Effect of "Time Scaling" on astonomical constants,
from the systems that use "TCB", "TCG", "TT", or "TDB".

The times, velocities, positions, and masses of the planets in our solar system are available from the fits to observational data conducted by JPL and recorded in their ephemeris files. The current JPL ephemerides provide the most accurate source for astronomical constants involving those measurements. However, they use a "scaled" system meaning that times, velocities, positions, and masses in the files are all divided by a constant factor of

$$1/(1 - L_B) = 1.0000000155051979175908466588 3829...$$

Two reasons why they do that are:

1. Many of the observational data points used in their integration are (or used to be) recorded from Earth, and so are tagged with a time coordinate in "Terestrial Time" (TT). But the equations of motion for integrating the forces between solar system bodies require a time coordinate steady versus a clock at the center of mass of the system, and with a rate equal to the SI second. "Barycentric Coordinate Time" (TCB) is such a time system (together with its corresponding set of position coordinates, its space-time coordinate system is called the "Barycentric Celestial Reference System" (BCRS)). However, "TCB" and "TT" run at very different rates, so if the two are mixed up in the input data, the error will cause serious effects in the integration. Thus a time scale which has the same long term scaling as "TT" but is steady versus "TCB" (called "Barycentic Dynamical Time" (TDB)) was devised for the time coordinate of the ephemerides.

When constructing the ephemeris files, "TDB" is used in the equations of motion which were intended for "TCB". The effect of that is only that the output data (the times, velocities, positions, and masses) of the integration are scaled by the factor stated above. The calculation of that will be shown below.

2. With "TDB" as the time coordinate, the ephemerides can be read directly by inputing the time from Earth clocks ("TT") without converting to "TCB". Although the output positions are then scaled by the factor above, the error in a scaled position coordinate does not increase over time as it does for a scaled time coordinate, which grows arbitrarily large over long enough time intervals. Thus, for many practical applications an ephemeris with a "TDB" time scale is slightly simpler to use than one versus "TCB", or at least was in the past.

However, note that for all general scientific purposes (such as retrieving values for astrophysical parameters), results read straight from the ephemeris files must have their scaling removed, in other words they must be multiplied by the factor above, the scaling is only a crutch for combining the observations reliably to make the ephemeris and to read it for predicting observations, any use external to those purposes must adjust them to their dynamically correct values (called "unscaled").

That explains the purpose and relation between the three time scales called "TCB", "TT", and "TDB". The remaining time scale called "Geocentric Coordinate Time" (TCG) is similar to "TCB" except it (and its corresponding space-time coordinate system called the "Geocentric Celestial Reference System" (GCRS)) is for integrating motions versus the Earth's center of mass rather than the solar system's (particularly useful for Earth orbits). Since such a system is not closed, forces from masses not included in the calculation of the system's center of mass (the effects of the rest of the solar system) are treated as tidal forces.

Here I will derive the numerical relation between the four time scales (the "scaling"):

The relativistic equations of motion that are integrated to fit the observations (since 1971) are:

1. for the relation between time, position, velocity and acceleration:

$$\ddot{\vec{x}}_j = -\sum_i \frac{G_N * m_i * (\vec{x}_j - \vec{x}_i)}{|(\vec{x}_j - \vec{x}_i)|^3} \ + \ \text{relativistic terms}$$

where i enumerates each known massive object in the solar system, except for the object j itself.

2. for calculating distance (from light propagation):

$$|(\vec{x}_j - \vec{x}_i)| = c * (t_j - t_i) \ + \ \text{relativistic terms} \ + \ \text{solar corona delay terms} \ + \ \text{ionosphere delay terms}$$

where $t_i, \vec{x}_i$ and $t_j, \vec{x}_j$ are any two space-time coordinates between which the light propagates.

(The correctional terms above (the "relativistic", "corona", and "ionosphere" terms) must have the same units as the first term and so are not needed here for balancing the units.)

