomenological damping rate $\gamma > 0$, while the domain curvature R_{AB} of the leaf Σ modulates the relaxation via the Bochner formula. This clarifies why the intrinsic Ricci tensor of Σ appears in the parabolic law (53) even though the mapping itself is between flat spaces. In other words, flat target metrics do not trivialize the Gauss maps because the domain curvature and phenomenological damping drive the dynamics.

14.6 Harmonic-map structures and their dynamics

Two distinct harmonic-map structures appear in the GBE:

1. **Map 1: the 2D base map.** Domain: the (t, r) base (2-dimensional Lorentzian). Target: the space of scale factors $(\mu, \nu) = (\ln F, \ln G)$ with metric (58). The map $\phi_1 : (t, r) \to N_1$ extremizes the effective action, yielding the GBE. Its Dirichlet energy density e_1 satisfies a **hyperbolic** Bochner-type identity on the Lorentzian base:

$$\Box e_1 = |\nabla^2\phi|^2 + \mathrm{Ric}^{(N_1)}_{\alpha\beta}\nabla^\mu\phi^\alpha\nabla_\mu\phi^\beta + \text{lower order},$$

   where $\Box = -\partial_t^2 + \partial_r^2$. The Hessian term $|\nabla^2\phi|^2$ is defined as

$$|\nabla^2\phi|^2 = g^{\mu\rho}g^{\nu\sigma}h_{\alpha\beta}(\nabla_\mu\nabla_\nu\phi^\alpha)(\nabla_\rho\nabla_\sigma\phi^\beta),$$

   with $g^{\mu\nu}$ the inverse Lorentzian metric of the domain and $h_{\alpha\beta}$ the positive-definite target metric (58). Because $g^{\mu\nu}$ is indefinite, this quantity is **not sign-definite**; it can be positive, zero, or negative depending on the components of $\nabla^2\phi$. Consequently, this identity does *not* imply dissipation; it reflects the conservative, wave-like character of the deterministic GBE.

2. **Map 2: the Gauss map from Σ.** Domain: a leaf Σ (constant t, r). Target: the space of extrinsic-curvature pairs (f, g) with metric given by the quadratic form $Q(f, g)$, (6). The fluctuation energy $\mathscr{E} = \frac{1}{2}\sigma_{AB}\sigma^{AB}$ is precisely the Dirichlet energy of the deviation map $\delta\phi_2 : \Sigma \to T_{(f,g)}N_2$.

In contrast to the Hessian term in the Bochner formula for Map 1, the base map, which is not sign-definite because the domain is Lorentzian, Map 2, the Gauss map, and the dissipation function $\mathscr{D}$, are in the domain Σ which is Riemannian, so the corresponding Hessian terms, such as $\nabla_A J_B \nabla^A J^B$, are non-negative and contribute positively to dissipation.

The interplay between the conservative dynamics of Map 1 and the dissipative fluctuations of Map 2 provides a complete picture: the mean geometry evolves via deterministic wave-like equations, while the fluctuations undergo curvature-driven relaxation toward the umbilic state.

15 Discussion: Novelty and limitations of GR

The presented stochastic geometric framework offers a new perspective on gravity. Its key novelties are:

1. **Intrinsic geometric fluctuations:** Instead of adding stochastic sources from matter fields, the geometry itself is treated as a random field. The Einstein equations emerge as first-moment equations of a more fundamental stochastic theory.

2. **Self-consistent closure:** The moment hierarchy derived from the GBE is closed by a geometric projector that respects the Codazzi constraint. No ad-hoc stress tensor is needed; the backreaction of fluctuations is encoded in the covariance of the extrinsic curvature.

3. **Fluctuation-dissipation relation:** The damping rate γ is introduced phenomenologically but is constrained by Gauss' principle and becomes curvature-driven. The framework naturally suggests a fluctuation-dissipation relation connecting the statistics of the forcing Ξ_{AB} to γ.

4. **Harmonic map analogy:** The identification of two harmonic map structures provides powerful mathematical tools (Jacobi operator, bubbling analysis[1]) to study stability and geometric transitions.

These advances highlight the limitations of classical GR:

- GR is deterministic and cannot describe the statistical properties of gravitational fields.
- It lacks a mechanism to drive the geometry toward equilibrium; the Einstein equations are conservative.
- The treatment of back-reaction from inhomogeneities or quantum fluctuations requires the introduction of effective stress tensors, which are often phenomenological and break geometric consistency.

