Merry-go-round
and
time-dependent symplectic forms

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Abstract

In the merry-go-round fictitious forces are acting like centrifugal force and Coriolis force. Like the Lorentz force Coriolis force is velocity dependent and, following Arnol′d [Arn61], can be modeled by twisting the symplectic form. If the merry-go-round is accelerated an additional fictitious force shows up, the Euler force.

In this article we explain how one deals symplectically with the Euler force by considering time-dependent symplectic forms. It will turn out that to treat the Euler force one also needs time-dependent primitives of the time-dependent symplectic forms.

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1 Introduction

1.1 Motivation and general perspective

On a merry-go-round fictitious forces appear. One of these fictitious forces is the centrifugal force. The centrifugal force only depends on position and is a conservative force, i.e. it is the gradient of a potential. If one moves on a merry-go-round an additional fictitious force is the Coriolis force. In contrast to the centrifugal force, the Coriolis force depends linearly on the velocity. There is a third fictitious force, the Euler force. The Euler force only appears if the merry-go-round is accelerated or decelerated. As the centrifugal force the Euler force only depends on position, but different from the centrifugal force the Euler force is not a conservative force, i.e. it is not a gradient vector field.

It was observed by Arnol′d [Arn61] that velocity dependent forces, as the Lorentz force of a magnetic field or the Coriolis force, can be modeled symplectically by twisting the standard symplectic form on the cotangent bundle. In this article we address the question how to model the Euler force symplectically. In an accelerated merry-go-round the Coriolis force itself is time-dependent and therefore modeling it as Arnol′d did, the symplectic form on the cotangent bundle gets time-dependent.

We explain in this paper that the Euler force can be modeled with the help of a time-dependent primitive of the time-dependent symplectic form. In particular, in contrast to the Coriolis force the choice of the time-dependent primitive matters for the Euler force. This is vaguely reminiscent of the Aharonov-Bohm effect which not only depends on the magnetic field, but also on the choice of the vector potential. On the other hand, we do not see a direct connection between the Aharonov-Bohm effect and the Euler force. The Aharonov-Bohm effect is a quantum mechanical phenomenon, while what we discuss here is a classical phenomenon where time-dependence of the magnetic field is crucial.

The study of accelerated merry-go-rounds is motivated by applying symplectic methods to find a gateway around Mars. In fact, the orbit of Mars is rather eccentric. Therefore the dynamics around Mars has to be modeled by the elliptic restricted three-body-problem and not just the circular one. In the elliptic restricted three-body-problem the rotational velocity is not constant as in accelerating and decelerating merry-go-round. However, time dependence of acceleration and deceleration is periodic with period 1 Martian year.
With this motivation in mind we are interested in detecting periodic orbits in periodically accelerating and decelerating merry-go-rounds. For that purpose

we are looking at symplectic forms depending periodically on time, together with a choice of time-dependent primitives. The time dependence of the primitive itself does not need to be periodic, but twisted-periodic in a sense specified in this paper in Definition 5.1.

1.2 Outline and main results

In Section 2 we treat a merry-go-round from the Hamiltonian point of view and derive its Hamilton equation in phase space and the force equation in configuration space. In the force equation we will discover the three fictitious forces, namely the centrifugal force, the Coriolis force, and the Euler force. This force equation fits into a more general class of equations which we refer to as the $(\boldsymbol{A}, \phi)$-equation. Here $\boldsymbol{A}$ is a time-dependent vector potential of a time-dependent magnetic field while ϕ is a time-dependent scalar potential.

In Section 3 we are considering periodic solutions of the $(\boldsymbol{A}, \phi)$-equation. In order to find periodic solutions to the $(\boldsymbol{A}, \phi)$-equation one needs some periodicity assumptions on our data. We do not need that the vector potential $\boldsymbol{A}$ itself is periodic in time, but only its differentials with respect to time as well as with respect to space. We are then discussing how periodic solutions can be detected variationally with the help of an action functional. Since we do not assume that the vector potential itself is periodic it requires some argument to show that the action functional is well defined. The advantage of considering vector potentials which are not necessary periodic in time is that this allows one to incorporate the potential ϕ in the vector potential as we will discuss in Section 3.2.

While in Section 3 we discuss a Lagrangian approach to periodic solutions of the $(\boldsymbol{A}, \phi)$-equation, we study in Section 4 a Hamiltonian approach. This Hamiltonian approach fits onto a more general class of functionals which we discuss in more detail in Section 5.

In Section 5 we are considering a time-dependent family of primitives of time-dependent symplectic forms. Again this family does not need to depend periodically on time, but twisted-periodically whose precise meaning is explained in (5.12). For such a twisted-periodic family of primitives we show how to associate a well defined action functional. We derive the critical point equation of this functional in two ways. In the proof of Theorem 5.5 we use Cartan's formula on the product of the underlying manifold with the circle. In Section 5.3 we give a different derivation of the critical point equation using Cartan's formula on the loop space. Since the loop space is infinite dimensional, strictly speaking there is no mathematical theory yet to use Cartan's formula there. It turns out that a new vector field Y is entering the critical point equation which similarly as the Hamiltonian vector field X is implicitly defined with the help of the time-derivative $\dot{\lambda}_t$ of the primitive. We refer to this vector field Y as the *Euler vector field* in view of its close relation with the Euler force. In the second interpretation using the Cartan formula on the loop space this derivative appears as the Lie derivative with respect to the vector field on the free loop space which generates the rotation.

2 Merry-go-round and fictitious forces

We consider a free particle in the plane. In an inertial system the free particle moves with constant velocity along a straight line. In particular, the Hamiltonian is just given by kinetic energy T. There are no accelerations, hence no forces acting. This changes as soon as we rotate the coordinate system with time-dependent angular velocity $\varpi\colon \mathbb{R} \to \mathbb{R}$. In this case apart from kinetic energy T the particle has angular momentum and three pseudo-forces appear, namely

- **centrifugal force** depending on the locus q and the angular velocity ϖ_t is a gradient, hence conservative;
- **Coriolis force** depending on the particle velocity $\dot q$ and ϖ_t;
- **Euler force** depending on the locus q and the angular acceleration $\dot\varpi_t$.

Let $\varpi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. The function whose Hamiltonian vector field generates rotation is angular momentum. Hence in a rotating coordinate system with time-dependent angular speed ϖ_t, say the xy-plane rotating anti-clockwise in space, the Hamiltonian of a free particle, say of mass $m = 1$, consists of kinetic energy minus angular momentum, in symbols

$$\begin{aligned} H^\varpi\colon \mathbb{R}^5 \to \mathbb{R}, \quad (t, q, \boldsymbol{p}) \mapsto &= \tfrac{1}{2}\left|\boldsymbol{p}\right|^2 - \left\langle \begin{pmatrix} 0\\ 0\\ \varpi_t \end{pmatrix}, \underbrace{\begin{pmatrix} q_1\\ q_2\\ 0 \end{pmatrix} \times \begin{pmatrix} p_1\\ p_2\\ 0 \end{pmatrix}}_{\text{angular momentum}} \right\rangle_{\mathbb{R}^3} \\ &= \tfrac{p_1^2+p_2^2}{2} - \varpi_t(q_1p_2 - q_2p_1) =: H^{\varpi_t}(q, \boldsymbol{p}). \end{aligned}$$