For equation #1 above to be valid the system enumerated by i must be approximately closed so that no significant forces are unaccounted for. It also requires the positional and time axises to mesh with its values at points far from the effects of the masses and be at rest with respect to the center of mass of the system so that there is no unaccounted for time-dialation or red-shift and so that the forces balance. "TCB" satisfies the latter requirement. "TDB" is "TCB" divided by the factor

$$1/(1 - L_B) = 1.000000015505197917590846658838 29...$$

When "TDB" is used in "TCB"'s place in equation #2, and c remains "299792458", the resulting *distances* are scaled by the same factor that TDB is: the distances that are input and output from the equation and thus the ephemerides are divided by the factor

$$1/(1 - L_B) = 1.00000001550519791759084665883829...$$

Those scalings of the time and distance coordinates then require a scaled mass to balance equation #1: G_N remains "$0.66743(7) \times 10^{-10}$". On the left hand side "$\ddot{\vec{x}}_j$" has units of $Distance^1 * Time^{-2}$ and thus the distance and time scaling multiplies it by $1/(1-L_B)$. The units of "$\frac{(\vec{x}_j - \vec{x}_i)}{|(\vec{x}_j - \vec{x}_i)|^3}$" are $Distance^{-2}$ and thus the distance scaling multiplies it by the factor $1/(1 - L_B)^2$. For the left and right hand side of the equation to balance, mass must then be divided by the factor $1/(1-L_B)$ since the value of G_N stays at "$0.66743(7) \times 10^{-10}$" and so there is nothing else to balance with except by changing the mass scaling. (Usually $G_N * m$ is of interest rather than m; they both have the same scaling factor.)

Thus the times, velocities, positions, and masses of all data input and output from the ephemerides of JPL are all divided by that same factor of $1/(1 - L_B)$ so that "TDB" can be used in place of "TCB" without changing its equations of motion or the numerical values of the fundamental constants (G_N and c).

Next I will derive the numbers used in the scalings:

To derived the rates of the four time scales I will use the equation for time dialation:

$$dt_v = dt * \sqrt{1 - \left(\frac{v}{c}\right)^2}$$

where dt_v are time intervals measured by clocks moving versus the center of mass.

And for redshift:

$$dt_g = dt * \sqrt{1 - \frac{2 * G_N * m}{r * c^2}}$$

where dt_g are time intervals measured by clocks in the gravitation potential of the mass m at distance r.

The ratio of TCG to TCB can be calculated as follows:

The velocity of the Earth in its orbit around the Sun can be estimated as follows:

$$149597870700\,\text{meters per au} * 2\pi \text{ divided by } 365.256363 \text{ days} = 29784 \text{ m/s}$$

which gives a time dialation of $1 - 4935085 \times 10^{-15}$

However by integrating DE405 I got the following more precise values which I use below:

$$\sqrt{< (\text{velocity of the Earth versus the solar system's barycenter})^2 >} = 29784.65(8)\ m/s$$

which gives a time dialation of $1 - 4935302 \times 10^{-15}$

The redshift of the Earth in its orbit around the Sun can be estimated as follows:

$$\sqrt{1 - \frac{2 * G_N * m_\odot}{r * c^2}} \approx 1 - 9870629 \times 10^{-15}$$

Where I used the value of the AU from the *Astronomical Almanac for the year 2011* for r and its value for $G_N\, M_\odot$.

However by integrating DE405 I got the following more precise value which I use below:

$$1 - 9870627 \times 10^{-15}$$

Combining those values with the redshifts from each of the planets and the moon I created the following list of effects:

	4935302	from Earth's velocity versus the solar system's barycenter
	9870627	from the gravitational potential of Sun
	1829	from the gravitational potential of Jupiter
	297	from the gravitational potential of Saturn
	142	from the gravitational potential of the Moon
	29	from the gravitational potential of Venus
	23	from the gravitational potential of Uranus
	17	from the gravitational potential of Neptune
+	2	from the gravitational potential of Mars

= 14808268

where each number in the list is the factor subtracted from 1 and multiplied times 10^{15}.