Our stochastic geometric framework addresses these limitations by making fluctuations intrinsic and incorporating dissipation via a geometric principle. It opens a new avenue to understand gravitational relaxation, the origin of cosmological anisotropy damping, and the possible statistical nature of spacetime at fundamental scales.

[1] In geometric analysis, 'bubbling' refers to energy-concentration phenomena that can lead to singularities or topological transitions, studied in contexts such as harmonic maps and Yang–Mills fields.

16 Conclusions and Perspectives

The stochastic geometric framework developed in this paper provides a self-consistent theory of gravitational fluctuations, rooted in the first-principles variation of the Einstein–Hilbert action with respect to sectoral scale factors. By promoting the extrinsic curvature to a random field, we derived a moment hierarchy from the GBE and closed it geometrically without introducing phenomenological stress tensors. The resulting fluctuation–dissipation structure is governed by a generalized Bochner formula, which links geometric relaxation to the mean extrinsic curvature and the intrinsic curvature of the foliation leaves.

Our central findings are:

1. The Einstein equations are not fundamental deterministic laws, but rather first-moment (mean-field) truncations of a more complete stochastic geometric theory. The celebrated "exact" vacuum solutions, such as Schwarzschild, are approximations that neglect the nonlinear term $K_{AB}K^{AB}$ and all higher fluctuations.

2. The GBE are more fundamental than the standard Einstein equations because they retain the full nonlinear extrinsic curvature structure, which is lost in the Riemann→Ricci reduction. This retention is essential in regimes where fluctuations become comparable to the mean, such as inside the photon sphere or near singularities.

3. Fluctuations are intrinsic to the geometry, arising from the randomness of extrinsic curvature. They obey a dissipative transport equation analogous to the harmonic map heat flow, with a curvature-driven damping rate γ determined by Gauss' principle of least constraint.

4. The dissipation function $\mathscr{D} = \frac{1}{2}J_A J^A$ serves as a Lyapunov functional whose monotonic decay—enforced by a parabolic maximum principle—drives the system toward umbilic (isotropic) equilibrium states. This irreversible relaxation is a geometric analogue of the second law of thermodynamics.

16.1 Thermodynamic Perspective and Maximum Principles

The parabolic structure of the fluctuation–dissipation dynamics reveals a deep connection between stochastic geometry and irreversible thermodynamics. The Bochner inequality

$$(\partial_t - \Delta)\mathscr{D} \leq -2\gamma\mathscr{D} + Q(J, J)$$

is a geometric counterpart of the maximum principles that govern dissipative processes in nonequilibrium statistical physics [19]. In particular, the monotonic decay of $\mathscr{D}$ under suitable curvature conditions mirrors the entropy increase in isolated systems or the free-energy minimization in driven systems.

This connection is not accidental: the formalism developed here is a geometric realization of [19], where irreversible evolution emerges from constrained variational principles and maximum principles for parabolic PDEs. In that framework, the dissipation function plays the role of a thermodynamic potential whose minimization yields the irreversible relaxation laws. Here, the dissipation function $\mathscr{D}$ is a purely geometric object, yet it obeys the same mathematical structure and enforces the same arrow of time.

16.2 Future Directions

The stochastic geometric framework opens several avenues for future research:

- **CMBR anisotropies and geometric fluctuations:** The stochastic geometric framework provides a natural mechanism for the generation and damping of primordial anisotropies. Applying the formalism to an FLRW background with stochastic extrinsic-curvature fluctuations could model the evolution of inhomogeneities and their imprint on the CMBR. The relaxation of the umbilic deviation energy $\mathscr{E}$ and the dissipation function $\mathscr{D}$ might produce specific signatures in the CMBR power spectrum—potentially explaining large-scale anomalies (such as the low-multipole suppression or hemispherical asymmetry) or providing novel constraints on the stochastic parameters. This approach would also clarify how geometric dissipation could lead to the observed isotropy of the CMBR without fine-tuned inflationary dynamics.

- **Cosmological back-reaction:** The GBE naturally incorporate nonlinear back-reaction terms ($K_{AB}K^{AB}$) that are omitted in standard cosmological averaging schemes. Applying the stochastic framework to inhomogeneous cosmology could yield effective Friedmann equations with fluctuation-driven corrections, potentially relevant for dark energy or structure formation.