After completing the squares H^ϖ is of the form

$$H^{\varpi_t}(q,\boldsymbol{p}) = \underbrace{\tfrac{1}{2}\left(\left(p_1 + \varpi_t q_2\right)^2 + \left(p_2 - \varpi_t q_1\right)^2\right)}_{T(\boldsymbol{p}-\boldsymbol{A}_t(q))} \underbrace{-\tfrac{1}{2}\varpi_t^2\left|q\right|^2}_{\phi_t(q)} = T(\boldsymbol{p} - \boldsymbol{A}_t|_q) + \phi_t|_q$$

where the time-dependent vector potential and the potential are

$$\boldsymbol{A}_t(q) = \boldsymbol{A}_{\varpi_t}(q) = \begin{pmatrix} -\varpi_t q_2\\ \varpi_t q_1 \end{pmatrix} = \varpi_t J_0 q, \qquad \phi_t(q) = -\tfrac{1}{2}\varpi_t^2\left|q\right|^2. \tag{2.1}$$

The anti-/clockwise quarter rotation in the plane is encoded by the matrices

$$J_0 := \begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix}, \qquad \bar{J}_0 := -J_0 = \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix}.$$

is the canonical complex structure on $\mathbb{R}^2$. The **Hamilton equations** for H^ϖ with respect to the canonical symplectic form $\omega_{\mathrm{can}} = d\lambda_{\mathrm{can}}$ are

$$\begin{cases} \dot q = \partial_p H^{\varpi_t} & = \boldsymbol{p} - \boldsymbol{A}_t(q)\\ \dot{\boldsymbol{p}} = -\partial_q H^{\varpi_t} & = \sum_{j=1}^{2}(p_j - A_t^j)\nabla \boldsymbol{A}_t^j(q) - \nabla\phi_t(q) \end{cases}. \tag{2.2}$$

This is a first order ODE for smooth maps $(q, \boldsymbol{p})\colon \mathbb{R} \to \mathbb{R}^4$. We eliminate $\boldsymbol{p}$ to get a second order ODE in q. For the following calculation we simplify notation $A_1 := A_t^1$ and $A_2 := A_t^2$ as well as $\partial_1 := \partial_{q_1}$ and $\partial_2 := \partial_{q_2}$. Differentiating the first Hamilton equation with respect to t and then using the second one for $\dot{\boldsymbol{p}}$ we obtain for q the second order ODE

$$\begin{aligned}
\ddot{q} &= \dot{\boldsymbol{p}} - \tfrac{d}{dt}\boldsymbol{A}_t(q)\\
&= -\partial_q\left(\tfrac{1}{2}\left|\boldsymbol{p} - \boldsymbol{A}_t(q)\right|^2\right) - \nabla\phi_t(q) - \dot{\boldsymbol{A}}_t(q) - d\boldsymbol{A}_t|_q\dot{q}\\
&= -\begin{pmatrix}\partial_1\frac{(p_1-A_1)^2+(p_2-A_2)^2}{2}\\ \partial_2\frac{(p_1-A_1)^2+(p_2-A_2)^2}{2}\end{pmatrix} - \nabla\phi_t - \dot{\boldsymbol{A}}_t - \begin{pmatrix}\partial_1 A_1 & \partial_2 A_1\\ \partial_1 A_2 & \partial_2 A_2\end{pmatrix}\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix}\\
&= \begin{pmatrix}\underline{(p_1-A_1)\partial_1 A_1} + (p_2-A_2)\partial_1 A_2\\ (p_1-A_1)\partial_2 A_1 + \underline{\underline{(p_2-A_2)\partial_2 A_2}}\end{pmatrix} - \begin{pmatrix}\underline{(\partial_1 A_1)\dot{q}_1} + (\partial_2 A_1)\dot{q}_2\\ (\partial_1 A_2)\dot{q}_1 + \underline{\underline{(\partial_2 A_2)\dot{q}_2}}\end{pmatrix} - \nabla\phi_t - \dot{\boldsymbol{A}}_t\\
&= \begin{pmatrix}(\partial_1 A_2 - \partial_2 A_1)\dot{q}_2\\ (\partial_2 A_1 - \partial_1 A_2)\dot{q}_1\end{pmatrix} - \nabla\phi_t - \dot{\boldsymbol{A}}_t \qquad , \partial_1 A_2 - \partial_2 A_1 =: \operatorname{rot}\boldsymbol{A}_t\\
&= (\operatorname{rot}\boldsymbol{A}_t)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix} - \nabla\phi_t - \dot{\boldsymbol{A}}_t\\
&= -\left(\operatorname{rot}\boldsymbol{A}_t|_q\right) J_0\dot{q} - \dot{\boldsymbol{A}}_t|_q - \nabla\phi_t|_q.
\end{aligned}$$

Underlined terms cancel due to the first Hamilton equation. We have shown

Lemma 2.1. *The first order ODE (2.2) is equivalent to the second order ODE*

$$\ddot{q} = -\left(\operatorname{rot}\boldsymbol{A}_t|_q\right) J_0\dot{q} - \dot{\boldsymbol{A}}_t|_q - \nabla\phi_t|_q. \tag{2.3}$$

We call (2.3) the $(\boldsymbol{A}, \phi)$***-equation****. It makes sense for smooth* $\phi\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$.

Corollary 2.2. *In the merry-go-round case—*$\boldsymbol{A}$, ϕ *given by (2.1)—we have*

$$\operatorname{rot}\boldsymbol{A}_t|_q := \partial_1\boldsymbol{A}_t^2 - \partial_2\boldsymbol{A}_t^1 = 2\varpi_t, \qquad \dot{\boldsymbol{A}}_t|_q = \dot{\varpi}_t J_0 q, \qquad \nabla\phi|_q = -\varpi_t^2 q,$$

so that the $(\boldsymbol{A}, \phi)$*-equation becomes*

$$\begin{aligned}
\ddot{q} &= -2\varpi_t J_0\dot{q} - \dot{\varpi}_t J_0 q + \varpi_t^2 q\\
&= \underbrace{2\varpi_t \bar{J}_0\dot{q}}_{\text{Coriolis}} + \underbrace{\dot{\varpi}_t\bar{J}_0 q}_{\text{Euler}} + \underbrace{\varpi_t^2 q}_{\text{centrifug.}} .
\end{aligned}$$

3 Lagrangian variational approach to periodic solutions of the (A, ϕ)-equation

In this section we are looking at periodic solutions of the equation (2.3), namely

$$\ddot{q} = -\left(\operatorname{rot}\boldsymbol{A}_t(q)\right) J_0\dot{q} - \dot{\boldsymbol{A}}_t(q) - \nabla\phi_t.$$

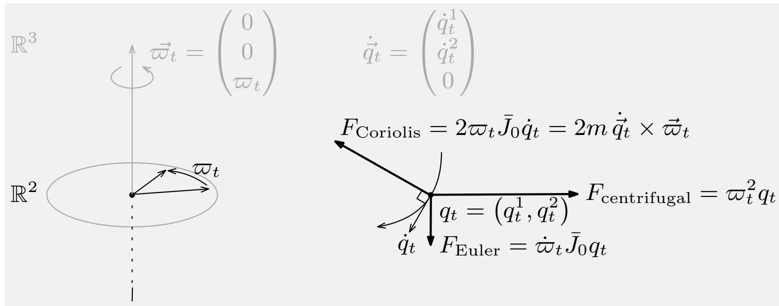

Figure 1: The three fictitious forces in the merry-go-round

In this article **periodic** means 1-periodic and we identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. To make sense of periodicity of solutions of the $(\boldsymbol{A}, \phi)$-equation we need that all three $(\operatorname{rot} \boldsymbol{A}_t)$, $\dot{\boldsymbol{A}}_t$, and ϕ_t are periodic in time t. We do not need that $\boldsymbol{A}_t$ itself is periodic in t, but we need some twisted-periodicity as explained next.