(Those redshifts from the planets were calculated by finding the mean of the reciprical distance from Earth through integrating DE405.)

The value for $1 - TCG/TCB$ from the almanac is "$1.48082686741(2) \times 10^{-8}$" which is equal to the number derived above to 7 digits thus the derivation above is verified.

The remaining ratio of TT to TCG can be estimated by the time dialation from the Earth's spin at the equator, and redshift at the equator from $G_N\, M_\oplus$:

The velocity of an observer at the Earth's equator versus the Earth's center of mass can be estimated as follows:

$$6378137 \text{ meters per } \mathrm{R}_{\odot} * 2\pi \text{ divided by } 86164.10 \text{ seconds} = 465.1011 \text{ m/s}$$

which gives a time dialation of $1 - 1203 \times 10^{-15}$

The redshift of a clock on the Earth's equator do to $G_N\, M_{\oplus}$ can be estimated as follows:

$$\sqrt{1 - \frac{2 * G_N * m_{\oplus}}{R_{\oplus} * c^2}} \approx 1 - 695349 \times 10^{-15}$$

Combining those values gives:

	695349	from the gravitational potential of the Earth
+	1203	from the spin of the Earth
=	696552	

The value for $1 - TT/TCG$ from the almanac is "$6.969290134(0) \times 10^{-10}$" which is slighly different from the value I derived above. However if I use the "Potential of the geoid" (W_0) from the almanac for the redshift in place of the calculations that used $G_N\, M_{\oplus}$ and $R_{\oplus}$, I get better agreement using the following equation:

$$\sqrt{1 - \frac{2 * W_0}{c^2}} = 1 - 696929 \times 10^{-15}$$

(which I use below)

Finally, combining the factors above for TT/TCG and TCG/TCB, gives the factor TT/TCB of:

	14808268	for TCG/TCB
+	696929	for TT/TCG
=	15505197	

The value for $1 - TT/TCB$ from the almanac is "$1.550519768(0) \times 10^{-8}$" which is equal to the number derived above to 7 digits thus the derivation above is verified.

The factors above only concern the secular differences between the time systems, but in addition periodic differences exist between all of them except "TDB" and "TCB" which have no periodic differences by definition. Similarly, they all had secular differences between them given by the factors calculated above, except "TT" and "TDB" which have (essentially) no secular variation by definition.

Next I will give some examples of conversions between scaled and unscaled values.

The value for the *au* given in the almanac is 149597870700(3) *m*. By definition $G_N M_\odot$ is calculated from that by

$$G_N * M_\odot(\text{scaled}) = \frac{\text{au}^3 * 0.01720209895^2}{86400^2} = 1327124400419 \times 10^8 \text{ m}^3 \text{ s}^{-2}$$

However those numbers are scaled. To get the unscaled mass of the Sun I must multiply the scaled value by $1/(1 - L_B)$. Thus

$$G_N * M_\odot(\text{unscaled}) = 1327124400419 \times 10^8 * 1.00000001550519791759084665883829... = 1327124420997 \times 10^8$$

For another example the $G_N\, M_\odot$ is given from satellite laser ranging as

$$G_N\, M_\odot(\text{unscaled}) = 3986004418 \times 10^5 \text{ m}^3 \text{ s}^{-2}$$

Since it is unscaled, that is the value for the mass of the Earth used in the TCB system, and also in the TCG system, since they both use unscaled masses. For a "TDB-compatible" value, the unscaled value must be multiplied by the corresponding factor:

$$G_N * M_\odot(\text{scaled to TDB}) = 3986004418 \times 10^5 \text{ m}^3 \text{ s}^{-2} / 1.00000001550519791759084665883829... = 3986004356 \times 10^5 \text{ m}^3 \text{ s}^{-2}$$

and

$$G_N * M_\odot(\text{scaled to TT}) = 3986004418 \times 10^5 \text{ m}^3 \text{ s}^{-2} * (1 - 6.969290134(0) \times 10^{-10}) = 3986004415 \times 10^5 \text{ m}^3 \text{ s}^{-2}$$

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