- **Quantum gravity connections:** The intrinsic randomness of extrinsic curvature may reflect underlying quantum gravitational fluctuations. The stochastic geometric theory could serve as a semiclassical bridge toward a full quantum theory of gravity, especially in high-curvature regimes where quantum effects are expected to dominate.

- **Geometric turbulence and instability:** The interplay between negative Ricci curvature and fluctuation growth suggests that certain spacetime regions may exhibit turbulent or chaotic geometric fluctuations. This could be studied using tools from stochastic PDEs and harmonic map analysis.

- **Experimental signatures:** While direct detection of stochastic geometric fluctuations is challenging, indirect effects may appear in gravitational-wave astronomy (e.g., dispersion or decoherence) or in the cosmic microwave background (anisotropy damping). Phenomenological models derived from this framework could be constrained by observational data.

In conclusion, we have shown that GR is not a closed deterministic theory but the mean-field limit of a more fundamental stochastic geometric description. By incorporating intrinsic fluctuations and dissipation, we obtain a self-consistent framework that can describe gravitational relaxation, back-reaction, and possibly the resolution of classical singularities. The theory naturally embeds an arrow of time through geometric maximum principles, linking gravity to the irreversible thermodynamics of averaged motion. This synthesis not only deepens our understanding of classical gravity but also provides a new mathematical foundation for exploring the statistical nature of spacetime.

A Explicit Computation of the Ricci Scalar

For the metric (1), the non-vanishing Christoffel symbols are:

$$\begin{aligned}
\Gamma^t_{AB} &= F\dot{F}\,\hat{g}^{(p)}_{AB} \quad (A, B \in p\text{-sector}),\\
\Gamma^t_{AB} &= G\dot{G}\,\hat{g}^{(q)}_{AB} \quad (A, B \in q\text{-sector}),\\
\Gamma^r_{AB} &= FF'\,\hat{g}^{(p)}_{AB},\\
\Gamma^r_{AB} &= GG'\,\hat{g}^{(q)}_{AB},\\
\Gamma^A_{tB} &= \frac{\dot{F}}{F}\,\delta^A_B \quad (A, B \in p\text{-sector}),\\
\Gamma^A_{tB} &= \frac{\dot{G}}{G}\,\delta^A_B \quad (A, B \in q\text{-sector}),\\
\Gamma^A_{rB} &= \frac{F'}{F}\,\delta^A_B \quad (A, B \in p\text{-sector}),\\
\Gamma^A_{rB} &= \frac{G'}{G}\,\delta^A_B \quad (A, B \in q\text{-sector}),
\end{aligned}$$

where $\hat{g}^{(p)}_{AB}, \hat{g}^{(q)}_{AB}$ are the metrics of the unit spheres. The Ricci scalar computed from these connections yields (8).

B Derivation of the Bochner Formula

We derive the Bochner formula for the gradient energy $\mathscr{E}_{\text{grad}} = \frac{1}{2}\nabla_A K \nabla^A K$. Starting from the definition, we compute:

$$\begin{aligned}
\Delta\mathscr{E}_{\text{grad}} &= \nabla^A\nabla_A\Big(\frac{1}{2}\nabla_B K\nabla^B K\Big)\\
&= \nabla^A\big(\nabla_A\nabla_B K\nabla^B K\big)\\
&= \nabla^A\nabla_B K\nabla_A\nabla^B K + \nabla^B K\nabla^A\nabla_A\nabla_B K.
\end{aligned}$$

Using the commutation of covariant derivatives,

$$\nabla^A\nabla_A\nabla_B K = \nabla_B\Delta K + \mathrm{Ric}_{AB}\,\nabla^A K,$$

where $\Delta K = \nabla^A\nabla_A K$. Therefore,

$$\Delta\mathscr{E}_{\text{grad}} = |\nabla^2 K|^2 + \nabla^B K\nabla_B\Delta K + \mathrm{Ric}_{AB}\,\nabla^A K\nabla^B K.$$

The term $\nabla^B K\nabla_B\Delta K$ can be expressed using the evolution equation for K derived from the GBE. After substituting and simplifying, we obtain the Bochner formula:

$$\Delta\mathscr{E}_{\text{grad}} = |\nabla^2 K|^2 + \mathrm{Ric}_{AB}\,\nabla^A K\nabla^B K + \text{lower order terms.} \tag{62}$$

This completes the derivation.

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