Definition 3.1 (Twisted-periodic 1-form)**.** Let $\mathfrak{Q} \subset \mathbb{R}^2$ be an open subset. A **twisted-periodic 1-form** $\{\theta_t\}$ is a real smooth family of 1-forms on $\mathfrak{Q}$

$$\theta_t = A_t^1\, dq_1 + A_t^2\, dq_2, \qquad A_t^1, A_t^2 \colon \mathfrak{Q} \to \mathbb{R}, \tag{3.4}$$

such that both derivatives, the derivative with respect to the parameter t as well as the exterior derivative with respect to the variable q, are periodic in time

$$\forall t \in \mathbb{R}\colon \qquad \dot{\theta}_{t+1} = \dot{\theta}_t, \qquad d\theta_{t+1} = d\theta_t.$$

Viewing the plane as a subspace of space $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ we use the coefficients of the twisted 1-form θ on the plane to define a vector field in space by

$$(\boldsymbol{A}_t, 0)\,|_{(q_1,q_2,q_3)} := \left(A_t^1|_{(q_1,q_2)}, A_t^2|_{(q_1,q_2)}, 0\right)$$

called a **vector potential**.

Remark 3.2. Magnetic fields are described by closed 2-forms σ, where closedness $d\sigma = 0$ encodes the fact that there are no magnetic charges; see e.g. [Web17, § 2.4.1]. For a twisted-periodic 1-form θ the 2-form

$$\sigma_t := d\theta_t = d\left(A_t^1\, dq_1 + A_t^2\, dq_2\right) = \underbrace{\left(\partial_1 A_t^2 - \partial_2 A_t^1\right)}_{=:\operatorname{rot} \boldsymbol{A}_t} dq_1 \wedge dq_2 \tag{3.5}$$

is time-periodic $\sigma_t := d\theta_t = d\theta_{t+1} = \sigma_{t+1}$ and closed $d\sigma_t = dd\theta_t = 0$. The magnetic vector field corresponding to the magnetic 2-form is $\boldsymbol{B}_t := (*\sigma)^\#$, i.e.

$$\boldsymbol{B}_t := \underbrace{\nabla \times}_{=\operatorname{rot}} \begin{pmatrix} \boldsymbol{A}_t \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_{q_1} \\ \partial_{q_2} \\ \partial_{q_3} \end{pmatrix} \times \begin{pmatrix} A_t^1 \\ A_t^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\partial_{q_3} A_t^2 \\ \partial_{q_3} A_t^1 \\ \partial_1 A_t^2 - \partial_2 A_t^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \operatorname{rot} \boldsymbol{A}_t \end{pmatrix}.$$

This shows that the magnetic field is perpendicular to the q_1-q_2-plane in $\mathbb{R}^3$.

Definition 3.3. Let $\mathfrak{Q} \subset \mathbb{R}^2$ be an open subset such that the first deRham cohomology $H^1_{dR}(\mathfrak{Q}) = 0$ vanishes. Let $\phi\colon \mathbb{S}^1 \times \mathfrak{Q} \to \mathbb{R}$ be a smooth map. Let θ be a twisted-periodic 1-form. The **classical action** functional is defined by

$$\begin{aligned} \mathcal{S}_L\colon \Lambda\mathfrak{Q} := W^{1,2}(\mathbb{S}^1, \mathfrak{Q}) &\to \mathbb{R} \\ q &\mapsto \int_0^1 \left(\tfrac{1}{2} \left|\dot q_t\right|^2 + \theta_t|_{q_t} \dot q_t - \phi_t(q_t) \right) dt \end{aligned} \tag{3.6}$$

where $L_t = L_{t+1}\colon T\mathfrak{Q} \to \mathbb{R}$ is the function, called **Lagrangian**, defined by

$$L_t(q, v) := \tfrac{1}{2} \left|v\right|^2 + \theta_t|_q v - \phi_t(q).$$

The following proposition shows that the action functional is well defined.

Proposition 3.4 (Magnetic part is well defined functional)**.** *Suppose that the first deRham cohomology $H^1_{dR}(\mathfrak{Q}) = 0$ of an open subset $\mathfrak{Q} \subset \mathbb{R}^2$ vanishes. Let θ be a twisted-periodic 1-form on $\mathfrak{Q}$. Then, for $k \in \mathbb{Z}$, the map defined by*

$$\mathcal{T}\colon \Lambda\mathfrak{Q} \to \mathbb{R}, \quad q \mapsto \int_k^{k+1} q^*\theta$$

is independent of k.

Proof. Since θ is twisted-periodic we have $d\theta_{t+1} = d\theta_t$ and therefore $\theta_{t+1} - \theta_t$ is a closed 1-form. Since $H^1_{dR}(\mathfrak{Q}) = 0$ it follows that $\theta_{t+1} - \theta_t$ is exact. So at every instant of time t there exists a smooth function $f_t\colon \mathfrak{Q} \to \mathbb{R}$ such that $\theta_{t+1} = \theta_t + df_t$. We may assume without loss of generality that $f_t = f(t, \cdot)$ for some smooth function $f\colon \mathbb{R} \times \mathfrak{Q} \to \mathbb{R}$; see Lemma 5.2. Using again that θ is twisted-periodic it follows that $d\dot f_t = \dot\theta_{t+1} - \dot\theta_t = 0$. Therefore $\dot f_t(q)$ is locally constant in q, but not necessarily in t. By definition of pull-back we obtain identity 1 in the following, while identity 2 uses periodicity $q_{t+1} = q_t$, hence $\dot q_{t+1} = \dot q_t$, namely

$$\begin{aligned} \int_{k+1}^{k+2} q^*\theta - \int_k^{k+1} q^*\theta &\overset{1}{=} \int_k^{k+1} \theta_{t+1}|_{q_{t+1}} \dot q_{t+1}\, dt - \int_k^{k+1} \theta_t|_{q_t} \dot q_t\, dt \\ &\overset{2}{=} \int_k^{k+1} \theta_{t+1}|_{q_t} \dot q_t\, dt - \int_k^{k+1} \theta_t|_{q_t} \dot q_t\, dt \\ &= \int_k^{k+1} \underbrace{(\theta_{t+1} - \theta_t)}_{=df_t}|_{q_t} \dot q_t\, dt \\ &\overset{4}{=} \int_k^{k+1} \left(\tfrac{d}{dt} f_t(q_t) - \dot f_t(q_t) \right) dt \\ &\overset{5}{=} f_{k+1}(q_1) - \underbrace{f_k(q_0)}_{=f_k(q_1)} - \int_k^{k+1} \underbrace{\dot f_t(q_t)}_{=\dot f_t(q_1)}\, dt \\ &\overset{6}{=} \int_k^{k+1} \dot f_t(q_1)\, dt - \int_k^{k+1} \dot f_t(q_1)\, dt \\ &= 0. \end{aligned} \tag{3.7}$$

Identity 4 is by the chain rule and identity 5 by the fundamental theorem of calculus. Identity 6 uses that $q_0 = q_1$ by periodicity and then we apply the fundamental theorem of calculus again. We also use that $\dot{f}_t(\cdot)$ is locally constant. This concludes the proof of Proposition 3.4. □

3.1 Critical points

Proposition 3.5. *The critical points of the classical action functional $\mathcal{S}_L \colon \Lambda\mathfrak{Q} \to \mathbb{R}$ in (3.6) are the solutions of the force equation*

$$\ddot{q} = -\left(\operatorname{rot} \boldsymbol{A}_t|_q\right) J_0 \dot{q} - \dot{\boldsymbol{A}}_t|_q - \nabla\phi_t(q) \tag{3.8}$$

for smooth loops $q \colon \mathbb{S}^1 \to \mathfrak{Q}$.

Proof. Let us move to $\mathbb{S}^1 \times \mathfrak{Q}$. Let D be the exterior derivative on $\mathbb{S}^1 \times \mathfrak{Q}$. The family of 1-forms θ_t on $\mathfrak{Q}$ induces a 1-form Θ on $\mathbb{S}^1 \times \mathfrak{Q}$, namely

$$\Theta|_{(t,x)} \begin{pmatrix} r \\ \xi \end{pmatrix} := \theta_t|_x \xi$$

whenever $(t, x) \in \mathbb{S}^1 \times \mathfrak{Q}$ and $(r, \xi) \in \mathbb{R} \times T_x\mathfrak{Q}$. The exterior derivative is

$$\begin{aligned} D\Theta|_{(t,x)} \left(\begin{pmatrix} r \\ \xi \end{pmatrix}, \begin{pmatrix} s \\ \eta \end{pmatrix} \right) &= d\theta_t|_x(\xi, \eta) + \left(dt \wedge \dot{\theta}_t|_x \right) \left(\begin{pmatrix} r \\ \xi \end{pmatrix}, \begin{pmatrix} s \\ \eta \end{pmatrix} \right) \\ &= d\theta_t|_x(\xi, \eta) + \dot{\theta}_t|_x \eta \cdot r - \dot{\theta}_t|_x \xi \cdot s. \end{aligned}$$

A loop q in $\mathfrak{Q}$ induces a loop Q in $\mathbb{S}^1 \times \mathfrak{Q}$, and a vector field ξ along q induces a vector field Ξ along Q, namely

$$Q_t = \begin{pmatrix} t \\ q_t \end{pmatrix}, \qquad \dot{Q}_t = \begin{pmatrix} 1 \\ \dot{q}_t \end{pmatrix}, \qquad \Xi = \begin{pmatrix} 0 \\ \xi \end{pmatrix}.$$

We calculate

$$(Q^* i_\Xi D\Theta)_t = D\Theta|_{(t,q_t)} \left(\begin{pmatrix} 0 \\ \xi \end{pmatrix}, \begin{pmatrix} 1 \\ \dot{q}_t \end{pmatrix} \right) dt = \left(d\theta_t|_{q_t}(\xi, \dot{q}_t) - \dot{\theta}_t|_{q_t} \xi \right) dt$$

for later application using that exterior derivative and pull-back commute. Then

$$\int_{\mathbb{S}^1} Q^*\Theta = \int_0^1 \Theta_{(t,q_t)} \begin{pmatrix} 1 \\ \dot{q}_t \end{pmatrix} dt = \int_0^1 \theta_t|_{q_t} \dot{q}_t \, dt = \int_{\mathbb{S}^1} q^*\theta.$$

Let $q_{(\tau)} := q + \tau\xi = (t, v + \tau\xi)$ for $\tau \in \mathbb{R}$. Then by the previous identity

$$
\begin{aligned}
d\mathcal{S}_L|_q\xi &= \tfrac{d}{d\tau}\big|_{\tau=0}\,\mathcal{S}_L(q+\tau\xi)\\
&= \int_0^1 \tfrac{d}{d\tau}\big|_0 \left(\tfrac{1}{2}|\dot q_t + \tau\dot\xi_t|^2 - \phi_t(q_t+\tau\xi_t)\right) dt + \tfrac{d}{d\tau}\big|_0 \int_{\mathbb{S}^1} q_{(\tau)}^*\theta\\
&= \int_0^1 \left(\langle \dot q_t, \dot\xi_t\rangle - d\phi_t|_{q_t}\xi_t\right) dt + \tfrac{d}{d\tau}\big|_0 \int_{\mathbb{S}^1} Q_{(\tau)}^*\Theta\\
&\stackrel{4}{=} \int_0^1 \left(\langle \dot q_t, \dot\xi_t\rangle - \langle\nabla\phi_t|_{q_t}, \xi_t\rangle\right) dt + \int_{\mathbb{S}^1} Q^* \underbrace{(Di_\Xi + i_\Xi D)}_{=L_\Xi}\Theta\\
&\stackrel{5}{=} \int_0^1 \left(\langle \dot q_t, \dot\xi_t\rangle - \langle\nabla\phi_t|_{q_t}, \xi_t\rangle + d\theta_t|_{q_t}(\xi, \dot q_t) - \dot\theta_t|_{q_t}\xi\right) dt\\
&\stackrel{6}{=} \int_0^1 \Big(\langle \dot q_t, \dot\xi_t\rangle - \langle\nabla\phi_t|_{q_t}, \xi_t\rangle + (\operatorname{rot}\boldsymbol{A}_t)\underbrace{(dq_1\wedge dq_2)(\xi,\dot q)}_{=\xi_1\dot q_2 - \xi_2\dot q_1} - \dot\theta_t|_{q_t}\xi\Big) dt\\
&\stackrel{7}{=} \int_0^1 \langle \dot q_t, \dot\xi_t\rangle dt + \left\langle \xi, -\nabla\phi_t|_q - (\operatorname{rot}\boldsymbol{A}_t)\, J_0\dot q - \dot{\boldsymbol{A}}_t\right\rangle_{L^2(\mathbb{S}^1,\mathbb{R}^2)}.
\end{aligned}
$$

Step 4 uses the definition of the Lie derivative (fisherman's derivative), see e.g. [Arn89, § 36 G], and Cartan's formula $L_\Xi = i_\Xi D + Di_\Xi$. Step 5 uses that the last but one summand vanishes since $Q^*D = DQ^*$, now apply Stokes theorem und use that the boundary of $\mathbb{S}^1$ is empty. Step 5 also uses the previously prepared formula for $Q^* i_\Xi D\Theta$. Step 6 uses formula (3.5) for $d\theta_t$.

Now suppose that q is a critical point of $\mathcal{S}_L$, i.e. $d\mathcal{S}_L|_q = 0$. Then the identity we just proved tells that $\dot q$ has a weak derivative, notation $\ddot q$, and

$$
\ddot q = -\left(\operatorname{rot}\boldsymbol{A}_t|_q\right) J_0\dot q - \dot{\boldsymbol{A}}_t|_q - \nabla\phi_t|_q.
$$

Note that the right hand side is in L^2, hence $q \in W^{2,2}(\mathbb{S}^1,\mathbb{R}^2)$. But then the right hand side is in $W^{1,2}$, hence $q \in W^{3,2}(\mathbb{S}^1,\mathbb{R}^2)$ and so on. Therefore $q \in \cap_{\ell\in\mathbb{N}_0} W^{\ell,2}(\mathbb{S}^1,\mathbb{R}^2) = C^\infty(\mathbb{S}^1,\mathbb{R}^2)$. This proves Proposition 3.5. □

3.2 Eliminate periodic scalar potentials

Definition 3.6 (Twisted-periodic vector potential)**.** A smooth map $\boldsymbol{A}\colon \mathbb{R}\times\mathfrak{Q} \to \mathbb{R}^2$, $(t,q)\mapsto \boldsymbol{A}(t,q) =: \boldsymbol{A}_t(q)$, is called a **twisted-periodic vector potential on $\mathfrak{Q}$** if at each time t it holds that

$$
\dot{\boldsymbol{A}}_{t+1} = \dot{\boldsymbol{A}}_t, \qquad \operatorname{rot}\boldsymbol{A}_{t+1} = \operatorname{rot}\boldsymbol{A}_t, \qquad \boldsymbol{A}_{t+1} - \boldsymbol{A}_t = df_t
$$

where $f_t = f(t,\cdot)$ for a smooth function $f\colon \mathbb{R}\times\mathfrak{Q}\to\mathbb{R}$; cf. Lemma 5.2.

Lemma 3.7. *Let $\phi\colon \mathbb{S}^1\times\mathfrak{Q}\to\mathbb{R}$ be a smooth function and $\boldsymbol{A}$ a twisted-periodic vector potential on $\mathfrak{Q}$. The $(\boldsymbol{A},\phi)$-**equation on $\mathfrak{Q}$** is the ODE*

$$
\ddot q = -\left(\operatorname{rot}\boldsymbol{A}_t\right) J_0\dot q - \dot{\boldsymbol{A}}_t - \nabla\phi_t(q) \tag{3.9}
$$

for smooth maps $t \mapsto q(t) \in \mathfrak{Q}$. Then the map defined by

$$(\boldsymbol{A}^\phi)_t := \boldsymbol{A}_t + \int_0^t \nabla\phi_s \, ds$$

is again a twisted-periodic vector potential. Moreover, a map q is a solution of the $(\boldsymbol{A}, \phi)$-equation iff q is a solution of the $(\boldsymbol{A}^\phi, 0)$-equation.

Proof. To see that $(\boldsymbol{A}^\phi)_t$ is a twisted-periodic vector potential note that

$$(\dot{\boldsymbol{A}^\phi})_{t+1} = \dot{\boldsymbol{A}}_{t+1} + \nabla\phi_{t+1} = \underline{\dot{\boldsymbol{A}}_t + \nabla\phi_t = (\dot{\boldsymbol{A}^\phi})_t}$$

and

$$\underbrace{\operatorname{rot}(\boldsymbol{A}^\phi)_{t+1}} = \operatorname{rot}\boldsymbol{A}_{t+1} + \int_0^{t+1} \underbrace{\operatorname{rot}(\nabla\phi_s)}_{=0} \, ds = \underbrace{\operatorname{rot}\boldsymbol{A}_{t+1}} = \underline{\operatorname{rot}\boldsymbol{A}_t = \operatorname{rot}(\boldsymbol{A}^\phi)_t}$$

where the last identity holds true by the underbraced one at time $t-1$. Now suppose that q solves (3.9). Then due to the two underlined identities above

$$-\left(\operatorname{rot}(\boldsymbol{A}^\phi)_t\right) J_0\dot{q} - (\dot{\boldsymbol{A}^\phi})_t - 0 = (\operatorname{rot}\boldsymbol{A}_t)\, J_0\dot{q} - \left(\dot{\boldsymbol{A}}_t + \nabla\phi_t(q)\right) = \ddot{q},$$

i.e. q satisfies the $(\boldsymbol{A}^\phi, 0)$-equation. Vice versa, in the $(\boldsymbol{A}^\phi, 0)$-equation for q insert $\partial_t(\boldsymbol{A}^\phi)_t = \dot{\boldsymbol{A}}_t + \nabla\phi_t$ and $\operatorname{rot}(\boldsymbol{A}^\phi)_t = \operatorname{rot}\boldsymbol{A}_t$ to get (3.9). This proves Lemma 3.7. □

4 Hamiltonian variational approach to periodic solutions

Let $\mathfrak{Q} \subset \mathbb{R}^2$ be an open subset such that the first deRham cohomology $H^1_{dR}(\mathfrak{Q}) = 0$ vanishes. Let $\phi \colon \mathbb{S}^1 \times \mathfrak{Q} \to \mathbb{R}$ be a smooth map. Consider the Hamiltonian given by kinetic plus potential energy by

$$H \colon \mathbb{S}^1 \times T^*\mathfrak{Q} \to \mathbb{R}, \quad (t, q, p) \mapsto \tfrac{1}{2}\left|p\right|^2 + \phi_t(q).$$

Let $\{\theta_t\}$ be a twisted-periodic 1-form on $\mathfrak{Q}$, see Definition 3.1, in particular

$$\theta_t = A_t^1 \, dq_1 + A_t^2 \, dq_2, \qquad \boldsymbol{A}_t = \left(A_t^1, A_t^2\right) : \mathfrak{Q} \to \mathbb{R}^2.$$

On the cotangent bundle $\pi \colon T^*\mathfrak{Q} \to \mathfrak{Q}$ we consider the time-dependent 1-form

$$\lambda_t := \lambda_{\mathrm{can}} + \pi^*\theta_t.$$

Then $\omega_t := d\lambda_t$ is a symplectic form at each time t and $\omega_{t+1} = \omega_t$ is periodic.

Definition 4.1. The **symplectic action functional** is defined by

$\mathcal{A} = \mathcal{A}_{\lambda,H} \colon \Lambda M \to \mathbb{R}$

$$v \mapsto \int_{\mathbb{S}^1} v^*\lambda - \int_0^1 H(t, v_t)\, dt = \int_0^1 \left(\lambda_t|_{v_t}\dot{v}_t - \tfrac{1}{2}\left|p\right|^2 - \phi_t(q)\right) dt.$$

That the term $\int_{\mathbb{S}^1} v^*\lambda$ is well defined follows from the assumption $H^1_{dR}(\mathfrak{Q}) = 0$. Since θ is twisted-periodic we have $d\theta_{t+1} = d\theta_t$ and therefore $\theta_{t+1} - \theta_t$ is closed. Since $H^1_{dR}(\mathfrak{Q})$ vanishes it follows that $\theta_{t+1} - \theta_t$ is exact. Therefore λ_t satisfies the twisted-periodicity condition (5.12). Hence the assertion follows from Proposition 5.4.

4.1 Critical points

Proposition 4.2. *If $v = (q, \boldsymbol{p})$ is a critical point of the symplectic action functional $\mathcal{A}_{\lambda,H}\colon \Lambda\mathfrak{Q} \to \mathbb{R}$, then q is a periodic solution of the $(\boldsymbol{A}, \phi)$-equation (2.3).*

Proof. By Theorem 5.5 the critical points of $\mathcal{A}_{\lambda,H}$ are smooth solutions $v = (q, \boldsymbol{p})\colon \mathbb{S}^1 \to \mathfrak{Q} \times \mathbb{R}^2$ of the (λ, H)-equation (5.14), namely $\dot{v} = X(v) + Y(v)$. It remains to calculate the Hamiltonian vector field X and the Euler vector field Y according to (5.13). To compute the Euler vector field Y we need the time derivative

$$\dot{\lambda}_t = \pi^*\dot{\theta}_t = \dot{A}^1_t dq_1 + \dot{A}^2_t dq_2$$

and the symplectic form

$$\omega_t = \omega_{\text{can}} + \pi^* d\theta_t = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \underbrace{\left(-\partial_{q_2} A^1_t + \partial_{q_1} A^2_t\right)}_{=\operatorname{rot} \boldsymbol{A}_t(q)} dq_1 \wedge dq_2.$$

Then Y_t, implicitly defined by $\dot{\lambda}_t = \omega_t(\cdot, Y_t)$, is explicitly given by the formula

$$Y_t = -\dot{A}^1_t \tfrac{\partial}{\partial p_1} - \dot{A}^2_t \tfrac{\partial}{\partial p_2}.$$

This is a vertical vector field, only depending on the base point q, but not on p. To compute the Hamilton vector field X observe that

$$dH_t = p_1\, dp_1 + p_2\, dp_2 + (\partial_{q_1}\phi_t)\, dq_1 + (\partial_{q_2}\phi_t)\, dq_2.$$

Then X_t, implicitly defined by $dH_t = \omega_t(\cdot, X_t)$, is explicitly given by the formula

$$X_t = p_1 \tfrac{\partial}{\partial q_1} + p_2 \tfrac{\partial}{\partial q_2} + \left(p_2 \operatorname{rot} \boldsymbol{A}_t - \tfrac{\partial \phi_t}{\partial q_1}\right) \tfrac{\partial}{\partial p_1} - \left(p_1 \operatorname{rot} \boldsymbol{A}_t - \tfrac{\partial \phi_t}{\partial q_2}\right) \tfrac{\partial}{\partial p_2}.$$

Summarizing

$$X_t(q, \boldsymbol{p}) + Y_t(q, \boldsymbol{p}) = \begin{pmatrix} p_1 \\ p_2 \\ p_2 \operatorname{rot} \boldsymbol{A}_t - \frac{\partial \phi_t}{\partial q_1} \\ -p_1 \operatorname{rot} \boldsymbol{A}_t - \frac{\partial \phi_t}{\partial q_1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\dot{\boldsymbol{A}}^1_t(q) \\ -\dot{\boldsymbol{A}}^2_t(q) \end{pmatrix}.$$

Therefore the (λ, H)-equation (5.14) is the following ODE system

$$\begin{cases} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} p_2 \operatorname{rot} \boldsymbol{A}_t - \frac{\partial \phi_t}{\partial q_1} - \dot{\boldsymbol{A}}^1_t \\ -p_1 \operatorname{rot} \boldsymbol{A}_t - \frac{\partial \phi_t}{\partial q_1} - \dot{\boldsymbol{A}}^2_t \end{pmatrix} \end{cases}$$

equivalently the (λ, H)-equation is given by

$$\begin{cases} \dot{q} = \boldsymbol{p} \\ \dot{\boldsymbol{p}} = -\left(\operatorname{rot} \boldsymbol{A}_t(q)\right) J_0 \boldsymbol{p} - \dot{\boldsymbol{A}}_t(q) - \nabla\phi_t(q) \end{cases}. \tag{4.10}$$

Writing the first order ODE system (4.10) as a second order ODE system we get the $(\boldsymbol{A}, \phi)$-equation

$$\ddot{q} = \underbrace{-\left(\operatorname{rot} \boldsymbol{A}_t(q)\right) J_0 \dot{q}}_{\text{Lorentz force}} \underbrace{-\dot{\boldsymbol{A}}_t(q)}_{\text{Euler f.}} \underbrace{-\nabla\phi_t(q)}_{\text{conservative}} \tag{4.11}$$

for smooth loops $q\colon \mathbb{S}^1 \to \mathbb{R}^2$. This proves Proposition 4.2. □

5 Euler vector field

Definition 5.1. A smooth[1] family of exact symplectic manifolds $(M, d\lambda_t)_{t\in\mathbb{R}}$ is called **twisted-periodic** if at each time t it holds that

$$d\lambda_{t+1} = d\lambda_t =: \omega_t, \qquad \dot{\lambda}_{t+1} = \dot{\lambda}_t, \qquad \lambda_{t+1} - \lambda_t = df_t \tag{5.12}$$

for a smooth function $f_t\colon M \to \mathbb{R}$.

Lemma 5.2. *There exists a smooth function $\tilde{f}\colon \mathbb{R} \times M \to \mathbb{R}$ such that the functions f_t in Definition 5.1 are given by $\tilde{f}(t, \cdot) = f_t$.*

Proof. We assume without loss of generality that M is connected, otherwise we apply the argument below to each component of M. Choose a base point $x_0 \in M$. Define $\tilde{f}\colon \mathbb{R} \times M \to \mathbb{R}$ by $\tilde{f}(t, x) := f_t(x) - f_t(x_0)$. Given $x \in M$, since M is connected there exists a smooth path $\gamma\colon [0, 1] \to M$ from x_0 to x. We obtain, by definition of $\tilde{f}$, the fundamental theorem of calculus, the chain rule, and the third property of λ_t being twisted-periodic, that

$$\begin{aligned} \tilde{f}(t, x) &= f_t(x) - f_t(x_0) \\ &= \int_0^1 \tfrac{d}{ds} f_t(\gamma_s)\, ds \\ &= \int_0^1 df_t|_{\gamma_s} \gamma'(s)\, ds \\ &= \int_0^1 \left(\lambda_{t+1} - \lambda_t\right)|_{\gamma_s} \gamma'(s)\, ds. \end{aligned}$$

Since λ is smooth in t, the function $\tilde{f}$ is smooth in t as well. This proves Lemma 5.2. □

[1] here smoothness refers to $\mathbb{R} \times TM \times \ni (t, (x, \xi)) \mapsto \lambda_t|_x \xi \in \mathbb{R}$ being smooth

Observe that the symplectic form depends periodically on time $\omega_{t+1} = \omega_t$. Let $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ be a smooth function on M depending periodically on time. At each time t non-degeneracy of the symplectic form ω_t allows for transforming the 1-forms dH_t and $\dot\lambda_t$ into vector fields along M, namely

$$dH_t = \omega_t(\cdot, X_t), \qquad \dot\lambda_t = \omega_t(\cdot, Y_t). \tag{5.13}$$

Note that both X_t and Y_t are periodic. While $X_t = X_{H_t}^{\omega_t}$ is called **Hamiltonian vector field**, we refer to

$$Y_t = Y_{\dot\lambda_t}^{\omega_t}$$

as **Euler vector field**, because of its relation to the Euler force as explained in the merry-go-round example in Section 2.

Definition 5.3. The **action functional** is defined by

$$\begin{aligned} \mathcal{A} = \mathcal{A}_{\lambda,H}\colon \Lambda M &\to \mathbb{R} \\ v &\mapsto \int_{\mathbb{S}^1} v^*\lambda - \int_0^1 H(t, v_t)\, dt = \int_0^1 \left(\lambda_t|_{v_t} \dot v_t - H(t, v_t)\right) dt. \end{aligned}$$

Proposition 5.4 (Well defined)**.** *The action functional $\mathcal{A}_{\lambda,H}$ is well defined.*

Proof. While H is periodic in time, the family of 1-forms λ_t is not necessarily. We need to show that for any loop $v \in \Lambda M$ the integral $\int_k^{k+1} \lambda_t|_{v_t} \dot v_t$ is independent of $k \in \mathbb{Z}$. To this end observe that $d\dot f_t = \dot\lambda_{t+1} - \dot\lambda_t = 0$ by property two in (5.12). Therefore $\dot f_t(q)$ is locally constant in q, but not necessarily in t. Now the proof proceeds repeating the calculation (3.7) just replacing u by v and θ by λ. This proves Proposition 5.4. □

Theorem 5.5 (Euler-Hamilton equation)**.** *Critical points of $\mathcal{A} = \mathcal{A}_{\lambda,H}$ are smooth solutions $v\colon \mathbb{S}^1 \to M$ of the **Euler-Hamilton** or **$(\boldsymbol{\lambda}, \boldsymbol{H})$-equation***

$$\dot v = X(v) + Y(v) \tag{5.14}$$

where $X = X_{H_t}^{\omega_t}$ and $Y = Y_{\dot\lambda_t}^{\omega_t}$ are determined by (5.13).

Proof. Let us move to $\mathbb{S}^1 \times M$. Let D be the exterior derivative on $\mathbb{S}^1 \times M$. The family of 1-forms λ_t on M induces a 1-form Λ on $\mathbb{S}^1 \times M$, namely

$$\Lambda|_{(t,x)} \begin{pmatrix} r \\ \xi \end{pmatrix} := \lambda_t|_x \xi$$

whenever $(t, x) \in \mathbb{S}^1 \times M$ and $(r, \xi) \in \mathbb{R} \times T_x M$. The exterior derivative is

$$\begin{aligned} D\Lambda|_{(t,x)} \left(\begin{pmatrix} r \\ \xi \end{pmatrix}, \begin{pmatrix} s \\ \eta \end{pmatrix} \right) &= d\lambda_t|_x(\xi, \eta) + \left(dt \wedge \dot\lambda_t|_x \right) \left(\begin{pmatrix} r \\ \xi \end{pmatrix}, \begin{pmatrix} s \\ \eta \end{pmatrix} \right) \\ &= d\lambda_t|_x(\xi, \eta) + \dot\lambda_t|_x \eta \cdot r - \dot\lambda_t|_x \xi \cdot s. \end{aligned}$$

A loop v in M induces a loop V in $\mathbb{S}^1 \times M$, and a vector field ξ along v induces a vector field Ξ along V, namely

$$V_t = \begin{pmatrix} t \\ v_t \end{pmatrix}, \qquad \dot V_t = \begin{pmatrix} 1 \\ \dot v_t \end{pmatrix}, \qquad \Xi = \begin{pmatrix} 0 \\ \xi \end{pmatrix}.$$

For later application we calculate

$$(V^* i_\Xi D\Lambda)_t = D\Lambda|_{(t,v_t)} \left(\begin{pmatrix} 0 \\ \xi \end{pmatrix}, \begin{pmatrix} 1 \\ \dot v_t \end{pmatrix} \right) dt = \left(d\lambda_t|_{v_t}(\xi, \dot v_t) - \dot\lambda_t|_{v_t}\xi \right) dt$$

using that exterior derivative and pull-back commute. Then

$$\begin{aligned} \mathcal{A}_{\Lambda,H}(V) &= \int_{\mathbb{S}^1} (V^*\Lambda - H \circ V) \\ &= \int_0^1 \left(\Lambda_{(t,v_t)} \begin{pmatrix} 1 \\ \dot v_t \end{pmatrix} - H(t, v_t) \right) dt \\ &= \int_0^1 (\lambda_t|_{v_t} \dot v_t - H(t, v_t))\, dt \\ &= \mathcal{A}(v). \end{aligned}$$

Pick a smooth map $\mathbb{R} \times \mathbb{S}^1 \to M$, $(\tau, t) \mapsto v_t^{(\tau)}$, with $v^{(0)} = v$ and $\frac{d}{d\tau}\big|_{\tau=0} v^{(\tau)} = \xi$. Hence $V_t^{(\tau)} := (t, v_t^{(\tau)})$ satisfies $V^{(0)} = V$ and $\frac{d}{d\tau}\big|_{\tau=0} V^{(\tau)} = (0, \xi) = \Xi$. Then by the previous displayed formula

$$\begin{aligned} d\mathcal{A}|_v \xi &= \tfrac{d}{d\tau}\big|_{\tau=0} \mathcal{A}(v^{(\tau)}) \\ &= \tfrac{d}{d\tau}\big|_{\tau=0} \mathcal{A}_{\Lambda,H}(V^{(\tau)}) \\ &= \int_{\mathbb{S}^1} \tfrac{d}{d\tau}\big|_{\tau=0} (V^{(\tau)})^*\Lambda - \int_0^1 \tfrac{d}{d\tau}\big|_{\tau=0} H(t, v_t^{(\tau)})\, dt \\ &\overset{4}{=} \int_{\mathbb{S}^1} V^* L_\Xi \Lambda - \int_0^1 dH_t|_{v_t}\xi_t\, dt \\ &\overset{5}{=} \int_{\mathbb{S}^1} (V^* D i_\Xi \Lambda + V^* i_\Xi D\Lambda) - \int_0^1 dH_t|_{v_t}\xi_t\, dt \\ &\overset{6}{=} \int_0^1 \left(d\lambda_t|_{v_t}(\xi_t, \dot v_t) - \dot\lambda_t|_{v_t}\xi_t - dH_t|_{v_t}\xi_t \right) dt \\ &= \int_0^1 \omega_t(\xi_t, \dot v_t - X_t(v_t) - Y_t(v_t))\, dt. \end{aligned}$$

Step 4 uses the definition of the Lie derivative (fisherman's derivative); see e.g. [Arn89, § 36 G]. Step 5 is Cartan's formula $L_\Xi = i_\Xi D + D i_\Xi$. Step 6 uses that summand one vanishes since $V^* D = D V^*$, now apply Stokes theorem und use that the boundary of $\mathbb{S}^1$ is empty. Step 6 also uses the previously prepared formula for summand two. This proves Theorem 5.5. □

5.1 Eliminating the Hamiltonian vector field

Proposition 5.6. *Let $(M, d\lambda_t)_{t\in\mathbb{R}}$ be a twisted-periodic exact symplectic manifold and $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ a smooth function. Then the family of 1-forms defined by*

$$\lambda_t^H := \lambda_t + \int_0^t dH_s\, ds$$

is twisted-periodic. Furthermore, the critical points of the functionals $\mathcal{A}_{\lambda,H}$ and $\mathcal{A}_{\lambda^H,0}$ coincide

$$\mathrm{Crit}\mathcal{A}_{\lambda,H} = \mathrm{Crit}\mathcal{A}_{\lambda^H,0}.$$

Proof. We verify the three conditions in (5.12): Condition one: We get that

$$\underline{d\lambda_t^H} = d\lambda_t + \int_0^1 ddH_s\, ds \underline{= d\lambda_t} \overset{(5.12))}{=} d\lambda_{t+1} = d\lambda_{t+1}^H$$

since $dd = 0$ and the final identity uses the underlined one at time $t+1$. To show condition two use the definition of λ_t^H to get the first and last identity in

$$\dot\lambda_t^H = \dot\lambda_t + dH_t = \dot\lambda_{t+1} + dH_{t+1} = \dot\lambda_{t+1}^H$$

where identity two uses periodicity of $\dot\lambda_t$ and of H_t. Condition three: Note that

$$\lambda_{t+1}^H - \lambda_t^H \overset{1}{=} \lambda_{t+1} - \lambda_t + \int_t^{t+1} dH_s\, ds \overset{2}{=} d(f_t + \bar H)$$

where equality 1 is by definition of λ_{t+1}^H and λ_t^H. Equality 2 uses the last hypothesis in (5.12) and the definition of the pointwise time-mean $\bar H := \int_t^{t+1} H_s\, ds$. This proves that the family λ_t^H is twisted-periodic.

To show that the critical points coincide it suffices to show that the (λ, H)- and the $(\lambda^H, 0)$-equation are equal, equivalently that the vector field identity

$$Y_t^H = Y_t + X_t, \qquad Y_t^H := Y_{\dot\lambda_t^H}^{d\lambda_t^H}, \quad Y_t := Y_{\dot\lambda_t}^{d\lambda_t}, \quad X_t := X_{H_t}^{d\lambda_t},$$

holds true. By definition (5.13) of the respective vector fields we get $1, 3$ in

$$d\lambda_t^H(\cdot, Y_t^H) \overset{1}{=} \dot\lambda_t^H \overset{2}{=} \dot\lambda_t + dH_t \overset{3}{=} d\lambda_t(\cdot, Y_t) + d\lambda_t(\cdot, X_t) = d\lambda_t(\cdot, Y_t + X_t).$$

Here equality 2 holds by definition of λ_t^H and the fundamental theorem of calculus. But $d\lambda_t^H = d\lambda_t$ since $dd = 0$. Now the equality $Y_t^H = Y_t + X_t$ follows by non-degeneracy of the symplectic form $d\lambda_t$. This proves Proposition 5.6. □

5.2 The Euler flow is symplectic

Let $(M, \lambda_t)_{t\in\mathbb{R}}$ be a twisted-periodic exact symplectic manifold. To simplify notation we assume in the following that the flow φ_Y^t of the Euler vector field Y_t globally exists, i.e. for any $t \in \mathbb{R}$ there is a diffeomorphism $\varphi_Y^t\colon M \to M$ such that $\varphi_Y^0 = \mathrm{id}_M$ and $\frac{d}{dt}\varphi_Y^t = Y_t \circ \varphi_Y^t$.

Proposition 5.7. *The Euler flow is symplectic, in symbols* $(\varphi_Y^t)^*\omega_0 = \omega_t$.

Proof. Since the inverse of φ_Y^t is φ_Y^{-t}, the identity

$$\omega_t = (\varphi_Y^t)^*\omega_0 = ((\varphi_Y^{-t})^{-1})^*\omega_0 = ((\varphi_Y^{-t})^*)^{-1}\omega_0$$

is equivalent to the identity $(\varphi_Y^{-t})^*\omega_t = \omega_0$. To prove this identity it suffices to show that the function $t \mapsto (\varphi_Y^{-t})^*\omega_t$ is constant since at time zero the value is $(\varphi_Y^0)^*\omega_0 = \mathrm{id}_M^*\omega_0 = \omega_0$. Indeed by Cartan's formula

$$L_{Y_t}\omega_t = di_{Y_t}\omega_t + i_{Y_t}d\omega_t = -d\dot{\lambda}_t = -\dot{\omega}_t$$

and then by the Leibniz rule the derivative vanishes

$$\tfrac{d}{dt}(\varphi_Y^{-t})^*\omega_t = (\varphi_Y^{-t})^*\bigl(-L_{Y_t}\omega_t\bigr) + (\varphi_Y^{-t})^*\dot{\omega}_t = 0.$$

This proves Proposition 5.7. □

5.3 Applying Cartan's formula on the loop space

In this section we give an alternative proof of Theorem 5.5 where we use Cartan's formula directly on the loop space. Since the loop space is infinite dimensional this alternative proof is not completely rigorous yet.

We first consider the following geometric setup. Assume that N is a manifold, $\Lambda \in \Omega^1(N)$ is a 1-form on N, and $\mathcal{V} \in \Gamma(TN)$ is a vector field on N. If N is finite dimensional the following discussion is completely rigorous. However, we want to apply the discussion below to the case where N is the loop space of a finite dimensional manifold. Plugging in the vector field into the 1-form we obtain a function

$$f := i_{\mathcal{V}}\Lambda\colon N \to \mathbb{R}.$$

By Cartan's formula the differential is given by

$$df = di_{\mathcal{V}}\Lambda = L_{\mathcal{V}}\Lambda - i_{\mathcal{V}}d\Lambda.$$

In the special case where $\Omega := d\Lambda$ is symplectic we can define a further vector field $\mathcal{Y}$ on N, called **Euler vector field**, by the requirement of equal 1-forms

$$i_{\mathcal{Y}}\Omega = L_{\mathcal{V}}\Lambda.$$

In this case the differential can be written as $df = i_{\mathcal{Y}-\mathcal{V}}\Omega$ and therefore critical points of f are points $z \in N$ satisfying the **abstract Euler equation**

$$\mathcal{V}(z) = \mathcal{Y}(z). \tag{5.15}$$

Example: Loop space

In the following we apply this observation to the loop space case $N = \mathcal{L}(M) := C^\infty(\mathbb{S}^1, M)$ with vector field and 1-form on $\mathcal{L}M$ given by

$$\mathcal{V}(z) = \dot{z} := \partial_t z, \qquad \Lambda := \int_0^1 \lambda_t \, dt,$$

where $\{\lambda_t\}_{t\in\mathbb{R}}$ is a twisted-periodic 1-form on M such that the periodic family $\omega_t := d\lambda_t$ consists of symplectic forms on M. The flow of $\mathcal{V} = \partial_t$ on $\mathcal{L}M$ is

$$\Phi^r_{\mathcal{V}} z = r_* z$$

where $(r_*z)(t) = z(t + r)$ for every time t. Hence the pull-back is given by $(\Phi^r_{\mathcal{V}})^*\Lambda = \int_0^1 \lambda_{t-r} \, dt$. The Lie derivative of Λ with respect to $\mathcal{V}$ is by definition

$$L_{\mathcal{V}}\Lambda := \tfrac{d}{dr}\big|_{r=0} (\Phi^r_{\mathcal{V}})^*\Lambda = -\int_0^1 \dot{\lambda}_t \, dt.$$

The exterior derivative of Λ is symplectic, namely

$$\Omega := d\Lambda = \int_0^1 \omega_t \, dt.$$

Therefore the Euler vector field localizes in the sense that

$$\omega_t|_{z_t} (\mathcal{Y}|_z(t), \cdot) = -\dot{\lambda}_t|_{z_t}.$$

Hence the loop space Euler vector field

$$\mathcal{Y}|_z(t) = Y_t|_{z_t}$$

coincides with the Euler vector field Y along M as defined by (5.13).

In particular, in this example the abstract Euler equation (5.15) gives rise to the manifold Euler equation

$$\dot{z}_t = Y_t|_{z_t}$$

as obtained earlier, see (5.14) with vanishing H.

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